Korean J. Math. **29** (2021), No. 4, pp. 715–723
http://dx.doi.org/10.11568/kjm.2021.29.4.715

SOME REMARKS ON THE GROWTH OF COMPOSITE p-ADIC ENTIRE FUNCTION

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Abstract. In this paper we wish to introduce the concept of generalized relative index-pair (α, β) of a p-adic entire function with respect to another p-adic entire function and then prove some results relating to the growth rates of composition of two p-adic entire functions with their corresponding left and right factors.

1. Introduction and preliminaries

Let us consider an algebraically closed field $\mathbb{K}$ of characteristic zero complete with respect to a p-adic absolute value $|\cdot|$ (example $\mathbb{C}_p$). For any $\Lambda \in \mathbb{K}$ and $R \in]0, +\infty[$, the closed disk $\{x \in \mathbb{K} : |x - \Lambda| \leq R\}$ and the open disk $\{x \in \mathbb{K} : |x - \Lambda| < R\}$ are denoted by $d(\Lambda, R)$ and $d(\Lambda, R^-)$ respectively. Also $C(\Lambda, r)$ denotes the circle $\{x \in \mathbb{K} : |x - \Lambda| = r\}$. Moreover $\mathcal{A}(\mathbb{K})$ represent the $\mathbb{K}$-algebra of analytic functions in $\mathbb{K}$ i.e., the set of power series with an infinite radius of convergence. For the most comprehensive study of analytic functions inside a disk or in the whole field $\mathbb{K}$, we refer the reader to the books [17–19, 22]. During the last several years the ideas of p-adic analysis have been studied from different aspects and many important results were gained (see [3] to [16]).

Let $f \in \mathcal{A}(\mathbb{K})$ and $r > 0$, then we denote by $|f|(r)$ the number $\sup\{|f(x)| : |x| = r\}$ where $|\cdot|(r)$ is a multiplicative norm on $\mathcal{A}(\mathbb{K})$. Moreover, if f is not a constant, the $|f|(r)$ is strictly increasing function of r and tends to $+\infty$ with r, therefore there exists its inverse function $\widehat{|f|} : (|f(0)|, \infty) \to (0, \infty)$ with $\lim\limits_{s\to\infty} \widehat{|f|}(s) = \infty$.

For $x \in [0, \infty)$ and $k \in \mathbb{N}$, we define $\log^{[k]} x = \log\left(\log^{[k-1]} x\right)$ and $\exp^{[k]} x = \exp\left(\exp^{[k-1]} x\right)$ where $\mathbb{N}$ is the set of all positive integers. We also denote $\log^{[0]} x = x$ and $\exp^{[0]} x = x$. Throughout the paper, log denotes the Neperian logarithm. Taking this into account the (p, q)-th order and (p, q)-th lower order of an entire function $f \in \mathcal{A}(\mathbb{K})$ are defined as follows:

Definition 1.1. [7] Let $f \in \mathcal{A}(\mathbb{K})$ and p, q be two positive integers. Then the (p, q)-th order $\varrho^{(p,q)}(f)$ and (p, q)-th lower order $\lambda^{(p,q)}(f)$ of f are respectively defined

Received July 17, 2021. Revised September 12, 2021. Accepted September 24, 2021.
2010 Mathematics Subject Classification: 12J25, 30D35, 30G06, 46S10.
Key words and phrases: p-adic entire function, growth, composition, generalized relative order (α, β), generalized relative index-pair (α, β).
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as:

$$\varrho^{(p,q)}(f) = \limsup_{r\to+\infty} \frac{\log^{[p]}|f|(r)}{\log^{[q]} r} \text{ and } \lambda^{(p,q)}(f) = \liminf_{r\to+\infty} \frac{\log^{[p]}|f|(r)}{\log^{[q]} r}.$$

Definition 1.1 avoids the restriction $p \geq q$ of the original definition of (p,q)-th order (respectively (p,q)-th lower order) of entire functions introduced by Juneja et al. [21] in complex context.

When $q = 1$, we get the definitions of generalized order and generalized lower order of an entire function $f \in \mathcal{A}(\mathbb{K})$ which symbolize as $\varrho^{(p)}(f)$ and $\lambda^{(p)}(f)$ respectively. If $p = 2$ and $q = 1$ then we write $\varrho^{(2,1)}(f) = \varrho(f)$ and $\lambda^{(2,1)}(f) = \lambda(f)$ where $\varrho(f)$ and $\lambda(f)$ are respectively known as order and lower order of $f \in \mathcal{A}(\mathbb{K})$ introduced by Boussaf et al. [13].

Now let L be a class of continuous non-negative functions α defined on $(-\infty,+\infty)$ such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ with $\alpha(x) \uparrow +\infty$ as $x \to +\infty$ and $\alpha((1+o(1))x) = (1+o(1))\alpha(x)$ as $x \to +\infty$. We say that $\alpha \in L^0$, if $\alpha \in L$ and $\alpha(cx) = (1+o(1))\alpha(x)$ as $x_0 \leq x \to +\infty$ for each $c \in (0,+\infty)$, i.e., α is slowly increasing function. Clearly $L^0 \subset L$.

The concept of generalized order (α,β) of entire function in complex context was introduced by Sheremeta [23] where $\alpha,\beta \in L$. In complex context, several authors made close investigations on the properties of entire functions related to generalized order (α,β) in some different direction. For the purpose of further applications of generalized order (α,β) of entire function in complex context, Biswas et al. [4,7] rewrite the definition of generalized order (α,β) of an entire function considering $\alpha,\beta \in L^0$. For details about generalized order (α,β) and generalized lower order (α,β), one may see [4,7]. Considering the ideas developed by Biswas et al. [4,7], one can define the generalized order (α,β) and generalized lower order (α,β) of an entire function $f \in \mathcal{A}(\mathbb{K})$ respectively in the following way:

Definition 1.2. [2] Let $f \in \mathcal{A}(\mathbb{K})$ and $\alpha,\ \beta \in L^0$. The generalized order (α,β) and generalized lower order (α,β) of f denoted by $\varrho_{(\alpha,\beta)}[f]$ and $\lambda_{(\alpha,\beta)}[f]$ respectively are defined as:

$$\varrho_{(\alpha,\beta)}[f] = \limsup_{r\to+\infty} \frac{\alpha(|f|(r))}{\beta(r)} \text{ and } \lambda_{(\alpha,\beta)}[f] = \liminf_{r\to+\infty} \frac{\alpha(|f|(r))}{\beta(r)}.$$

If $\alpha(r) = \log^{[p]} r$ and $\beta(r) = \log^{[q]} r$, then Definition 1.1 is a special case of Definition 1.2.

In this connection one may give the following definition:

Definition 1.3. An entire function $f \in \mathcal{A}(\mathbb{K})$ is said to have generalized index-pair (α,β) if $b < \varrho_{(\alpha,\beta)}[f] < \infty$ and $\varrho_{(\exp\alpha,\exp\beta)}[f]$ is not a non-zero finite number, where $b = 1$ if $\alpha = \beta$ and $b = 0$ for otherwise. Moreover if $0 < \varrho_{(\alpha,\beta)}[f] < +\infty$, then

$$\begin{cases} \varrho_{(\alpha(\gamma_1^{-1}),\ \beta)}[f] = \infty \quad \text{when } \alpha\left(\gamma_1^{-1}\right) \in L^0 \text{ and } \lim\limits_{r\to+\infty} \frac{\alpha(\gamma_1^{-1}(r))}{\alpha(r)} = +\infty, \\ \varrho_{(\alpha,\ \beta(\gamma_1^{-1}))}[f] = 0 \quad \text{when } \beta\left(\gamma_1^{-1}\right) \in L^0 \text{ and } \lim\limits_{r\to+\infty} \frac{\beta(\gamma_1^{-1}(r))}{\beta(r)} = +\infty, \\ \varrho_{(\alpha(\gamma_1),\ \beta(\gamma_1))}[f] = 1 \quad \text{when } \lim\limits_{r\to+\infty} \frac{\alpha(\gamma_1(r))}{\alpha(r)} = 0 \text{ and } \lim\limits_{r\to+\infty} \frac{\beta(\gamma_1(r))}{\beta(r)} = 0. \end{cases}$$

Similarly for $0 < \lambda_{(\alpha,\beta)}[f] < +\infty$, one can easily verify that

$$\begin{cases} \lambda_{(\alpha(\gamma_1^{-1}),\,\beta)}[f] = \infty \quad \text{when } \alpha\left(\gamma_1^{-1}\right) \in L^0 \text{ and } \lim\limits_{r\to+\infty} \frac{\alpha(\gamma_1^{-1}(r))}{\alpha(r)} = +\infty, \\ \lambda_{(\alpha,\,\beta(\gamma_1^{-1}))}[f] = 0 \quad \text{when } \beta\left(\gamma_1^{-1}\right) \in L^0 \text{ and } \lim\limits_{r\to+\infty} \frac{\beta(\gamma_1^{-1}(r))}{\beta(r)} = +\infty, \\ \lambda_{(\alpha(\gamma_1),\,\beta(\gamma_1))}[f] = 1 \quad \text{when } \lim\limits_{r\to+\infty} \frac{\alpha(\gamma_1(r))}{\alpha(r)} = 0 \text{ and } \lim\limits_{r\to+\infty} \frac{\beta(\gamma_1(r))}{\beta(r)} = 0. \end{cases}$$

Definition 1.3 extends the definition of index-pair (p, q) of p-adic entire function f introduced by Biswas [7] which is analogous to a definition of Juneja et al. [20, 21] in complex context. For details about index-pair (p, q) of p-adic entire function f, one may see [7].

The notion of relative order was first introduced by Bernal [1]. In order to make some progress in the study of p-adic analysis, Biswas [6] has introduced the definitions of relative order $\varrho_g(f)$ and relative lower order $\lambda_g(f)$ of entire function $f \in \mathcal{A}(\mathbb{K})$ with respect to another entire function $g \in \mathcal{A}(\mathbb{K})$ in the following way:

$$\varrho_g(f) = \limsup_{r\to+\infty} \frac{\log \widehat{|g|}\left(|f|(r)\right)}{\log r} \text{ and } \lambda_g(f) = \liminf_{r\to+\infty} \frac{\log \widehat{|g|}\left(|f|(r)\right)}{\log r}.$$

In the case of relative order, it therefore seems reasonable to define suitably the generalized relative order (α, β) of entire function belonging to $\mathcal{A}(\mathbb{K})$. With this in view one may introduce the definitions of generalized relative order (α, β) and generalized relative lower order (α, β) of an entire function $f \in \mathcal{A}(\mathbb{K})$ with respect to another entire function $g \in \mathcal{A}(\mathbb{K})$ denoted by $\varrho_{(\alpha,\beta)}[f]_g$ and $\lambda_{(\alpha,\beta)}[f]_g$ respectively, in the follows way:

Definition 1.4. Let $f,\ g \in \mathcal{A}(\mathbb{K})$ and $\alpha,\ \beta \in L^0$. The generalized relative order (α, β) and generalized relative lower order (α, β) of f with respect to g denoted by $\varrho_{(\alpha,\beta)}[f]_g$ and $\lambda_{(\alpha,\beta)}[f]_g$ respectively are defined as:

$$\varrho_{(\alpha,\beta)}[f]_g = \limsup_{r\to+\infty} \frac{\alpha(\widehat{|g|}(|f|(r)))}{\beta(r)} \text{ and } \lambda_{(\alpha,\beta)}[f]_g = \liminf_{r\to+\infty} \frac{\alpha(\widehat{|g|}(|f|(r)))}{\beta(r)}.$$

Now we introduce the following definition which will be needed in the sequel:

Definition 1.5. Let $f,\ g \in \mathcal{A}(\mathbb{K})$. f is said to have generalized relative index-pair (α, β) with respect to g, if $b < \varrho_{(\alpha,\beta)}[f]_g < \infty$ and $\varrho_{(\exp\alpha,\exp\beta)}[f]_g$ is not a non-zero finite number, where $b = 1$ if $\alpha = \beta$ and $b = 0$ for otherwise. Moreover if $0 < \varrho_{(\alpha,\beta)}[f]_g < +\infty$, then

$$\begin{cases} \varrho_{(\alpha(\gamma_1^{-1}),\,\beta)}[f]_g = \infty \quad \text{when } \alpha\left(\gamma_1^{-1}\right) \in L^0 \text{ and } \lim\limits_{r\to+\infty} \frac{\alpha(\gamma_1^{-1}(r))}{\alpha(r)} = +\infty, \\ \varrho_{(\alpha,\,\beta(\gamma_1^{-1}))}[f]_g = 0 \quad \text{when } \beta\left(\gamma_1^{-1}\right) \in L^0 \text{ and } \lim\limits_{r\to+\infty} \frac{\beta(\gamma_1^{-1}(r))}{\beta(r)} = +\infty, \\ \varrho_{(\alpha(\gamma_1),\,\beta(\gamma_1))}[f]_g = 1 \quad \text{when } \lim\limits_{r\to+\infty} \frac{\alpha(\gamma_1(r))}{\alpha(r)} = 0 \text{ and } \lim\limits_{r\to+\infty} \frac{\beta(\gamma_1(r))}{\beta(r)} = 0. \end{cases}$$

Similarly for $0 < \lambda_{(\alpha,\beta)}[f]_g < +\infty$, one can easily verify that

$$\begin{cases} \lambda_{(\alpha(\gamma_1^{-1}),\,\beta)}[f]_g = \infty \quad \text{when } \alpha\left(\gamma_1^{-1}\right) \in L^0 \text{ and } \lim\limits_{r\to+\infty} \frac{\alpha(\gamma_1^{-1}(r))}{\alpha(r)} = +\infty, \\ \lambda_{(\alpha,\,\beta(\gamma_1^{-1}))}[f]_g = 0 \quad \text{when } \beta\left(\gamma_1^{-1}\right) \in L^0 \text{ and } \lim\limits_{r\to+\infty} \frac{\beta(\gamma_1^{-1}(r))}{\beta(r)} = +\infty, \\ \lambda_{(\alpha(\gamma_1),\,\beta(\gamma_1))}[f]_g = 1 \quad \text{when } \lim\limits_{r\to+\infty} \frac{\alpha(\gamma_1(r))}{\alpha(r)} = 0 \text{ and } \lim\limits_{r\to+\infty} \frac{\beta(\gamma_1(r))}{\beta(r)} = 0. \end{cases}$$

In this paper we wish to prove some results relating to the growth rates of composition of two p-adic entire functions with their corresponding left and right factors on the basis of their generalized relative order (α,β) and generalized relative lower order (α,β) where α, $\beta \in L^0$. Further we assume that throughout the present paper α_1, α_2, β_1, and β_2 always denote the functions belonging to L^0.

2. Results

First of all, we recall one related known property which can be found in [12] or [13] and will be needed in order to prove our results, as we see in the following lemma:

LEMMA 2.1. *Let f, $g \in \mathcal{A}(\mathbb{K})$. Then for all sufficiently large positive numbers of r the following equality holds*

$$|f(g)|(r) = |f|(|g|(r)).$$

We now prove

THEOREM 2.2. *Let f, g, $h \in \mathcal{A}(\mathbb{K})$. Also let the generalized relative index-pair of f with respect to h be (α_1,β_1) and the generalized index-pair of g be (α_2,β_2). Then* ***(i)*** *the generalized relative index-pair of $f(g)$ is (α_1,β_2) when $\beta_1(r) = \alpha_2(r)$ and either $\lambda_{(\alpha_1,\beta_1)}[f]_h > 0$ or $\lambda_{(\alpha_2,\beta_2)}[g] > 0$. Also*

$$\begin{aligned} (a)\ \lambda_{(\alpha_1,\beta_1)}[f]_h\varrho_{(\alpha_2,\beta_2)}[g] &\le \varrho_{(\alpha_1,\beta_2)}[f(g)]_h \leqslant \varrho_{(\alpha_1,\beta_1)}[f]_h\varrho_{(\alpha_2,\beta_2)}[g] \\ \text{if } \lambda_{(\alpha_1,\beta_1)}[f] &> 0 \text{ and} \\ (b)\ \lambda_{(\alpha_1,\beta_1)}[f]_h\varrho_{(\alpha_2,\beta_2)}[g] &\le \varrho_{(\alpha_1,\beta_2)}[f(g)]_h \leqslant \varrho_{(\alpha_1,\beta_1)}[f]_h\varrho_{(\alpha_2,\beta_2)}[g] \\ \text{if } \lambda_{(\alpha_2,\beta_2)}[g] &> 0; \end{aligned}$$

(ii) *the generalized relative index-pair of $f(g)$ is $(\alpha_1,\beta_1(\alpha_2^{-1}(\beta_2)))$ when $\beta_1(\alpha_2^{-1}(r)) \in L^0$ and either $\lambda_{(\alpha_1,\beta_1)}[f]_h > 0$ or $\lambda_{(\alpha_2,\beta_2)}[g] > 0$. Also*

(a) $\lambda_{(\alpha_1,\beta_1)}[f]_h \le \varrho_{(\alpha_1,\ \beta_1(\alpha_2^{-1}(\beta_2)))}[f(g)]_h \leqslant \varrho_{(\alpha_1,\beta_1)}[f]_h$ *if* $\lambda_{(\alpha_1,\beta_1)}[f]_h > 0$ *and*

(b) $\varrho_{(\alpha_1,\ \beta_1(\alpha_2^{-1}(\beta_2)))}[f(g)]_h = \varrho_{(\alpha_1,\beta_1)}[f]_h$ *if* $\lambda_{(\alpha_2,\beta_2)}[g] > 0$;

(iii) *the generalized relative index-pair of $f(g)$ is $(\alpha_2(\beta_1^{-1}(\alpha_1)),\beta_2)$ when $\alpha_2(\beta_1^{-1}(r)) \in L^0$ and either $\lambda_{(\alpha_1,\beta_1)}[f]_h > 0$ or $\lambda_{(\alpha_2,\beta_2)}[g] > 0$. Also*

(a) $\varrho_{(\alpha_2(\beta_1^{-1}(\alpha_1)),\ \beta_2)}[f(g)]_h = \varrho_{(\alpha_2,\beta_2)}[g]$ *if* $\lambda_{(\alpha_1,\beta_1)}[f]_h > 0$ *and*

(b) $\lambda_{(\alpha_2,\beta_2)}[g] \le \varrho_{(\alpha_2(\beta_1^{-1}(\alpha_1)),\ \beta_2)}[f(g)]_h \leqslant \varrho_{(\alpha_2,\beta_2)}[g]$ *if* $\lambda_{(\alpha_2,\beta_2)}[g] > 0$.

Proof. Since $\widehat{|h|}(r)$ is an increasing function of r, it follows from Lemma 2.1 and for all sufficiently large values r that

$$\alpha_1(\widehat{|h|}(|f(g)|(r))) \ge \left(\lambda_{(\alpha_1,\beta_1)}[f]_h - \varepsilon\right)\beta_1(|g|(r)) \tag{1}$$

and also for a sequence of values of r tending to infinity that

$$\alpha_1(\widehat{|h|}(|f(g)|(r))) \ge \left(\varrho_{(\alpha_1,\beta_1)}[f]_h - \varepsilon\right))\beta_1(|g|(r)). \tag{2}$$

Similarly, in view of Lemma 2.1, we have for all sufficiently large values of r that

$$\alpha_1(\widehat{|h|}(|f(g)|(r))) \leqslant \left(\varrho_{(\alpha_1,\beta_1)}[f]_h + \varepsilon\right)\beta_1(|g|(r)). \tag{3}$$

Now the following two cases may arise:
Case I. Let $\beta_1(r) = \alpha_2(r)$.
Now we have from (3) for all sufficiently large values of r that

$$\alpha_1(\widehat{|h|}(|f(g)|(r))) \leqslant \left(\varrho_{(\alpha_1,\beta_1)}[f]_h + \varepsilon\right)\left(\varrho_{(\alpha_2,\beta_2)}[g] + \varepsilon\right)\beta_2(r)$$

$$\text{(4)} \qquad i.e.,\ \limsup_{r\to+\infty} \frac{\alpha_1(\widehat{|h|}(|f(g)|(r)))}{\beta_2(r)} \leqslant \varrho_{(\alpha_1,\beta_1)}[f]_h \varrho_{(\alpha_2,\beta_2)}[g].$$

Also from (1), we obtain for a sequence of values of r tending to infinity that

$$\alpha_1(\widehat{|h|}(|f(g)|(r))) \geq \left(\lambda_{(\alpha_1,\beta_1)}[f]_h - \varepsilon\right)\left(\varrho_{(\alpha_2,\beta_2)}[g] - \varepsilon\right)\beta_2(r)$$

$$\text{(5)} \qquad i.e.,\ \limsup_{r\to+\infty} \frac{\alpha_1(\widehat{|h|}(|f(g)|(r)))}{\beta_2(r)} \geq \lambda_{(\alpha_1,\beta_1)}[f]_h \varrho_{(\alpha_2,\beta_2)}[g].$$

Moreover, we have from (2) for a sequence of values of r tending to infinity that

$$\alpha_1(\widehat{|h|}(|f(g)|(r))) \geq \left(\varrho_{(\alpha_1,\beta_1)}[f]_h - \varepsilon\right)\left(\lambda_{(\alpha_2,\beta_2)}[g] - \varepsilon\right)\beta_2(r)$$

$$\text{(6)} \qquad i.e.,\ \limsup_{r\to+\infty} \frac{\alpha_1(\widehat{|h|}(|f(g)|(r)))}{\beta_2(r)} \geq \varrho_{(\alpha_1,\beta_1)}[f]_h \lambda_{(\alpha_2,\beta_2)}[g].$$

Therefore from (4) and (5), we get for $\lambda_{(\alpha_1,\beta_1)}[f]_h > 0$ that

$$\lambda_{(\alpha_1,\beta_1)}[f]_h \varrho_{(\alpha_2,\beta_2)}[g] \leq \limsup_{r\to+\infty} \frac{\alpha_1(\widehat{|h|}(|f(g)|(r)))}{\beta_2(r)} \leqslant \varrho_{(\alpha_1,\beta_1)}[f]_h \varrho_{(\alpha_2,\beta_2)}[g]$$

$$\text{(7)} \qquad i.e.,\ \lambda_{(\alpha_1,\beta_1)}[f]_h \varrho_{(\alpha_2,\beta_2)}[g] \leq \varrho_{(\alpha_1,\beta_2)}[f(g)]_h \leqslant \varrho_{(\alpha_1,\beta_1)}[f]_h \varrho_{(\alpha_2,\beta_2)}[g].$$

Likewise, from (4) and (6), we obtain for $\lambda_{(\alpha_2,\beta_2)}[g] > 0$ that

$$\varrho_{(\alpha_1,\beta_1)}[f]_h \lambda_{(\alpha_2,\beta_2)}[g] \leq \limsup_{r\to+\infty} \frac{\alpha_1(\widehat{|h|}(|f(g)|(r)))}{\beta_2(r)} \leqslant \varrho_{(\alpha_1,\beta_1)}[f]_h \varrho_{(\alpha_2,\beta_2)}[g]$$

$$\text{(8)} \qquad i.e.,\ \varrho_{(\alpha_1,\beta_1)}[f]_h \lambda_{(\alpha_2,\beta_2)}[g] \leq \varrho_{(\alpha_1,\beta_2)}[f(g)]_h \leqslant \varrho_{(\alpha_1,\beta_1)}[f]_h \varrho_{(\alpha_2,\beta_2)}[g].$$

Also from (7) and (8) one can easily verify that

(i) $\varrho_{(\alpha_1(\gamma_1^{-1}),\, \beta_2)}[f(g)]_h = \infty$
when $\alpha_1(\gamma_1^{-1}) \in L^0$ and $\lim\limits_{r\to+\infty} \frac{\alpha_1(\gamma_1^{-1}(r))}{\alpha_1(r)} = +\infty$,
(ii) $\varrho_{(\alpha_1,\, \beta_2(\gamma_1^{-1}))}[f(g)]_h = 0$
when $\beta_2\left(\gamma_1^{-1}\right) \in L^0$ and $\lim\limits_{r\to+\infty} \frac{\beta_2(\gamma_1^{-1}(r))}{\beta_2(r)} = +\infty$ and
(iii) $\varrho_{(\alpha_1(\gamma_1),\, \beta_2(\gamma_1))}[f(g)]_h = 1$
when $\lim\limits_{r\to+\infty} \frac{\alpha_1(\gamma_1(r))}{\alpha_1(r)} = 0$ and $\lim\limits_{r\to+\infty} \frac{\beta_2(\gamma_1(r))}{\beta_2(r)} = 0$.

Therefore we obtain that the generalized relative index-pair of $f(g)$ is (α_1, β_2) when $\beta_1(r) = \alpha_2(r)$ and either $\lambda_{(\alpha_1,\beta_1)}[f]_h > 0$ or $\lambda_{(\alpha_2,\beta_2)}[g] > 0$ and thus the first part of the theorem is established.
Case II. Let $\beta_1(\alpha_2^{-1}(r)) \in L^0$.
Now we obtain from (3) for all sufficiently large values of r that

$$\alpha_1(\widehat{|h|}(|f(g)|(r))) \leqslant \left(\varrho_{(\alpha_1,\beta_1)}[f]_h + \varepsilon\right)\beta_1\left(\alpha_2^{-1}\left(\alpha_2\left(|g|\,(r)\right)\right)\right)$$

$$i.e.,\ \alpha_1(\widehat{|h|}(|f(g)|(r))) \leqslant (1+o(1))\left(\varrho_{(\alpha_1,\beta_1)}[f]_h+\varepsilon\right)\beta_1\left(\alpha_2^{-1}\left(\beta_2\left(r\right)\right)\right)$$

$$i.e.,\ \alpha_1(\widehat{|h|}(|f(g)|(r))) \leqslant (1+o(1))\left(\varrho_{(\alpha_1,\beta_1)}[f]_h+\varepsilon\right)\beta_1\left(\alpha_2^{-1}\left(\beta_2\left(r\right)\right)\right)$$

$$(9)\qquad i.e.,\ \lim_{r\to+\infty}\frac{\alpha_1(\widehat{|h|}(|f(g)|(r)))}{\beta_1\left(\alpha_2^{-1}\left(\beta_2\left(r\right)\right)\right)} \leqslant \varrho_{(\alpha_1,\beta_1)}[f]_h.$$

Also from (1), we have for a sequence of values of r tending to infinity that

$$\alpha_1(\widehat{|h|}(|f(g)|(r))) \geq \left(\lambda_{(\alpha_1,\beta_1)}[f]_h-\varepsilon\right)\beta_1(\alpha_2^{-1}(\left(\varrho_{(\alpha_2,\beta_2)}[g]+\varepsilon\right)\beta_2(r)))$$

$$i.e.,\ \alpha_1(\widehat{|h|}(|f(g)|(r))) \geq (1+o(1))\left(\lambda_{(\alpha_1,\beta_1)}[f]_h-\varepsilon\right)\beta_1\left(\alpha_2^{-1}\left(\beta_2\left(r\right)\right)\right)$$

$$(10)\qquad i.e.,\ \limsup_{r\to+\infty}\frac{\alpha_1(\widehat{|h|}(|f(g)|(r)))}{\beta_1\left(\alpha_2^{-1}\left(\beta_2\left(r\right)\right)\right)} \geq \lambda_{(\alpha_1,\beta_1)}[f]_h.$$

Further, we get from (2) for a sequence of values of r tending to infinity that

$$\alpha_1(\widehat{|h|}(|f(g)|(r))) \geq \left(\varrho_{(\alpha_1,\beta_1)}[f]_h-\varepsilon\right)\beta_1(\alpha_2^{-1}(\left(\lambda_{(\alpha_2,\beta_2)}[g]-\varepsilon\right)\beta_2(r)))$$

$$i.e.,\ \alpha_1(\widehat{|h|}(|f(g)|(r))) \geq (1+o(1))\left(\varrho_{(\alpha_1,\beta_1)}[f]_h-\varepsilon\right)\beta_1\left(\alpha_2^{-1}\left(\beta_2\left(r\right)\right)\right)$$

$$(11)\qquad i.e.,\ \limsup_{r\to+\infty}\frac{\alpha_1(\widehat{|h|}(|f(g)|(r)))}{\beta_1\left(\alpha_2^{-1}\left(\beta_2\left(r\right)\right)\right)} \geq \varrho_{(\alpha_1,\beta_1)}[f]_h.$$

Therefore from (9) and (10), we get for $\lambda_{(\alpha_1,\beta_1)}[f]_h > 0$ that

$$\lambda_{(\alpha_1,\beta_1)}[f]_h \leq \limsup_{r\to+\infty}\frac{\alpha_1(\widehat{|h|}(|f(g)|(r)))}{\beta_1\left(\alpha_2^{-1}\left(\beta_2\left(r\right)\right)\right)} \leqslant \varrho_{(\alpha_1,\beta_1)}[f]_h$$

$$(12)\qquad i.e.,\ \lambda_{(\alpha_1,\beta_1)}[f]_h \leq \varrho_{\left(\alpha_1,\beta_1\left(\alpha_2^{-1}(\beta_2)\right)\right)}\left[f(g)\right]_h \leqslant \varrho_{(\alpha_1,\beta_1)}[f]_h.$$

Likewise, from (9) and (11) we get for $\lambda_{(\alpha_2,\beta_2)}[g] > 0$ that

$$\varrho_{(\alpha_1,\beta_1)}[f]_h \leq \limsup_{r\to+\infty}\frac{\alpha_1(\widehat{|h|}(|f(g)|(r)))}{\beta_1\left(\alpha_2^{-1}\left(\beta_2\left(r\right)\right)\right)} \leqslant \varrho_{(\alpha_1,\beta_1)}[f]_h$$

$$(13)\qquad i.e.,\ \varrho_{\left(\alpha_1,\beta_1\left(\alpha_2^{-1}(\beta_2)\right)\right)}\left[f(g)\right]_h = \varrho_{(\alpha_1,\beta_1)}[f]_h.$$

Further from (12) and (13) one can easily verify that

(i) $\varrho_{(\alpha_1(\gamma_1^{-1}),\ \beta_1(\alpha_2^{-1}(\beta_2)))}[f(g)]_h = \infty$
when $\alpha_1(\gamma_1^{-1}) \in L^0$ and $\lim_{r\to+\infty}\frac{\alpha_1(\gamma_1^{-1}(r))}{\alpha_1(r)} = +\infty$,
(ii) $\varrho_{(\alpha_1,\ \beta_1(\alpha_2^{-1}(\beta_2(\gamma_1^{-1}))))}[f(g)]_h = 0$
when $\beta_1(\alpha_2^{-1}(\beta_2(\gamma_1^{-1}))) \in L^0$ and $\lim_{r\to+\infty}\frac{\beta_1(\alpha_2^{-1}(\beta_2(\gamma_1^{-1}(r))))}{\beta_1(\alpha_2^{-1}(\beta_2(r)))} = +\infty$ and
(iii) $\varrho_{(\alpha_1(\gamma_1),\ \beta_1(\alpha_2^{-1}(\beta_2(\gamma_1))))}[f(g)]_h = 1$
when $\lim_{r\to+\infty}\frac{\alpha_1(\gamma_1(r))}{\alpha_1(r)} = 0$ and $\lim_{r\to+\infty}\frac{\beta_1(\alpha_2^{-1}(\beta_2(\gamma_1(r))))}{\beta_1(\alpha_2^{-1}(\beta_2(r)))} = 0$.

Therefore we get that the generalized index-pair of $f(g)$ is $\left(\alpha_1,\beta_1\left(\alpha_2^{-1}\left(\beta_2\right)\right)\right)$ when $\beta_1\left(\alpha_2^{-1}\left(r\right)\right) \in L^0$ and either $\lambda_{(\alpha_1,\beta_1)}[f]_h > 0$ or $\lambda_{(\alpha_2,\beta_2)}[g] > 0$ and thus the second part of the theorem follows.

Case III. Let $\alpha_2(\beta_1^{-1}(r)) \in L^0$
Then we obtain from (3) for all sufficiently large values of r that

$$\alpha_2(\beta_1^{-1}(\alpha_1(\widehat{|h|}(|f(g)|(r))))) \leq \alpha_2\left(\beta_1^{-1}\left(\left(\varrho_{(\alpha_1,\beta_1)}[f]_h + \varepsilon\right)\beta_1(|g|(r))\right)\right)$$

$$i.e.,\ \alpha_2(\beta_1^{-1}(\alpha_1(\widehat{|h|}(|f(g)|(r))))) \leqslant (1+o(1))\,\alpha_2(|g|(r))$$

$$i.e.,\ \alpha_2(\beta_1^{-1}(\alpha_1(\widehat{|h|}(|f(g)|(r))))) \leqslant (1+o(1))\left(\varrho_{(\alpha_2,\beta_2)}[g] + \varepsilon\right)\beta_2(r)$$

$$i.e.,\ \limsup_{r\to+\infty}\frac{\alpha_2(\beta_1^{-1}(\alpha_1(\widehat{|h|}(|f(g)|(r)))))}{\beta_2(r)} \leqslant \varrho_{(\alpha_2,\beta_2)}[g]. \tag{14}$$

Also from (1) we have for a sequence of values of r tending to infinity that

$$\alpha_2(\beta_1^{-1}(\alpha_1(\widehat{|h|}(|f(g)|(r))))) \geq (1+o(1))\,\alpha_2(|g|(r))$$

$$i.e.,\ \alpha_2(\beta_1^{-1}(\alpha_1(\widehat{|h|}(|f(g)|(r))))) \geq (1+o(1))\left(\varrho_{(\alpha_2,\beta_2)}[g] - \varepsilon\right)\beta_2(r)$$

$$i.e.,\ \limsup_{r\to+\infty}\frac{\alpha_2(\beta_1^{-1}(\alpha_1(\widehat{|h|}(|f(g)|(r)))))}{\beta_2(r)} \geq \varrho_{(\alpha_2,\beta_2)}[g]. \tag{15}$$

Similarly, we get from (2) for a sequence of values of r tending to infinity that

$$\alpha_2(\beta_1^{-1}(\alpha_1(\widehat{|h|}(|f(g)|(r))))) \geq (1+o(1))\left(\lambda_{(\alpha_2,\beta_2)}[g] - \varepsilon\right)\beta_2(r)$$

$$i.e.,\ \limsup_{r\to+\infty}\frac{\alpha_2(\beta_1^{-1}(\alpha_1(\widehat{|h|}(|f(g)|(r)))))}{\beta_2(r)} \geq \lambda_{(\alpha_2,\beta_2)}[g]. \tag{16}$$

Therefore from (14) and (15), we obtain for $\lambda_{(\alpha_1,\beta_1)}[f]_h > 0$ that

$$\varrho_{(\alpha_2,\beta_2)}[g] \leq \limsup_{r\to+\infty}\frac{\alpha_2\left(\beta_1^{-1}\left(\alpha_1\left(\widehat{|h|}(|f(g)|(r))\right)\right)\right)}{\beta_2(r)} \leqslant \varrho_{(\alpha_2,\beta_2)}[g]$$

$$i.e.,\ \varrho_{\left(\alpha_2\left(\beta_1^{-1}(\alpha_1)\right),\beta_2\right)}[f(g)]_h = \varrho_{(\alpha_2,\beta_2)}[g]. \tag{17}$$

Similarly, from (14) and (16) we get for $\lambda_{(\alpha_2,\beta_2)}[g] > 0$ that

$$\lambda_{(\alpha_2,\beta_2)}[g] \leq \limsup_{r\to+\infty}\frac{\alpha_2(\beta_1^{-1}(\alpha_1(\widehat{|h|}(|f(g)|(r)))))}{\beta_2(r)} \leqslant \varrho_{(\alpha_2,\beta_2)}[g]$$

$$i.e.,\ \lambda_{(\alpha_2,\beta_2)}[g] \leq \varrho_{\left(\alpha_2\left(\beta_1^{-1}(\alpha_1)\right),\beta_2\right)}[f(g)]_h \leqslant \varrho_{(\alpha_2,\beta_2)}[g]. \tag{18}$$

So from (17) and (18) one can easily verify that

(i) $\varrho_{(\alpha_2(\beta_1^{-1}(\alpha_1(\gamma_1^{-1}))),\ \beta_2)}[f(g)]_h = \infty$
when $\alpha_2(\beta_1^{-1}(\alpha_1(\gamma_1^{-1}))) \in L^0$ and $\lim_{r\to+\infty}\frac{\alpha_2(\beta_1^{-1}(\alpha_1(\gamma_1^{-1}(r))))}{\alpha_2(\beta_1^{-1}(\alpha_1(r)))} = +\infty$,
(ii) $\varrho_{(\alpha_2(\beta_1^{-1}(\alpha_1)),\ \beta_2(\gamma_1^{-1}))}[f(g)]_h = 0$
when $\beta_2\left(\gamma_1^{-1}\right) \in L^0$ and $\lim_{r\to+\infty}\frac{\beta_2\left(\gamma_1^{-1}(r)\right)}{\beta_2(r)} = +\infty$ and
(iii) $\varrho_{(\alpha_2(\beta_1^{-1}(\alpha_1(\gamma_1))),\ \beta_2(\gamma_1))}[f(g)]_h = 1$
when $\lim_{r\to+\infty}\frac{\alpha_2(\beta_1^{-1}(\alpha_1(\gamma_1(r))))}{\alpha_2(\beta_1^{-1}(\alpha_1(r)))} = 0$ and $\lim_{r\to+\infty}\frac{\beta_2(\gamma_1(r))}{\beta_2(r)} = 0$.

So we obtain that the generalized index-pair of $f(g)$ is $(\alpha_2(\beta_1^{-1}(\alpha_1)),\beta_2)$ when $\alpha_2(\beta_1^{-1}(r)) \in L^0$ and either $\lambda_{(\alpha_1,\beta_1)}[f]_h > 0$ or $\lambda_{(\alpha_2,\beta_2)}[g] > 0$ and thus the third part of the theorem is established. □

THEOREM 2.3. *Let f, g, $h \in \mathcal{A}(\mathbb{K})$. Also let the generalized relative index-pair of f with respect to h be (α_1,β_1) and the generalized index-pair of g be (α_2,β_2). Then*

(i) *If $\beta_1(r) = \alpha_2(r)$, $\lambda_{(\alpha_1,\beta_1)}[f]_h > 0$ and $\lambda_{(\alpha_2,\beta_2)}[g] > 0$, then*

$$\begin{aligned}\lambda_{(\alpha_1,\beta_1)}[f]_h\lambda_{(\alpha_2,\beta_2)}[g] &\leq \lambda_{(\alpha_1,\beta_2)}[f(g)]_h \\ &\leq \min\left\{\varrho_{(\alpha_1,\beta_1)}[f]_h\lambda_{(\alpha_2,\beta_2)}[g],\ \lambda_{(\alpha_1,\beta_1)}[f]_h\varrho_{(\alpha_2,\beta_2)}[g]\right\}.\end{aligned}$$

(ii) *If $\beta_1\left(\alpha_2^{-1}(r)\right) \in L^0$, $\lambda_{(\alpha_1,\beta_1)}[f]_h > 0$ and $\lambda_{(\alpha_2,\beta_2)}[g] > 0$, then*

$$\lambda_{(\alpha_1,\beta_1(\alpha_2^{-1}(\beta_2)))}[f(g)]_h = \lambda_{(\alpha_1,\beta_1)}[f]_h.$$

(iii) *If $\alpha_2\left(\beta_1^{-1}(r)\right) \in L^0$, $\lambda_{(\alpha_1,\beta_1)}[f]_h > 0$ and $\lambda_{(\alpha_2,\beta_2)}[g] > 0$, then*

$$\lambda_{(\alpha_2(\beta_1^{-1}(\alpha_1)),\beta_2)}[f(g)]_h = \lambda_{(\alpha_2,\beta_2)}[g].$$

In the line of Theorem 2.2 one can easily deduce the conclusion of Theorem 2.3 and so its proof is omitted.

Acknowledgement

The authors are extremely grateful to the anonymous learned referee for his/her keen reading, valuable suggestions and constructive comments for the improvement of the paper.

References

[1] L. Bernal, *Orden relativo de crecimiento de funciones enteras*, Collect. Math., **39** (1988), 209–229.

[2] T. Biswas and C. Biswas, *On the Growth Properties of Composite p-adic Entire Functions*, Noor Publishing, Chisinau-2068, Republic of Moldova Europe, 133p., 2021.

[3] T. Biswas, *Relative (p,q)-φ order oriented some growth properties of p-adic entire functions*, J. Fract. Calc. Appl., **11** (1) (2020), 161–169.

[4] T. Biswas, C. Biswas and R. Biswas, *A note on generalized growth analysis of composite entire functions*, Poincare J. Anal. Appl., **7** (2) (2020), 257–266.

[5] T. Biswas and C. Biswas, *Generalized (α,β) order based on some growth properties of wronskians*, Mat. Stud., **54** (1) (2020), 46–55.

[6] T. Biswas, *Some growth properties of composite p-adic entire functions on the basis of their relative order and relative lower order*, Asian-Eur. J. Math., **12** (3) (2019), 1950044, 15p., https://doi.org/10.1142/S179355711950044X.

[7] T. Biswas, *Some growth aspects of composite p-adicentire functions in the light of their (p,q)-th relative order and (p,q)-th relative type*, J. Chungcheong Math. Soc., **31** (4) (2018), 429–460.

[8] T. Biswas, *On some growth analysis of p-adic entire functions on the basis of their (p,q)-th relative order and (p,q)-th relative lower order*, Uzbek Math. J., **2018** (4) (2018), 160–169.

[9] T. Biswas, *Relative order and relative type based growth properties of iterated p-adic entire functions*, Korean J. Math., **26** (4) (2018), 629–663.

[10] T. Biswas, *A note on (p,q)-th relative order and (p,q)-th relative type of p-adic entire functions*, Honam Math. J., **40** (4)(2018), 621–659.

[11] T. Biswas, *(p,q)-th order oriented growth measurement of composite p-adic entire functions*, Carpathian Math. Publ., **10** (2) (2018), 248–272.

[12] K. Boussaf, A. Boutabaa and A. Escassut, *Order, type and cotype of growth for p-Adic entire functions, A survey with additional properties, p-Adic Numbers,* Ultrametric Anal. Appl., **8** (4), (2016), 280–297.

[13] K. Boussaf, A. Boutabaa and A. Escassut, *Growth of p-adic entire functions and applications*, Houston J. Math., **40** (3) (2014), 715–736.

[14] A. Escassut, K. Boussaf and A. Boutabaa, *Order, type and cotype of growth for p-adic entire functions*, Sarajevo J. Math., Dedicated to the memory of Professor Marc Krasner, **12(25)** (2) (2016), 429–446, suppl.

[15] A. Escassut, *Value Distribution in p-adic Analysis*, World Scientific Publishing Co. Pte. Ltd., Singapore, 2015.

[16] A. Escassut and J. Ojeda, *Exceptional values of p-adic analytic functions and derivative*, Complex Var. Elliptic Equ., **56** (1-4) (2011), 263–269.

[17] A. Escassut, *p-adic Value Distribution. Some Topics on Value Distribution and Differentability in Complex and P-adic Analysis*, Math. Monogr., Series 11, Science Press, Beijing, (2008), 42–138.

[18] A. Escassut, *Analytic Elements in p-adic Analysis*, World Scientific Publishing Co. Pte. Ltd. Singapore, 1995.

[19] P. C. Hu and C. C. Yang, *Meromorphic Functions over non-Archimedean Fields*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2000.

[20] O. P. Juneja, G. P. Kapoor and S. K. Bajpai, *On the (p,q)-type and lower (p,q)-type of an entire function*, J. Reine Angew. Math., **290** (1977), 180-190.

[21] O. P. Juneja, G. P. Kapoor and S. K. Bajpai, *On the (p,q)-order and lower (p,q)-order of an entire function*, J. Reine Angew. Math., **282** (1976), 53–67.

[22] A. Robert, *A Course in p-adic analysis*, Graduate texts, Springer, New York, 2000.

[23] M. N. Sheremeta, *Connection between the growth of the maximum of the modulus of an entire function and the moduli of the coefficients of its power series expansion*, Izv. Vyssh. Uchebn. Zaved Mat., **2** (1967), 100–108 (in Russian).

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