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FSAL MONO-IMPLICIT NORDSIECK GENERAL LINEAR METHODS WITH INHERENT RUNGE-KUTTA STABILITY FOR DAES

P. O. OLATUNJI [1†] AND M. N. O. IKHILE [2]

[1] DEPARTMENT OF MATHEMATICAL SCIENCES, ADEKUNLE AJASIN UNIVERSITY, NIGERIA
Email address: †peter.olatunji@aaua.edu.ng

[2] ADVANCED RESEARCH LABORATORY, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BENIN, BENIN CITY, NIGERIA

ABSTRACT. This paper introduces mono-implicit general linear methods, a special class of general linear methods, which are implicit in the output solution for the numerical integration of differential algebraic equations. We show how L-stable inherent Runge-Kutta members can be derived. The procedures for implementation have been discussed. The numerical test on the problem considered shows that the methods have improved accuracy when compared to RADAU IIA and the results from MATLAB ode15s, which have been taken as our reference solution.

1. INTRODUCTION

Differential algebraic equations are equations involving unknown functions and their derivatives in an implicit way. The autonomous general form of a differential algebraic system of equation is the implicit differential equation [1]

$$F(y'(x), y(x)) = 0, \qquad y(x_0) = y_0, \tag{1.1}$$

where $F : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$, $y \in \mathbb{R}^m$ and $\frac{\partial F}{\partial y'}$ may be singular.

Definition 1.1. The DAE (1.1) has differentiation index k, if k is the minimal number of analytical differentiations

$$F(y', y) = 0,\ \frac{d}{dx}F(y', y) = 0, \cdots, \frac{d^k}{dx^k}F(y', y) = 0, \tag{1.2}$$

such that (1.2) allow us to extract through algebraic manipulations an explicit ordinary differential system $y' = f(y(x))$.

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†Corresponding author.

In this paper, we are interested in the development of first same as last (FSAL) general linear methods (GLMs) in Nordsieck form with inherent Runge-Kutta stability for the numerical solution of DAEs of index-1

$$\begin{aligned} y' &= f(y,z) \\ 0 &= g(y,z), \end{aligned} \tag{1.3}$$

with initial conditions $g(y_0, z_0) = 0;\ y_0 = a;\ z_0 = b$ and $\frac{\partial g}{\partial z}$ is non-singular, and index-2

$$\begin{aligned} y' &= f(y,z) \\ 0 &= g(y), \end{aligned} \tag{1.4}$$

having initial conditions $g(y_0) = 0; g_y(y_0)f(y_0, z_0) = 0;\ y_0 = a;\ z_0 = b$, where f and g are assured to be smooth and $\frac{\partial g}{\partial y}\frac{\partial f}{\partial z}$ is non-singular. Here, the solution of the DAE (1.4) must satisfy the equation

$$0 = \frac{\partial g}{\partial z} f(y,z).$$

General linear methods were first proposed by [2] as a unifying framework for studying consistency, convergence and stability of Runge-Kutta and linear multistep methods (LMMs) [3]. A class of GLM called Diagonally Implicit Multistage Integration Methods (DIMSIMs) introduced by Butcher in [4] have gained popular interest. DIMSIMs have been constructed and presented as type 1, 2, 3 and 4 in [4] among others, with the implementation procedures discussed in [5, 6]. Second derivative GLMs have also been developed by several authors, some of which includes [7, 8, 9]. Recently, [10] introduced a new family of GLMs with strong regularity property. In implementing numerical methods, Nordsieck vectors were first introduced by Nordsieck in [11] to implement the Adams methods, [12] used this Nordsieck vector in the code DIFSUB for the solution of ODEs. This has also been applied to GLMs (see [13]).
Implicit Runge-Kutta methods (RKMs) have been known to have good stability properties (that is, having wide region of stability) as compared to the linear multistep methods, which is inhibited by the Dahlquist order barrier [14]; this has been the motivation for constructing GLMs with *Runge-Kutta Stabiliy* (i.e. GLMs having same stability domains equivalent to RKMs). Further to this, the desire to create a matrix structure of GLMs having Runge-Kutta stability properties gave rise to methods with ***inherent Runge-Kutta stability***. GLMs having this property was introduced in [15], and several authors have developed GLMs having inherent Runge-Kutta stability, some of which include the works of [3, 16, 17, 18, 19].

2. FSAL Mono-Implicit Nordsieck GLMs with Inherent Runge-Kutta Stability

For the numerical integration of (1.1), we consider the general linear methods in Nordsieck form defined as

$$\begin{aligned} Y &= h(A \otimes I)F + (U \otimes I)y^{[n-1]} \\ y^{[n]} &= h(B \otimes I)F + (V \otimes I)y^{[n-1]}, \end{aligned} \tag{2.1}$$

$n = 1, 2, ..., N$, h is the step size, and

$$Y = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ \vdots \\ Y_s \end{bmatrix}; \quad F = \begin{bmatrix} f(Y_1) \\ f(Y_2) \\ f(Y_3) \\ \vdots \\ f(Y_s) \end{bmatrix}; \quad y^{[n]} = \begin{bmatrix} y_1^{[n]} \\ y_2^{[n]} \\ y_3^{[n]} \\ \vdots \\ y_{p+1}^{[n]} \end{bmatrix} = \begin{bmatrix} y_{n+1} \\ hy'_{n+1} \\ h^2y''_{n+1} \\ \vdots \\ h^py_{n+1}^{(p)} \end{bmatrix} \approx \begin{bmatrix} y(x_{n+1}) \\ hy'(x_{n+1}) \\ h^2y''(x_{n+1}) \\ \vdots \\ h^py^{(p)}(x_{n+1}) \end{bmatrix},$$

with $x_{n+1} = x_n + h$, $Y_i = y_{n+c_i} \approx y(x_n + c_ih), F_i = f_{n+c_i} \approx y'(x_n + c_ih);\ i = 1(1)s$. Here, c_i is chosen as $c_i = \frac{i}{s}$, and I is the $s \times s$ identity matrix, where s is the number of stages. The matrices U, V have the form

$$U = \begin{bmatrix} 1 & u_{12} & u_{13} & \cdots & u_{1(s-1)} & u_{1s} \\ 1 & u_{22} & u_{23} & \cdots & u_{2(s-1)} & u_{2s} \\ 1 & u_{32} & u_{33} & \cdots & u_{3(s-1)} & u_{3s} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & u_{(s-1)2} & u_{(s-1)3} & \cdots & u_{(s-1)(s-1)} & u_{(s-1)s} \\ 1 & u_{s2} & u_{s3} & \cdots & u_{s(s-1)} & u_{ss} \end{bmatrix},$$

$$V = \begin{bmatrix} 1 & u_{s2} & u_{s3} & \cdots & u_{s(s-1)} & u_{ss} \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & v_{32} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & v_{(s-1)2} & v_{(s-1)3} & \cdots & 0 & 0 \\ 0 & v_{s2} & v_{s3} & \cdots & v_{s(s-1)} & 0 \end{bmatrix}.$$

Two cases of this new method will be considered, we classify these cases by the structure of A and B matrices having the form in table 1. The method (2.1) can be represented in the compact form

$$\left[\begin{array}{c} Y \\ \hline y^{[n]} \end{array}\right] = \left[\begin{array}{c|c} A \otimes I & U \otimes I \\ \hline B \otimes I & V \otimes I \end{array}\right] \left[\begin{array}{c} hF \\ \hline y^{[n-1]} \end{array}\right].$$

The ***first same as last*** (FSAL) property of (2.1) is that the first output solution in $y^{[n]}$ is the same as the last stage in Y. Here, the first row of matrices B and V is the same as the last row of matrices A and U respectively (i.e. $a_{si} = b_{1i}$ and $u_{si} = v_{1i},\ i = 1, 2, ..., s$).

This family of method is what we call FSAL Mono-Implicit Nordsieck General Linear Methods (MiGLMs). The MiGLMs are implicit on the last stage Y_s; this has been done to improve stability as in the case of backward difference formula (BDF) (see [1, 12, 20, 21] and also to avoid high computational cost as in the case of fully implicit Runge-Kutta methods) (an example is the RADAU, Guass and Lobatto RKMs). Another advantage of FSAL MiGLMs is that they are readily amenable to be stiffly stable [1]. The structure of the A matrix gives the GLM nature of the nested methods proposed by [22]. The idea of nesting was first proposed by

[23, 24] by extending the mono-implicit RKMs of [25, 26], where nested implicit Runge-Kutta formulas based on Gauss quadrature formula were developed. This idea has been extended to a new way of nesting in the hybrid multistep methods by [22, 27, 28, 29, 30]. In fact, the case II MiGLM is the nested general linear method (NGLM) proposed by [31] that are implicit in their stages.

Case	Structure of A	Structure of B
case I	$\begin{bmatrix} 0 & 0 & \cdots & 0 & a_{1s} \\ a_{21} & 0 & \cdots & 0 & a_{2s} \\ a_{31} & a_{32} & \cdots & 0 & a_{3s} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{(s-1)1} & a_{(s-1)2} & \cdots & 0 & a_{(s-1)s} \\ a_{s1} & a_{s2} & \cdots & a_{s(s-1)} & a_{ss} \end{bmatrix}$	$\begin{bmatrix} a_{s1} & a_{s2} & \cdots & a_{s(s-1)} & a_{ss} \\ 0 & 0 & \cdots & 0 & 1 \\ b_{31} & b_{32} & \cdots & b_{3(s-1)} & b_{3s} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{(s-1)1} & b_{(s-1)2} & \cdots & b_{(s-1)(s-1)} & b_{(s-1)s} \\ b_{s1} & b_{s2} & \cdots & b_{s(s-1)} & b_{ss} \end{bmatrix}$
case II	$\begin{bmatrix} \lambda & 0 & 0 & \cdots & 0 & a_{1s} \\ a_{21} & \lambda & 0 & \cdots & 0 & a_{2s} \\ a_{31} & a_{32} & \lambda & \cdots & 0 & a_{3s} \\ 0 & a_{42} & a_{43} & \cdots & 0 & a_{4s} \\ 0 & 0 & a_{53} & \cdots & 0 & a_{5s} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & a_{(s-1)s} \\ 0 & 0 & 0 & \cdots & a_{s(s-1)} & \lambda \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & \cdots & a_{s(s-1)} & \lambda \\ 0 & 0 & \cdots & 0 & 1 \\ b_{31} & b_{32} & \cdots & b_{3(s-1)} & b_{3s} \\ b_{41} & b_{42} & \cdots & b_{4(s-1)} & b_{4s} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{(s-1)1} & b_{(s-1)2} & \cdots & b_{(s-1)(s-1)} & b_{(s-1)s} \\ b_{s1} & b_{s2} & \cdots & b_{s(s-1)} & b_{ss} \end{bmatrix}$

TABLE 1. Two cases of the FSAL GLM in Nordsieck form; $\lambda \geq 0$

Theorem 2.1. [3, 16, 17, 18] *The method (2.1) has order p and stage order $q = p$, if and only if*

$$\begin{aligned} e^{cz} &= zAe^{cz} + UZ + O(z^{p+1}) \\ e^{z}Z &= zBe^{cz} + VZ + O(z^{p+1}), \quad p \geq 1 \end{aligned} \tag{2.2}$$

where $Z = [1,\ z,\ z^2,\ \cdots\ z^p]^T$ *and* $e^{cz} = [e^{c_1 z},\ e^{c_2 z},\ e^{c_3 z},\ \cdots\ e^{c_1 z}]^T$.

Proof. Since the stage order $q = p$, then the stage values satisfy

$$\begin{aligned} Y_i &= y(x_n + c_i h) + O(h^{p+1}) \\ &= y(x_n) + c_i h y'(x_n) + \cdots + \frac{c_i^p h^p}{p!} y^{(p)}(x_n) + O(h^{p+1}). \end{aligned} \tag{2.3}$$

In Nordsieck form, the incoming approximation $y^{[n-1]}$ is

$$y^{[n-1]} = \begin{bmatrix} y_1^{[n-1]} \\ y_2^{[n-1]} \\ y_3^{[n-1]} \\ \vdots \\ y_{p+1}^{[n-1]} \end{bmatrix} = \begin{bmatrix} y_n \\ hy_n' \\ h^2 y_n'' \\ \vdots \\ h^p y_n^{(p)} \end{bmatrix} \approx \begin{bmatrix} y(x_n) \\ hy'(x_n) \\ h^2 y''(x_n) \\ \vdots \\ h^p y^{(p)}(x_n) \end{bmatrix}.$$

Thus, (2.3) can be written as

$$Y = Cy^{[n-1]} + O(h^{p+1}), \tag{2.4}$$

where C is given by

$$C = \begin{bmatrix} 1 & c_1 & \frac{c_1^2}{2!} & \cdots & \frac{c_1^p}{p!} \\ 1 & c_2 & \frac{c_2^2}{2!} & \cdots & \frac{c_2^p}{p!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & c_s & \frac{c_s^2}{2!} & \cdots & \frac{c_s^p}{p!} \end{bmatrix}.$$

For each stage derivative, we have

$$\begin{aligned} hf(Y_i) &= hy'(x_n + c_i h) + O(h^{p+2}) \\ &= \sum_{k=1}^{p+1} \frac{c_i^{k-1}}{(k-1)!} y^{(k)}(x_n)h^k + +O(h^{p+2}) \\ &= \sum_{k=1}^{p} \frac{c_i^{k-1}}{(k-1)!} y^{(k)}(x_n)h^k + +O(h^{p+1}), \end{aligned}$$

which can be expressed in matrix form as

$$hF = CKy^{[n-1]} + O(h^{p+1}), \tag{2.5}$$

where K is the shifting matrix given by

$$K = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

The output approximation of order p at the step n has the form

$$y_j^{[n]} = h^j y^{(j)}(x_n + h) + O(h^{p+1}), \qquad j = 1(1)p, \tag{2.6}$$

with the incoming approximation having the form

$$y_j^{[n-1]} = h^{j-1} y^{(j-1)}(x_n) + O(h^{p+1}), \qquad j = 1(1)p.$$

Expanding (2.6) by Taylor series about x_n yields

$$y^{[n]} = Ey^{[n-1]} + O(h^{p+1}), \tag{2.7}$$

where

$$E = \begin{bmatrix} 1 & \frac{1}{1!} & \frac{1}{2!} & \cdots & \frac{1}{p!} \\ 0 & 1 & \frac{1}{1!} & \cdots & \frac{1}{(p-1)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{1!} \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix},$$

substituting (2.3), (2.5) and (2.7) into (2.1) gives

$$\begin{aligned} Cy^{[n-1]} &= ACKy^{[n-1]} + Uy^{[n-1]} + O(h^{p+1}) \\ Ey^{[n-1]} &= BCKy^{[n-1]} + Vy^{[n-1]} + O(h^{p+1}). \end{aligned} \tag{2.8}$$

Let $z^k = h^k y^{(k)}(x_n)$, then $y^{[n-1]}$ can be represented by $Z = [1,\ z,\ z^2,\ \cdots\ z^p]^T$, therefore, (2.8) becomes

$$\begin{aligned} CZ &= ACKZ + UZ + O(z^{p+1}) \\ EZ &= BCKZ + VZ + O(z^{p+1}). \end{aligned} \tag{2.9}$$

The matrices K, C, E have the properties

$$\begin{aligned} KZ &= zZ + O(z^{p+1}) \\ CZ &= e^{cz} + O(z^{p+1}) \\ EZ &= e^{z}Z + O(z^{p+1}), \end{aligned}$$

substituting these properties into (2.9), then (2.2) is obtained. □

Equating the coefficients of the powers of z in (2.2), we get

$$\begin{aligned} U &= C - ACK \\ V &= E - BCK. \end{aligned} \tag{2.10}$$

The stability region of general linear methods for ODEs is determined using the stability matrix

$$M(z) = V + zB(I - zA)^{-1}U, \tag{2.11}$$

provided $(I - zA)^{-1}$ is non-singular. However, in the case of DAEs, the limit of this matrix at infinity is essential [32], therefore

$$\lim_{z\to\infty} M(z) = M(\infty) = V - BA^{-1}U.$$

When numerical integrators are used at each mesh point, there are differences between the numerical solution and exact solution, this is the global error. Instability sets in when this error accumulates and the numerical solution does not converge to the true solution. Sequel to this, a method must posses a wide region of stability. We therefore give some definitions:

Definition 2.2. [33] For a non-singular $(I - zA)^{-1}$, the GLM (2.1) is A-stable if the stability matrix $M(z)$ is a stable matrix for all $z \in C^-$.

Definition 2.3. [33] The GLM (2.1) is L-stable if it is A-stable and $\rho(M(\infty)) = 0$.

Definition 2.4. [32] The GLM (2.1) is stable at infinity if and only if

$$\begin{cases} \rho(M(\infty)) \leq 1 \\ \forall\ w \in S_p\left\{M(\infty)\right\}, |w| = 1 \Longrightarrow \lambda\ is\ non-defective. \end{cases}$$

Taking the advantage of the implicit Runge-Kutta method of having good stability properties, we thus impose this stability condition (RK-stability) on the family of methods to be constructed. The stability function of $M(z)$ is defined by

$$\Pi(w,z) = det(wI - M(z)). \tag{2.12}$$

Definition 2.5. [16] If $\Pi(w,z)$ defined in (2.12) has the special form

$$\Pi(w,z) = w^{s-1}(w - R(z))$$

then the GLM (2.1) is said to possess Runge-Kutta (RK) stability.

The key property of the new family of methods studied in this paper is not just having the advantages of linear multistep methods and Runge-Kutta methods, but of the property of *inherent Runge-Kutta stability*.

Definition 2.6. [18] If the GLM (2.1) satisfies $Ve_1 = e_1$, then (2.1) has the property of inherent Runge-Kutta stabiliy if

$$\begin{aligned} det(wI - V) = w^{s-1}(w-1) \\ BA \equiv XB \\ BU \equiv XV - VX, \end{aligned} \tag{2.13}$$

where the notation $\equiv$ denotes equality of two matrices except for the first row.

The matrix X is the doubly companion matrix [18] defined by,

$$X = \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & \cdots & -\alpha_{p-1} & -\alpha_p & -\alpha_{p+1} - \beta_{p+1} \\ 1 & 0 & 0 & \cdots & 0 & 0 & -\beta_p \\ 0 & 1 & 0 & \cdots & 0 & 0 & -\beta_{p-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & -\beta_3 \\ 0 & 0 & 0 & \cdots & 1 & 0 & -\beta_2 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -\beta_1 \end{bmatrix};$$

where α_i and β_i are real coefficients. In this article we use the shifting matrix

$$J = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix},$$

as a special case of X, following [17].

Theorem 2.7. [17] *If (2.1) has the property of inherent Runge-Kutta stability (IRK), then the stability matrix $M(z)$ and V matrix have a single non-zero eigenvalue, and the stability function of $M(z)$ has the form*

$$\Pi(w,z) = w^{s-1}(w - R(z)).$$

The proof is that of Theorem 3.1 of [16] and Theorem 2.6 of [3]. For the GLM (2.1), V has only one non-zero eigenvalue which equals 1.

3. Finding Mono-Implicit GLMs with IRK stability

The approach used to derive FSAL mono-implicit GLM (MiGLM) with IRK stability properties is discussed in this section. As earlier stated in section 2, there are order conditions that should be satisfied as well as conditions for IRK stability. The approach explained in this section has been used to find the coefficients of A, U, B, V of the two classes of FSAL MiGLM with IRK stability.

Step I. Parameters of the method:

- s: the number of stages. Here, we choose $s = p + 1$.
- c: the abscissae vector. Here, we choose $c_i = \frac{i}{s};\ i = 1(1)s$.
- a_{ij} and λ: constant coefficients of the A matrix, chosen to achieve desired stability properties.

Step II. Using the order conditions in equation (2.10), obtain the resulting system of equations.

Step III. In order to ensure that these methods satisfy the IRK stability, set $X = J$ in (2.15) and obtain the system of equations

$$\begin{aligned} BA - JB &\equiv 0, \\ BU - JV + VJ &\equiv 0. \end{aligned} \tag{3.1}$$

Step IV. Solve the obtained system of equations from steps II and III to obtain values for other a_{ij}, and u_{ij}, b_{ij}, v_{ij} in terms of the real coefficient(s) a_{ij} and λ chosen in step **I**.

Step V. In order to obtain methods with wide range of stability (e.g. A-stability and L-stability), values of a_{1s} and λ are chosen to achieve desired results. The approach used to achieve this, is to use the result in step IV to compute for $R(z)$ (this is done by finding the only non-zero eigenvalue of $M(z)$), express as a ratio of two polynomials in the form

$$R(z) = \frac{N(z)}{D(z)}, \qquad D(z) \neq 0.$$

To satisfy L-stability requirement, we ensure that the coefficient of the highest degree of z in $N(z)$ is zero. That is; for

$$R(z) = \frac{N(z)}{D(z)} = \frac{\gamma_1 + \gamma_2 z + \gamma_3 z + \cdots + \gamma_{s-1} z^p}{\delta_1 + \delta_2 z + \delta_3 z + \cdots + \delta_{s-1} z^p + \delta_s z^{p+1}}, \qquad p \geq 1, \tag{3.2}$$

γ_s is fixed to be zero.

However, for A-stability, $R(z)$ must satisfy $|R(z)| \leq 1$ for all z in the left half of the complex plane. Replacing $z = iy$, then $|R(z)| \leq 1$ is equivalent to the E-polynomial greater or equal to zero [33], where

$$E(y) = |D(iy)|^2 - |N(iy)|^2 = D(iy)D(-iy) - N(iy)N(-iy); \qquad y \in \mathbb{R}. \tag{3.3}$$

Theorem 3.1. *(Compare Theorem 351C in* [33]*) The FSAL MiGLM with IRK stability, having stability matrix $M(z)$ with only one non-zero eigenvalue given as $R(z) = \frac{N(z)}{D(z)}$ is A-stable, if and only if*

(i) *all poles of $R(z)$ are in the right half plane.*
(ii) *$E(y) = D(iy)D(-iy) - N(iy)N(-iy) \geq 0$ for all real y.*

By this theorem, the selected a_{ij} and λ chosen in step **I** can be computed.

The FSAL MiGLM with IRK stability that is A-stable and obeys (3.2) is L-stable. This can be clearly seen. If the FSAL MiGLM with IRK stability has stability matrix $M(z)$ as defined in (2.11) and if (2.1) posses IRK stability, then the stability matrix $M(z)$ has only one non-zero eigenvalue $R(z)$. If (2.1) is also L-stable, then $R(z)$ has the form in (3.2), and taking the limit as $z \to \infty$ yields

$$\rho(M(\infty)) = R(\infty) = 0.$$

3.1. **Examples of the mono-implicit GLM (2.1).** Some example methods (case I and II) are presented below.
Method $s = 2$, case I.
Step I For $s = 2, p = 1$, the A, U, B, V for case I is given as

$$\left[\begin{array}{c} y_{n+c_1} \\ y_{n+c_2} \\ \hline y_{n+1} \\ hy'_{n+1} \end{array}\right] = \left[\begin{array}{c|c} A & U \\ \hline B & V \end{array}\right] \left[\begin{array}{c} hf_{n+c_1} \\ hf_{n+c_2} \\ \hline y_n \\ hy'_n \end{array}\right] \Rightarrow \left[\begin{array}{c} y_{n+c_1} \\ y_{n+c_2} \\ \hline y_{n+1} \\ hy'_{n+1} \end{array}\right] \left[\begin{array}{cc|cc} 0 & a_{12} & 1 & u_{12} \\ a_{21} & a_{22} & 1 & u_{22} \\ \hline a_{21} & a_{22} & 1 & u_{22} \\ 0 & 1 & 0 & 0 \end{array}\right] \left[\begin{array}{c} hf_{n+c_1} \\ hf_{n+c_2} \\ \hline y_n \\ hy'_n \end{array}\right]. \tag{3.4}$$

Step II Using the order conditions (2.10) with the abscissae values $c_1 = \frac{1}{2}, c_2 = 1$, we obtain

$$\begin{aligned} u_{12} &= \frac{1}{2} - a_{12}, \\ u_{22} &= 1 - a_{21} - a_{22}. \end{aligned} \tag{3.5}$$

Step III Using the conditions necessary for IRK stability, it is observed that

$$BA - JB \equiv 0$$
$$BU - JV + VJ \equiv 0$$

with $J = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.
Step IV and V Substituting u_{12} and u_{22} as obtained from (3.5) into (3.4), the stability matrix

is given as

$$M(z)=\begin{bmatrix} -\frac{za_{21}+1}{a_{12}a_{21}z^2+a_{22}z-1} & \frac{-za_{21}+2za_{12}a_{21}+2a_{21}+2a_{22}-2}{2(a_{12}a_{21}z^2+a_{22}z-1)} \\ -\frac{z(za_{21}+1)}{a_{12}a_{21}z^2+a_{22}z-1} & \frac{z(-za_{21}+2za_{12}a_{21}+2a_{21}+2a_{22}-2)}{2(a_{12}a_{21}z^2+a_{22}z-1)} \end{bmatrix},$$

having the only non-zero eigenvalue

$$R(z)=\frac{N(z)}{D(z)}=\frac{(2a_{12}a_{21}-a_{21})\,z^2+2a_{22}z-2z-2}{2\,(a_{12}a_{21}z^2+a_{22}z-1)}.$$

To achieve L-stability, the coefficient of z^2 in $N(z)$ is set to zero, thus letting $a_{12}=\frac{1}{2}$, $R(z)$ becomes

$$R(z)=\frac{N(z)}{D(z)}=\frac{2\,(a_{22}z-z-1)}{a_{21}z^2+2a_{22}z-2}. \tag{3.6}$$

Also, to ensure A-stability, we replace $z=iy$ in (3.6) and check E-polynomial using (3.3). The E-polynomial for the method (3.4) is given as

$$E(y)=a_{21}^2y^4+(4a_{21}+8a_{22}-4)\,y^2\geq 0. \tag{3.7}$$

The (3.7) suggests that for A-stability, a_{21} and a_{22} should be chosen such that $E\geq 0$. Therefore A-stability is achieved whenever $a_{21}\geq 1-2a_{22}$. Choosing $a_{21}=\frac{1}{2}$, then $a_{22}\geq\frac{1}{4}$.
As an example, choose $a_{21}=\frac{1}{2}, a_{22}=\frac{1}{4}$, the E-polynomial is

$$E(y)=\frac{y^4}{4}>0,\quad \forall y\in\mathbb{R},$$

with resulting method (3.4) defined as

$$\begin{bmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \\ \hline y_{n+1} \\ hy'_{n+1} \end{bmatrix}=\left[\begin{array}{cc|cc} 0 & \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & \frac{1}{4} & 1 & \frac{1}{4} \\ \hline \frac{1}{2} & \frac{1}{4} & 1 & \frac{1}{4} \\ 0 & 1 & 0 & 0 \end{array}\right]\begin{bmatrix} hf_{n+\frac{1}{2}} \\ hf_{n+1} \\ \hline y_n \\ hy'_n \end{bmatrix}. \tag{3.8}$$

The stability matrix is given as

$$M(z)=\begin{bmatrix} -\frac{2(z+2)}{z^2+z-4} & -\frac{1}{z^2+z-4} \\ -\frac{2z(z+2)}{z^2+z-4} & -\frac{z}{z^2+z-4} \end{bmatrix}.$$

The only non-zero eigenvalue of this GLM (3.8) is

$$R(z)=\frac{-4-3z}{-4+z+z^2}.$$

The method (3.8) is L-stable. However, the last stage (same as the output method y_{n+1}) gained order, to give $p=2$.

Method $s = 3$, case I.

Step I For $s = 3, p = q = 2$, the A, U, B, V for case I is given as

$$\left[\begin{array}{c} Y_1 \\ Y_2 \\ Y_3 \\ \hline y_{n+1} \\ hy'_{n+1} \\ h^2y''_{n+1} \end{array}\right] = \left[\begin{array}{ccc|ccc} 0 & 0 & a_{13} & 1 & u_{12} & u_{13} \\ a_{21} & 0 & a_{23} & 1 & u_{22} & u_{23} \\ a_{31} & a_{32} & a_{33} & 1 & u_{32} & u_{33} \\ \hline a_{31} & a_{32} & a_{33} & 1 & u_{32} & u_{33} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ b_{31} & b_{32} & b_{33} & 0 & v_{32} & 0 \end{array}\right] \left[\begin{array}{c} hF_1 \\ hF_2 \\ hF_3 \\ \hline y_n \\ hy'_n \\ h^2y''_n \end{array}\right], \tag{3.9}$$

with $Y_1 = y_{n+c_1}, Y_2 = y_{n+c_2}, Y_3 = y_{n+c_3}$ and $F_1 = y'_{n+c_1}, F_2 = y'_{n+c_2}, F_3 = y'_{n+c_3}$.

Step II Applying the order conditions in (2.10) with the abscissae values $c_1 = \frac{1}{3}, c_2 = \frac{2}{3}, c_3 = 1$, we obtain

$$\begin{gathered} a_{13} + u_{12} = \frac{1}{3},\ a_{13} + u_{13} = \frac{1}{18},\ a_{21} + a_{23} + u_{22} = \frac{2}{3}, \\ \frac{a_{21}}{3} + a_{23} + u_{23} = \frac{2}{9},\ a_{31} + a_{32} + a_{33} + u_{32} = 1,\ \frac{a_{31}}{3} + \frac{2a_{32}}{3} + a_{33} + u_{33} = \frac{1}{2}, \\ b_{31} + b_{32} + b_{33} + v_{32} = 0,\ \frac{b_{31}}{3} + \frac{2b_{32}}{3} + b_{33} = 1. \end{gathered} \tag{3.10}$$

Step III Using the conditions necessary for IRK stability defined in (3.1), with $J = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, the system of equations

$$\begin{gathered} a_{21}b_{32} + a_{31}b_{33} = 0,\ a_{32}b_{33} = 0, a_{13}b_{31} + a_{23}b_{32} + a_{33}b_{33} - 1 = 0 \\ b_{31}u_{12} + b_{32}u_{22} + b_{33}u_{32} = 0,\ b_{31}u_{13} + b_{32}u_{23} + b_{33}u_{33} = 0. \end{gathered} \tag{3.11}$$

are obtained.

Step IV Solving the system of equations (3.10) and (3.11) in terms of a_{23}, a_{31}, a_{32} and a_{33} yields

$$\begin{gathered} b_{31} = -12,\ b_{32} = \frac{15}{2},\ b_{33} = 0,\ a_{13} = \frac{1}{24}\left(15a_{23} - 2\right),\ a_{21} = 0, \\ v_{32} = \frac{9}{2},\ u_{12} = \frac{-5}{24}\left(3a_{23} - 2\right),\ u_{13} = \frac{-5}{72}\left(9a_{23} - 2\right),\ u_{22} = \frac{1}{3}\left(2 - 3a_{23}\right), \\ u_{23} = \frac{1}{9}\left(2 - 9a_{23}\right),\ u_{32} = -a_{31} - a_{32} - a_{33} + 1,\ u_{33} = \frac{1}{6}\left(-2a_{31} - 4a_{32} - 6a_{33} + 3\right). \end{gathered} \tag{3.12}$$

Step V Substituting the real coefficients (3.12) into (3.9), then $M(z)$ can be obtained. The only non-zero eigenvalue of $M(z)$ is given as

$$R(z) = \frac{N(z)}{D(z)} = \frac{-72 + \left(72a_{33} - 72\right)z + \varsigma_1 z^2 + \varsigma_2 z^3}{3\left(15a_{23}a_{31}z^2 - 2a_{31}z^2 + 24a_{23}a_{32}z^2 + 24a_{33}z - 24\right)},$$

where $\varsigma_1 = 45a_{23}a_{31} - 6a_{31} + 72a_{23}a_{32} + 72a_{33} - 36$ and $\varsigma_2 = 45a_{23}a_{31} - 10a_{31} + 72a_{23}a_{32} - 16a_{32}$. In order to ensure L-stability, ς_1 and ς_2 in $N(z)$ is set to zero. Thus,

$$a_{31} = 6\left(2a_{33} - 1\right), \; a_{32} = \frac{-15}{4}\left(2a_{33} - 1\right).$$

Also, to ensure A-stability, we look for suitable values of a_{23} and a_{33} by enforcing the condition that the E-polynomial (3.3) is greater or equal to zero. Here, we obtain the values, $a_{23} = \frac{1}{4}$ and $a_{33} = 0$. Thus, the resulting method is given as

$$\left[\begin{array}{c} Y_1 \\ Y_2 \\ Y_3 \\ \hline y_{n+1} \\ hy'_{n+1} \\ h^2y''_{n+1} \end{array}\right] = \left[\begin{array}{ccc|ccc} 0 & 0 & \frac{7}{96} & 1 & \frac{25}{96} & -\frac{5}{288} \\ 0 & 0 & \frac{1}{4} & 1 & \frac{5}{12} & -\frac{1}{36} \\ -6 & \frac{15}{4} & 0 & 1 & \frac{13}{4} & 0 \\ \hline -6 & \frac{15}{4} & 0 & 1 & \frac{13}{4} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -12 & \frac{15}{2} & 0 & 0 & \frac{9}{2} & 0 \end{array}\right] \left[\begin{array}{c} hF_1 \\ hF_2 \\ hF_3 \\ \hline y_n \\ hy'_n \\ h^2y''_n \end{array}\right]. \tag{3.13}$$

The stability matrix is

$$M(z) = \left[\begin{array}{ccc} \frac{9z-4}{2(z^2-2)} & -\frac{13}{2(z^2-2)} & 0 \\ \frac{z(9z-4)}{2(z^2-2)} & -\frac{13z}{2(z^2-2)} & 0 \\ -\frac{z(2z-9)}{z^2-2} & -\frac{2z^2+9}{z^2-2} & 0 \end{array}\right],$$

and the only non-zero eigenvalue is

$$R(z) = -\frac{2(z+1)}{z^2-2}.$$

The method (3.13) is L-stable.

Methods $s = 4, 5$, case I.

Higher order methods can also obtained following the procedures above. The third order L-stable method with abscissae vector $c = \left[\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right]$ is given as

$$\left[\begin{array}{c} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ \hline y_{n+1} \\ hy'_{n+1} \\ h^2y''_{n+1} \\ h^3y^{(3)}_{n+1} \end{array}\right] \left[\begin{array}{cccc|cccc} 0 & 0 & 0 & \frac{1}{192} & 1 & \frac{47}{192} & \frac{5}{192} & 0 \\ -\frac{8}{3} & 0 & 0 & \frac{1}{24} & 1 & \frac{25}{8} & \frac{3}{4} & \frac{1}{12} \\ -\frac{76}{13} & 0 & 0 & \frac{9}{64} & 1 & \frac{5371}{832} & \frac{1333}{832} & \frac{19}{104} \\ -\frac{14}{3} & \frac{19}{3} & -\frac{26}{9} & \frac{2}{3} & 1 & \frac{14}{9} & 0 & 0 \\ \hline -\frac{14}{3} & \frac{19}{3} & -\frac{26}{9} & \frac{2}{3} & 1 & \frac{14}{9} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 28 & -38 & \frac{52}{3} & 0 & 0 & -\frac{22}{3} & 0 & 0 \\ 112 & -152 & \frac{208}{3} & -6 & 0 & -\frac{70}{3} & 2 & 0 \end{array}\right] \left[\begin{array}{c} hF_1 \\ hF_2 \\ hF_3 \\ hF_4 \\ \hline y_n \\ hy'_n \\ h^2y''_n \\ h^3y^{(3)}_n \end{array}\right], \tag{3.14}$$

with the stability matrix given as

$$M(z) = \begin{bmatrix} -\frac{2(11z-9)}{3(z^2-4z+6)} & \frac{28}{3(z^2-4z+6)} & 0 & 0 \\ -\frac{2z(11z-9)}{3(z^2-4z+6)} & \frac{28z}{3(z^2-4z+6)} & 0 & 0 \\ -\frac{2z(35z-66)}{3(z^2-4z+6)} & \frac{2\left(3z^2+44z-66\right)}{3(z^2-4z+6)} & 0 & 0 \\ -\frac{4z(37z-105)}{3(z^2-4z+6)} & \frac{14\left(3z^2+8z-30\right)}{3(z^2-4z+6)} & 2 & 0 \end{bmatrix}.$$

The only non-zero eigenvalue of $M(z)$ is

$$R(z) = \frac{2(z+3)}{z^2-4z+6}.$$

For the method $s = 5$, with abscissae vector $c = \left[\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1\right]$, the method A, U, B, V constructed is given as

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1}{10} \\ 0 & 0 & 0 & 0 & \frac{1408}{2125} \\ \frac{2156000}{17169} & -\frac{327250}{17169} & 0 & 0 & -\frac{2290758641}{24322750} \\ \frac{5992000}{17169} & -\frac{909500}{17169} & 0 & 0 & -\frac{3205443984}{12161375} \\ \frac{305}{72} & -\frac{65}{8} & \frac{535}{72} & -\frac{385}{144} & \frac{1}{2} \end{bmatrix},$$

$$U = \begin{bmatrix} 1 & \frac{1}{10} & -\frac{2}{25} & -\frac{73}{1500} & -\frac{83}{5000} \\ 1 & -\frac{558}{2125} & -\frac{1238}{2125} & -\frac{2044}{6375} & -\frac{1162}{10625} \\ 1 & -\frac{856130627}{72968250} & \frac{934855868}{12161375} & \frac{6733539637}{145936500} & \frac{7655942327}{486455000} \\ 1 & -\frac{1154793248}{36484125} & \frac{2618160624}{12161375} & \frac{4711234288}{36484125} & \frac{2678304424}{60806875} \\ 1 & -\frac{53}{144} & \frac{1}{12} & 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} \frac{305}{72} & -\frac{65}{8} & \frac{535}{72} & -\frac{385}{144} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \\ -\frac{305}{6} & \frac{195}{2} & -\frac{535}{6} & \frac{385}{12} & 0 \\ -\frac{33595}{177} & \frac{67685}{118} & -\frac{95765}{177} & \frac{68915}{354} & -12 \\ -\frac{9620855}{236} & \frac{5970243}{944} & \frac{1}{4} & \frac{1}{4} & -\frac{4296}{59} \end{bmatrix},$$

$$V = \begin{bmatrix} 1 & -\frac{53}{144} & \frac{1}{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{125}{12} & 0 & 0 & 0 \\ 0 & -\frac{4501}{177} & -\frac{625}{59} & 0 & 0 \\ 0 & \frac{32581441}{944} & \frac{13442481}{2360} & \frac{4078737}{11800} & 0 \end{bmatrix},$$

where, $Y = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \end{bmatrix}$, $y^{[n]} = \begin{bmatrix} y_{n+1} \\ hy'_{n+1} \\ h^2 y''_{n+1} \\ h^3 y^{(3)}_{n+1} \\ h^4 y^{(4)}_{n+1} \end{bmatrix}$, $hF = \begin{bmatrix} hF_1 \\ hF_2 \\ hF_3 \\ hF_4 \\ hF_5 \end{bmatrix}$ and $y^{[n-1]} = \begin{bmatrix} y_n \\ hy'_n \\ h^2 y''_n \\ h^3 y^{(3)}_n \\ h^4 y^{(4)}_n \end{bmatrix}$.

The stability matrix $M(z)$ has only one non-zero eigenvalue given as

$$R(z) = \frac{z^2 + 6z + 12}{z^2 - 6z + 12}.$$

From $R(z)$, it is deduced that the fourth order method for this case is A-stable but not L-stable.
Method $s = 2$, case II.

Step I In this case, for $s = 2$, the A, U, B, V is given as

$$\begin{bmatrix} y_{n+c_1} \\ y_{n+c_2} \\ \hline y_{n+1} \\ hy'_{n+1} \end{bmatrix} = \left[\begin{array}{c|c} A & U \\ \hline B & V \end{array}\right] \begin{bmatrix} hf_{n+c_1} \\ hf_{n+c_2} \\ \hline y_n \\ hy'_n \end{bmatrix} \Rightarrow \begin{bmatrix} y_{n+c_1} \\ y_{n+c_2} \\ \hline y_{n+1} \\ hy'_{n+1} \end{bmatrix} \left[\begin{array}{cc|cc} \lambda & a_{12} & 1 & u_{12} \\ a_{21} & \lambda & 1 & u_{22} \\ \hline a_{21} & \lambda & 1 & u_{22} \\ 0 & 1 & 0 & 0 \end{array}\right] \begin{bmatrix} hf_{n+c_1} \\ hf_{n+c_2} \\ \hline y_n \\ hy'_n \end{bmatrix}. \tag{3.15}$$

Step II Using the order conditions (2.10) with the abscissae values $c_1 = \frac{1}{2}, c_2 = 1$ and $a_{21} = \frac{\lambda}{2}$ then

$$u_{12} = \frac{1}{2}\left(-2a_{12} - 2\lambda + 1\right)$$

$$u_{22} = \frac{1}{2}(2 - 3\lambda).$$

Step III In this case, $p = 1$.

Step IV and V By substituting a_{21}, u_{12} and u_{22} into (3.15), the stability matrix can computed, and $M(z)$ is given as

$$M(z) = \begin{bmatrix} -\frac{z\lambda-2}{2\lambda^2 z^2 - \lambda a_{12} z^2 - 4\lambda z + 2} & \frac{4z\lambda^2 - 3z\lambda - 2za_{12}\lambda - 6\lambda + 4}{2(2\lambda^2 z^2 - \lambda a_{12} z^2 - 4\lambda z + 2)} \\ -\frac{z(z\lambda-2)}{2\lambda^2 z^2 - \lambda a_{12} z^2 - 4\lambda z + 2} & \frac{z\left(4z\lambda^2 - 3z\lambda - 2za_{12}\lambda - 6\lambda + 4\right)}{2(2\lambda^2 z^2 - \lambda a_{12} z^2 - 4\lambda z + 2)} \end{bmatrix},$$

with the only non-zero eigenvalue

$$R(z) = \frac{N(z)}{D(z)} = \frac{z^2\left(-2a_{12}\lambda + 4\lambda^2 - 3\lambda\right) + (4 - 8\lambda)z + 4}{2\left(-a_{12}\lambda z^2 + 2\lambda^2 z^2 - 4\lambda z + 2\right)}.$$

For L-stability of (3.15), the coefficient of z^2 in $N(z)$ is set to zero, that is, we equate

$$-2a_{12}\lambda + 4\lambda^2 - 3\lambda = 0,$$

which gives $a_{12} = \frac{1}{2}(4\lambda - 3)$, and thus $R(z)$ yields

$$R(z) = \frac{N(z)}{D(z)} = -\frac{4(2\lambda z - z - 1)}{3\lambda z^2 - 8\lambda z + 4}. \tag{3.16}$$

What is now left is to ensure A-stability, this is done by requiring $E(y) \geq 0$. Replacing $z = iy$ in (3.16), thus

$$E(y) = 9\lambda^2 y^4 + (40\lambda - 16)y^2 \geq 0 \tag{3.17}$$

The inequality (3.18) suggests that for A-stability to be achieved, $\lambda \geq \frac{2}{5}$. As an example, choosing $\lambda = \frac{2}{5}$, the resulting A-stable method is given as

$$\begin{bmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \\ \hline y_{n+1} \\ hy'_{n+1} \end{bmatrix} = \left[\begin{array}{cc|cc} \frac{2}{5} & -\frac{7}{10} & 1 & \frac{4}{5} \\ \frac{1}{5} & \frac{2}{5} & 1 & \frac{2}{5} \\ \hline \frac{1}{5} & \frac{2}{5} & 1 & \frac{2}{5} \\ 0 & 1 & 0 & 0 \end{array}\right] \begin{bmatrix} hf_{n+\frac{1}{2}} \\ hf_{n+1} \\ \hline y_n \\ hy'_n \end{bmatrix}, \tag{3.18}$$

having stability matrix,

$$M(z) = \begin{bmatrix} -\frac{2(z-5)}{3z^2-8z+10} & \frac{4}{3z^2-8z+10} \\ -\frac{2(z-5)z}{3z^2-8z+10} & \frac{4z}{3z^2-8z+10} \end{bmatrix},$$

with the only non-zero eigenvalue of $M(z)$ being

$$R(z) = \frac{2(z+5)}{3z^2 - 8z + 10}.$$

The method (3.18) is seen to be L-stable. This method gained an order to make $p = 2$.
Method $s = 3$, case II.
Step I With $s = 3$, the method for case II is in general

$$\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ \hline y_{n+1} \\ hy'_{n+1} \\ h^2y''_{n+1} \end{bmatrix} = \left[\begin{array}{ccc|ccc} \lambda & 0 & a_{13} & 1 & u_{12} & u_{13} \\ a_{21} & \lambda & a_{23} & 1 & u_{22} & u_{23} \\ 0 & a_{32} & \lambda & 1 & u_{32} & u_{33} \\ \hline 0 & a_{32} & \lambda & 1 & u_{32} & u_{33} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ b_{31} & b_{32} & b_{33} & 0 & v_{32} & 0 \end{array}\right] \begin{bmatrix} hF_1 \\ hF_2 \\ hF_3 \\ \hline y_n \\ hy'_n \\ h^2y''_n \end{bmatrix}. \tag{3.19}$$

Step II Using the order conditions in (2.10) with the abscissae values $c_1 = \frac{1}{3}, c_2 = \frac{2}{3}, c_3 = 1$ and $a_{21} = \frac{\lambda}{2}$, we obtain

$$\begin{gathered} a_{13} + \lambda + u_{12} = \frac{1}{3},\ a_{13} + \frac{\lambda}{3} + u_{13} = \frac{1}{18},\ a_{23} + \frac{3\lambda}{2} + u_{22} = \frac{2}{3}, \\ a_{23} + \frac{5\lambda}{6} + u_{23} = \frac{2}{9},\ a_{32} + \lambda + u_{32} = 1,\ \frac{2a_{32}}{3} + \lambda + u_{33} = \frac{1}{2}, \\ b_{31} + b_{32} + b_{33} + v_{32} = 0, \frac{b_{31}}{3} + \frac{2b_{32}}{3} + b_{33} = 1. \end{gathered} \tag{3.20}$$

Step III In order to achieve IRK stability, the condition (3.1) must be satisfied, So, with $J = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, IRK stability will be achieved if

$$\begin{gathered} b_{31}\lambda + \frac{b_{32}\lambda}{2} = 0,\ a_{32}b_{33} + b_{32}\lambda = 0,\ a_{13}b_{31} + a_{23}b_{32} + b_{33}\lambda - 1 = 0, \\ b_{31}u_{12} + b_{32}u_{22} + b_{33}u_{32} = 0,\ b_{31}u_{13} + b_{32}u_{23} + b_{33}u_{33} = 0. \end{gathered} \tag{3.21}$$

Step IV Solving the systems of equations in (3.21) and (3.22) in terms of a_{13} and λ yields

$$\begin{aligned} b_{31} = \frac{9}{2},\ b_{32} = -9,\ b_{33} = \frac{11}{2},\ a_{23} = \frac{1}{18}\left(9a_{13} + 11\lambda - 2\right), \\ a_{32} = \frac{18\lambda}{11},\ v_{32} = -1,\ u_{12} = \frac{1}{3}\left(-3a_{13} - 3\lambda + 1\right), \\ u_{13} = \frac{1}{18}\left(-18a_{13} - 6\lambda + 1\right),\ u_{22} = \frac{1}{18}\left(-9a_{13} - 38\lambda + 14\right), \\ u_{23} = \frac{1}{18}\left(-9a_{13} - 26\lambda + 6\right),\ u_{32} = \frac{1}{11}(11 - 29\lambda),\ u_{33} = \frac{1}{22}(11 - 46\lambda). \end{aligned}$$

Step V Substituting the real coefficients into (3.19), then $M(z)$ can be obtained. The only non-zero eigenvalue of $M(z)$ is given as

$$R(z) = \frac{N(z)}{D(z)} = \frac{z^3\left(18a_{13}\lambda - 40\lambda^2 + 10\lambda\right) + z^2\left(18a_{13}\lambda - 44\lambda^2 + 62\lambda - 11\right) + (66\lambda - 22)z - 22}{2\left(9a_{13}\lambda z^2 + 2\lambda^2 z^3 - 22\lambda^2 z^2 - 2\lambda z^2 + 33\lambda z - 11\right)}.$$

In order to ensure L-stability, the coefficient of z^3 in $N(z)$ is equated to zero, to obtain a_{13}. Thus,

$$18a_{13}\lambda - 40\lambda^2 + 10\lambda = 0,$$

then

$$a_{13} = \frac{5}{9}(4\lambda - 1).$$

To also ensure A-stability, we look for suitable values of λ by enforcing the condition that the E-polynomial (3.3) is greater or equal to zero. Then $R(z)$ becomes

$$R(z) = \frac{-4\lambda^2 z^2 + 52\lambda z^2 - 11z^2 + 66\lambda z - 22z - 22}{2\left(2\lambda^2 z^3 - 2\lambda^2 z^2 - 7\lambda z^2 + 33\lambda z - 11\right)}.$$

The E-polynomial from this is given as

$$E = 16\lambda^4 y^6 + \left(-2596\lambda^2 + 1144\lambda - 121\right) y^4.$$

Therefore, the second order method is A-stable if $0.176304 \leq \lambda \leq 0.264374$. Choosing $\lambda = \frac{1}{5}$ the resulting method is

$$\left[\begin{array}{c} Y_1 \\ Y_2 \\ Y_3 \\ \hline y_{n+1} \\ hy'_{n+1} \\ h^2y''_{n+1} \end{array}\right] = \left[\begin{array}{ccc|ccc} \frac{1}{5} & 0 & -\frac{1}{9} & 1 & \frac{11}{45} & \frac{1}{10} \\ \frac{1}{10} & \frac{1}{5} & -\frac{2}{45} & 1 & \frac{37}{90} & \frac{1}{10} \\ 0 & \frac{18}{55} & \frac{1}{5} & 1 & \frac{26}{55} & \frac{9}{110} \\ \hline 0 & \frac{18}{55} & \frac{1}{5} & 1 & \frac{26}{55} & \frac{9}{110} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{9}{2} & -9 & \frac{11}{2} & 0 & -1 & 0 \end{array}\right] \left[\begin{array}{c} hF_1 \\ hF_2 \\ hF_3 \\ \hline y_n \\ hy'_n \\ h^2y''_n \end{array}\right]. \tag{3.22}$$

The stability matrix of this is given as

$$M(z) = \left[\begin{array}{ccc} -\frac{2z^2-20z+275}{2z^3-37z^2+165z-275} & \frac{5(3z-26)}{2z^3-37z^2+165z-275} & -\frac{45}{2(2z^3-37z^2+165z-275)} \\ -\frac{z\left(2z^2-20z+275\right)}{2z^3-37z^2+165z-275} & \frac{5z(3z-26)}{2z^3-37z^2+165z-275} & -\frac{45z}{2(2z^3-37z^2+165z-275)} \\ -\frac{z\left(17z^2+110z+275\right)}{2z^3-37z^2+165z-275} & \frac{13z^3-93z^2-165z+275}{2z^3-37z^2+165z-275} & -\frac{45z^2}{2(2z^3-37z^2+165z-275)} \end{array}\right],$$

and its only non-zero eigenvalue is

$$R(z) = \frac{-19z^2 - 220z - 550}{2\left(2z^3 - 37z^2 + 165z - 275\right)}.$$

The method (3.22) is L-stable.
Methods $s = 4, 5$, case II.
Following the procedures discussed above, we choose a_{14} and λ to ensure L-stability and A-stability. The L-stable third order method with abscissae vector $c = \left[\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right]$ and $\lambda = \frac{27}{20}$, is

$$\left[\begin{array}{c} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ \hline y_{n+1} \\ hy'_{n+1} \\ h^2y''_{n+1} \\ h^3y^{(3)}_{n+1} \end{array}\right] = \left[\begin{array}{cccc|cccc} \frac{27}{20} & 0 & 0 & \frac{139}{9920} & 1 & -\frac{11051}{9920} & -\frac{3177}{9920} & -\frac{2773}{59520} \\ \frac{279}{260} & \frac{27}{20} & 0 & -\frac{439}{3120} & 1 & -\frac{5561}{3120} & -\frac{1057}{1560} & -\frac{2773}{24960} \\ 0 & \frac{1053}{220} & \frac{27}{20} & -\frac{3617}{3520} & 1 & -\frac{15343}{3520} & -\frac{671}{320} & -\frac{2773}{7040} \\ 0 & 0 & -\frac{594}{215} & \frac{27}{20} & 1 & \frac{415}{172} & \frac{1051}{860} & \frac{2773}{10320} \\ \hline 0 & 0 & -\frac{594}{215} & \frac{27}{20} & 1 & \frac{415}{172} & \frac{1051}{860} & \frac{2773}{10320} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{248}{15} & -\frac{104}{5} & \frac{88}{15} & \frac{43}{15} & 0 & -\frac{67}{15} & 0 & 0 \\ \frac{31496}{405} & -\frac{11128}{135} & \frac{2552}{135} & \frac{2881}{405} & 0 & -\frac{961}{45} & \frac{13}{27} & 0 \end{array}\right] \left[\begin{array}{c} hF_1 \\ hF_2 \\ hF_3 \\ hF_4 \\ \hline y_n \\ hy'_n \\ h^2y''_n \\ h^3y^{(3)}_n \end{array}\right], \tag{3.23}$$

whose stability matrix $(M(z))$ has only one non-zero eigenvalue given as

$$R(z) = \frac{314903z^3 + 659670z^2 - 908160z + 206400}{3\left(59049z^4 - 277749z^3 + 557010z^2 - 371520z + 68800\right)}.$$

The fourth order L-stable method with inherent Runge-Kutta stability constructed with abscissae vector $c = \left[\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1\right]$ and $\lambda = \frac{21}{50}$, is

$$A = \begin{bmatrix} \frac{21}{50} & 0 & 0 & 0 & -\frac{28758947}{3228284375} \\ \frac{3099153}{5273800} & \frac{21}{50} & 0 & 0 & -\frac{49619763617451}{9534182636650000} \\ \frac{79152743673}{173954183480} & \frac{3876243}{4123075} & \frac{21}{50} & 0 & -\frac{1637379807351255878O7}{786203001492113975000} \\ 0 & \frac{3306303}{613075} & \frac{3463383}{613075} & \frac{21}{50} & -\frac{205406762}{76634375} \\ 0 & 0 & 0 & -\frac{514983}{3358370} & \frac{21}{50} \end{bmatrix},$$

$$U = \begin{bmatrix} 1 & -\frac{1362927231}{6456568750} & -\frac{177851253}{3228284375} & -\frac{101204833}{38739412500} & \frac{76812901}{77478825000} \\ 1 & -\frac{264862822585687}{476709131832500} & -\frac{73168226144087}{476709131832500} & -\frac{24782153724907}{28602547909950000} & \frac{64028853073669}{143012739549750000} \\ 1 & -\frac{3958101225050562144O9}{393101500746056987500} & -\frac{130035411699607374309}{393101500746056987500} & -\frac{155503142590446172890}{786203001492113975000} & \frac{256515143216797401}{178682500339116812500} \\ 1 & -\frac{1223365851}{153268750} & -\frac{220888263}{76634375} & -\frac{72275759}{459806250} & \frac{76812901}{459806250} \\ 1 & \frac{6157094}{8395925} & \frac{340328}{1679185} & \frac{2889779}{503755500} & -\frac{76812901}{5037555000} \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 & 0 & -\frac{514983}{3358370} & \frac{21}{50} \\ 0 & 0 & 0 & 0 & 1 \\ -\frac{235123}{11232} & \frac{211723}{5616} & -\frac{164923}{5616} & \frac{24523}{11232} & \frac{335837}{56160} \\ -\frac{26111078281}{55194048} & \frac{14470724881}{27597024} & -\frac{7907563081}{27597024} & \frac{888885181}{55194048} & \frac{8243790839}{275970240} \\ -\frac{1375814052394645}{271223551872} & \frac{572238025434445}{135611775936} & -\frac{245066080909045}{135611775936} & \frac{226828557553 45}{271223551872} & \frac{336423112807 51}{271223551872} \end{bmatrix},$$

$$V = \begin{bmatrix} 1 & \frac{6157094}{8395925} & \frac{340328}{1679185} & \frac{2889779}{503755500} & -\frac{76812901}{5037555000} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{249163}{56160} & 0 & 0 & 0 \\ 0 & \frac{52235556661}{275970240} & \frac{137971}{9828} & 0 & 0 \\ 0 & \frac{665144996307749}{271223551872} & \frac{10623769825}{48294792} & \frac{38929}{103194} & 0 \end{bmatrix},$$

with, $Y = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \end{bmatrix}$, $y^{[n]} = \begin{bmatrix} y_{n+1} \\ hy'_{n+1} \\ h^2 y''_{n+1} \\ h^3 y^{(3)}_{n+1} \\ h^4 y^{(4)}_{n+1} \end{bmatrix}$, $hF = \begin{bmatrix} hF_1 \\ hF_2 \\ hF_3 \\ hF_4 \\ hF_5 \end{bmatrix}$ and $y^{[n-1]} = \begin{bmatrix} y_n \\ hy'_n \\ h^2 y''_n \\ h^3 y^{(3)}_n \\ h^4 y^{(4)}_n \end{bmatrix}$.

The stability matrix $M(z)$ can be verified to have only one non-zero eigenvalue using (2.12).

4. Implementation and Numerical Experiments

There are efficient ways of implementing general linear methods in literature, some of which include: the implementation of diagonally implicit multistage integration methods, a class of GLMs of [6], implementation of GLMs with IRK stability of [3, 16], just to mention a few, in what follows, we follow the ideas of [3].

4.1. **Convergence of MiGLMs with IRK stability for index 1 and 2 DAEs.** As earlier stated, we are interested in solving differential algebraic equations of index 1 and 2.

Index 1 DAEs.
Applying (2.1) to (1.3), and for every invertible matrix A, we obtain

$$\begin{aligned} Y_i &= hAF(Y_i, Z_i) + Uy^{[n-1]}, \\ 0 &= g(Y_i, Z_i), \\ y^{[n]} &= hBF(Y_i, Z_i) + Vy^{[n-1]}, \\ z^{[n]} &= M(\infty)z^{[n-1]} + BA^{-1}Z_i, \end{aligned} \tag{4.1}$$

with respect to the constructed methods $M(\infty) = V - BA^{-1}U$.

Theorem 4.1. [1, 32] *For the index 1 problem (1.3) with consistent initial conditions, the global error of the integration procedure (4.1) when (2.1) of order* p, *stage order* $q, (p \geq q)$ *and having non-singular A matrix, satisfies*

$$Y_i - y(x_n + c_i h) = O(h^{q+1}),$$
$$Z_i - z(x_n + c_i h) = O(h^{q+1}),$$

with the output solution having global error satisfying

$$y_n - y(x_n) = O(h^p),$$
$$z_n - z(x_n) = O(h^r),$$

for $nh \leq Constant.$

(a) *the MiGLM (2.1) is stiffly accurate and* $z_n - z(x_n) = O(h^p)$.
(b) $r = min(p, q + 1)$ *if* $\rho(M(\infty)) < 1$.
(c) $r = min(p - 1, q)$ *if* $\rho(M(\infty)) = 1$.
(d) *if the method is not stable at infinity, then the solution* z_n *diverges.*

Here, $\rho(M(\infty))$ *means the spectral radius of* $M(\infty)$.

Proof. For the first case, the MiGLM (2.1) is stiffly accurate, thus the numerical solutions arising from (4.1) are equivalent to those of the equation

$$y' = f(y, g(y)),$$

where $z = g(y)$ is the locally unique solution of the second equation of (1.3), thus, the convergence of the solution is described as

$$z_n - z(x_n) = O(h^p).$$

For the remaining case, we have

$$Z(x_n + c_i h) = hAZ'(x_n + c_i h) + Uz(x_n) + O(h^{q+1}), \tag{4.2}$$

$$z(x_n + h) = hBZ'(x_n + c_i h) + Vz(x_n) + O(h^{p+1}), \tag{4.3}$$

since A is non-singular, we compute $Z(x_n + c_i h)$ from (4.2) and substitute into (4.3). This gives

$$z(x_n + h) = M(\infty)z(x_n) + BA^{-1}Z'(x_n + c_i h) + O(h^{q+1}) + O(h^{p+1}), \tag{4.4}$$

and denote the global error $\triangle z_n = z_n - z(x_n)$ and $\triangle Z_n = Z_{n+c_i} - Z(x_n + c_i h)$, subtracting (4.4) from the last expression of (4.1) yields,

$$\triangle z_{n+1} = M(\infty)\triangle z(x_n) + BA^{-1}\triangle Z_n' + O(h^{q+1}) + O(h^{p+1}). \tag{4.5}$$

Since the first three equations of (4.1) define y_{n+1} independently of z_n, y_{n+1} coincides with the solution of (ii) by the same method. Thus, we have

$$\triangle y_n = y_n - y(x_n) = e_p(x_n)h^p + O(h^{p+1}).$$

Also, for the y-component, (4.2) is written as

$$Y(x_n + c_i h) = hAY'(x_n + c_i h) + Uy(x_n) + O(h^{q+1}),$$

and subtracting from the first equation of (4.1) yields

$$\triangle Y_n = U\triangle y_n + hA\left[f\left(Y_{n+c_i}, g(Y_{n+c_i})\right) - f\left(y(x_n + c_i h), g(y(x_n + c_i h))\right)\right] + O(h^{q+1}), \tag{4.6}$$

where $\triangle Y_n = Y_{n+c_i} - y(x_n + c_i h)$ and $\triangle y_n = y_n - y(x_n)$, (4.6) implies that

$$\triangle Y_n = Y_{n+c_i} - y(x_n + c_i h) = O(h^v),$$

with $v = min(p, p+1)$ and the second equation of (4.1) also gives

$$\triangle Z_n = Z_{n+c_i} - z(x_n + c_i h) = O(h^v).$$

Then (4.5) becomes

$$\triangle z_{n+1} = M(\infty)\triangle z(x_n) + O(h^v).$$

Recursion of this formula leads to

$$\triangle z_n = M(\infty)^n\delta_0 + \sum_{i=1}^{n} M(\infty)^{n-i}\delta_i, \qquad (\triangle z_0 = 0), \tag{4.7}$$

with $\delta_0 = O(h^{q+1})$ and $\delta_i = O(h^v)$.
If $\rho(M(\infty)) = 1$, the statement is proved, and if $\rho(M(\infty)) \neq 1$, (4.7) is written as

$$\sum_{i=0}^{n} M(\infty)^{n-i}\delta_i = \left[(I - M(\infty)^{n+1})(I - M(\infty)^{-1})\right]\delta_0 + \sum_{i=1}^{n}\left[(I - M(\infty)^{n-i+1})(I - M(\infty)^{-1})\right](\delta_i - \delta_{i-1}),$$

and the result follows that

$$\delta_i - \delta_{i-1} = O(h^{v+1}).$$

□

Index 2 DAEs.
When (2.1) is applied to (1.4), we obtain

$$\begin{aligned} Y_i &= hAF(Y_i, Z_i) + Uy^{[n-1]}, \\ 0 &= g(Y_i), \\ y^{[n]} &= hBF(Y_i, Z_i) + Vy^{[n-1]}, \\ z^{[n]} &= M(\infty)z^{[n-1]} + BA^{-1}Z_i. \end{aligned} \tag{4.8}$$

The first two equations of (4.8) must have a unique solution, in order to compute for the output solution y_{n+1} and z_{n+1} arising from the last two equations of (4.8).

Theorem 4.2. [1, 32] *Let the index 2 problem (1.4) have consistent initial conditions, whose solution is given as $(y(x), z(x))$, the MiGLM of order p and stage order q with $p \geq 2$ and $q \geq 1$ applied on (1.4), and obtaining (4.8) is convergent with order $r = min(p, q+1)$, i.e.*

$$\begin{aligned} y_n - y(x_n) &= O(h^r), \\ z_n - z(x_n) &= O(h^r), \qquad r \geq 2, \end{aligned}$$

for $x_n - x_0 = nh \leq Constant$.

Proof. The proof is divided into several parts containing

(a) Existence and Uniqueness of the numerical solution.
(b) Influence of perturbation.
(c) Local error.
(d) Convergence for the y and z component.

The proof to (a), (b), and (c) are similar to that of the proof to Theorems 2, 3 and 4 of [32]. For the case (d), let the auxiliary form of (4.8) be given as

$$\begin{aligned} \hat{Y}_i &= hAF(\hat{Y}_i, \hat{Z}_i) + U\hat{y}^{[n-1]}, \\ \hat{y}^{[n]} &= hBF(\hat{Y}_i, \hat{Z}_i) + V\hat{y}^{[n-1]}, \\ 0 &= g(\hat{Y}_i). \end{aligned}$$

Convergence of the y-component: denoting the global error $\triangle y_n = y_n - y(x_n)$, and suppose that $\| \triangle y_n \| \leq C_0 h^2$ and let δ_y denote the local error, i.e. $\delta_y = y_n - y(x_n) = O(h^r)$. If C_1 also denotes the constant involved in the $O(h^r)$ term then

$$\| y_{n+1} - \hat{y}_{n+1} \| = \| \triangle y_{n+1} - \delta_y \| \leq \left(C_0 + C_1 h^{r-2} \right) h^2 \leq C_2 h^2.$$

This comes further to be

$$\begin{aligned} \triangle y_{n+1} &= (y_{n+1} - \hat{y}_{n+1}) + (\hat{y}_{n+1} - y(x_n + h)) \\ &= V \triangle y_n + hB \left(f(Y_i, Z_i) - f(\hat{Y}_i, \hat{Z}_i) \right) + \delta_y. \end{aligned}$$

Denoting $\triangle F_n = f(Y_i, Z_i) - f(\hat{Y}_i, \hat{Z}_i)$, the Lipschitz condition of F_n satisfies

$$\| \triangle F_n \| \leq L \| B \| \left(\| y_n - \hat{y}_n \| + \| z_n - \hat{z}_n \| \right).$$

For the stiffly accurate MiGLM,

$$g(Y_i) - g(Y(x_n + c_i h)) = 0 \qquad\qquad i = 1, 2, ..., s.$$

So that the following estimations are obtained

$$\| y_n - \hat{y}_n \| \leq C_3 \| \triangle y_n \|,$$

and

$$\| \triangle F_n \| \leq C_4 \| \triangle y_n \| .$$

Consequently a recursion of the form

$$\triangle y_{n+1} = V \triangle y_n + h \triangle F_n \delta_y,$$

We can also say there exist another constant C_5 independent of C_0 such that for all n,

$$nh \leq Constant \Longrightarrow \| \triangle y_n \| \leq C_5 h^r.$$

It is seen that C_6 is independent of C_0

$$\triangle y_{n+1} \leq C_6 h^r.$$

This same relation holds for all point x_n (by induction).
Convergence of the z-component: the global error of the z-component can be written as

$$z_{n+1} - z(x_n + h) = (z_{n+1} - \hat{z}_{n+1}) + (\hat{z}_{n+1} - z(x_n + h)),$$

where $\hat{z}_{n+1}$ satisfies the auxiliary system

$$\hat{z}(x_n + h) = M(\infty)\hat{z}(x_n) + BA^{-1} Z_i$$
$$\hat{y}_n = hAF(\hat{Y}_i, \hat{Z}_i) + U\hat{y}(x_n).$$

Then

$$z_{n+1} - z(x_n + h) = M(\infty)(z_n - z(x_n)) + BA^{-1}(z_n - \hat{z}_n) + \delta_y,$$

where $\delta_z = z_n - z(x_n) = O(h^r)$. This can also be shown as in the case of the y-component above that

$$\| z_n - \hat{z}_n \| \leq C (\| y_n - y(x_n) \| + h^r).$$

The result follows from the hypothesis. □

4.2. **Implementation procedures.** To implement the MiGLM (2.1), the following procedures are considered.

General linear method starters: In implementation, starter is required for finding an approximation to the Nordsieck vector

$$y^{[0]} = \begin{bmatrix} y_1^{[0]} \\ y_2^{[0]} \\ y_3^{[0]} \\ \vdots \\ y_{p+1}^{[0]} \end{bmatrix} \approx \begin{bmatrix} y(x_0) \\ hy'(x_0) \\ h^2y''(x_0) \\ \vdots \\ h^p y^{(p)}(x_0) \end{bmatrix}.$$

A good estimate we use here is constructing explicit RKM with abscissae values $c_i = \frac{i-1}{s},\ i = 1, 2, ..., s$ given as

$$\begin{aligned} Y &= h\bar{A}F + \mathbf{e}y_0 \\ y^{[0]} &= h\bar{B}F + \mathbf{e}_1 y_0 \end{aligned} \tag{4.9}$$

where $\mathbf{e} = (1, 1, 1, ..., 1)^T \in \mathbb{R}^{(s\times1)}$, $\mathbf{e}_1 = (1, 0, 0, ..., 0)^T \in \mathbb{R}^{(s\times1)}$ and y_0 is an approximation to the initial solution $y(x_0)$.

Lemma 4.3. [31] *The stage order q of (4.9) satisfies*

$$\hat{c}_i^k = k\sum_{j=1}^{s} \bar{a}_{ij}\hat{c}_j^{k-1}; \quad i = 1, 2, ..., s;\ k = 1, 2, ..., q, \tag{4.10}$$

and the output order p of (4.9) satisfies

$$\frac{1}{k} = \sum_{j=1}^{s} \bar{b}_{ij}\hat{c}_j^{k-1} \quad i = 1, 2, ..., s;\ k = 1, 2, ..., p. \tag{4.11}$$

The stages of (4.9) is obtained by writing the first equation of (4.9) as

$$Y_i = h\sum_{j=1}^{s-1} \bar{a}_{ij}F(Y_j) + y(x_0), \tag{4.12}$$

where $Y_i \approx y(x_0 + \hat{c}_i h)$ and $\hat{c}_i \in [0, 1]$. Expanding (4.12) by Taylor series about x_0 and equating powers of h yields the stage order condition in (4.10).
In similar sense, the component of the output method in (4.9) is obtained by expanding

$$y(x_0 + h) = h\sum_{j=1}^{s-1} \bar{b}_{1j}F(Y_j) + y(x_0),$$

and

$$h^{k-1}y^{(k-1)}(x_0 + h) = h\sum_{j=1}^{s-1} \bar{b}_{ij}F(Y_j), \qquad i = 2, 3, ..., s;\ k = 2, 3, ..., p,$$

by Taylor series about x_0 and equating powers of h, yielding (4.11). Thus, the first stage order condition yields

$$\bar{a}_{1j} = 0,$$

the second stage order condition yields

$$\hat{c}_2 = \sum_{j=1}^{1} \hat{a}_{2j},$$

the third stage order condition yields

$$\hat{c}_3 = \sum_{j=1}^{2} \hat{a}_{3j}, \quad \hat{c}_3^2 = 2\sum_{j=1}^{2} \hat{a}_{3j},$$

the fourth stage order condition yields

$$\hat{c}_4 = \sum_{j=1}^{3} \hat{a}_{4j}, \quad \hat{c}_4^2 = 2\sum_{j=1}^{3} \hat{a}_{4j} \quad \hat{c}_4^3 = 3\sum_{j=1}^{3} \hat{a}_{4j}.$$

Example: For the method of $s = 2, q = 1, p = 2$, $c_1 = 0$, $c_2 = \frac{1}{2}$ the predictor method is given as

$$\left[\begin{array}{c} Y_1 \\ Y_2 \\ \hline y_{n+1} \\ hy'_{n+1} \end{array}\right] = \left[\begin{array}{cc|c} 0 & 0 & 1 \\ \frac{1}{2} & 0 & 1 \\ \hline 0 & 1 & 1 \\ -1 & 2 & 0 \end{array}\right] \left[\begin{array}{c} hF_1 \\ hF_2 \\ \hline y_n \end{array}\right].$$

In the case of $s = 3, q = 2, p = 3$, $c_1 = 0$, $c_2 = \frac{1}{3}, c_3 = \frac{2}{3}$, the predictor method is given as

$$\left[\begin{array}{c} Y_1 \\ Y_2 \\ Y_3 \\ \hline y_{n+1} \\ hy'_{n+1} \\ hy''_{n+1} \end{array}\right] = \left[\begin{array}{ccc|c} 0 & 0 & 0 & 1 \\ \frac{1}{3} & 0 & 0 & 1 \\ 0 & \frac{2}{3} & 0 & 1 \\ \hline \frac{1}{4} & 0 & \frac{3}{4} & 1 \\ 1 & -3 & 3 & 0 \\ \frac{9}{2} & -12 & \frac{15}{2} & 0 \end{array}\right] \left[\begin{array}{c} hF_1 \\ hF_2 \\ hF_3 \\ \hline y_n \end{array}\right].$$

For $s = 4, q = 3, p = 4$, $c_1 = 0$, $c_2 = \frac{1}{4}, c_3 = \frac{1}{2}, c_4 = \frac{3}{4}$, the predictor method is given as

$$\left[\begin{array}{c} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ \hline y_{n+1} \\ hy'_{n+1} \\ hy''_{n+1} \\ hy^{(3)}_{n+1} \end{array}\right] = \left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 1 \\ \frac{1}{4} & 0 & 0 & 0 & 1 \\ 0 & \frac{1}{2} & 0 & 0 & 1 \\ \frac{3}{16} & 0 & \frac{9}{16} & 0 & 1 \\ \hline 0 & \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} & 1 \\ -1 & 4 & -6 & 4 & 0 \\ -\frac{22}{3} & 28 & -38 & \frac{52}{3} & 0 \\ -32 & 112 & -128 & 48 & 0 \end{array}\right] \left[\begin{array}{c} hF_1 \\ hF_2 \\ hF_3 \\ hF_4 \\ \hline y_n \end{array}\right].$$

Iterative Procedure to resolve implicitness in the methods: The Newton's method is employed to resolve the implicitness of the MiGLMs (2.1) . In implementing fully implicit Runge-Kutta methods, the implicitness of each stages must be resolved, however, in the case of the MiGLMs proposed, the implicitness is only on the last stage (Y_s), thus making MiGLMs less computational intensive to that of implicit RKMs. The procedure employed is to first predict the initial Nordsieck vectors $y^{[0]}$ and the last stage Y_s and denote as $\hat{Y}_s$ using the starters discussed earlier. To solve the non-linear problem (1.1), the stages $Y_i,\ i = 1, 2, ..., s-1$ of the MiGLMs (2.1) have to be solved iteratively using

$$Y_i - ha_{ii}f(Y_i) = h\sum_{j=1}^{i-1} a_{ij}f(Y_i) + ha_{ss}f(\hat{Y}_s) + \sum_{j=1}^{r} u_{ij}y_j^{[n-1]}, \qquad i = 1, 2, ..., s-1, \quad (4.13)$$

and the last stage $\hat{Y}_s$ is improved using

$$Y_s - ha_{ss}f(Y_s) = h\sum_{j=1}^{s-1} a_{sj}f(Y_i) + \sum_{j=1}^{r} u_{sj}y_j^{[n-1]}. \quad (4.14)$$

Denoting the right hand side of (4.13) as ϕ_i, (4.13) is expressed as

$$Y_i - ha_{ii}f(Y_i) = \phi_i, \qquad i = 1, 2, ..., s-1, \quad (4.15)$$

equation (4.15) can be expressed as

$$\varphi_i = Y_i - ha_{ii}f(Y_i) - \phi_i = 0, \qquad i = 1, 2, ..., s-1.$$

The Newton's method for (4.13) is defined as

$$Y_i^{[m+1]} = Y_i^{[m]} - J^{-1}\varphi_i^{[m]}, \qquad i = 1, 2, ..., s-1, \qquad m = 0, 1, 2, ..., M,$$

where m is the m-th Newton's iteration and J is the Jacobian defined as

$$J = I - ha_{ii}\frac{\partial f}{\partial y}(Y_i), \qquad i = 1, 2, ..., s-1,$$

also, denoting the right hand side of (4.14) as ϕ_s, (4.14) is expressed as

$$Y_s - ha_{ss}f(Y_s) = \phi_s, \quad (4.16)$$

equation (4.16) is given as

$$\varphi_s = Y_s - ha_{ss}f(Y_s) - \phi_s = 0.$$

The iterative procedure for (4.14) is thus defined as

$$Y_s^{[m+1]} = Y_s^{[m]} - \chi^{-1}\varphi_s^{[m]}, \qquad m = 0, 1, 2, ..., M,$$

where χ is the Jacobian of φ_s. After getting the stages, Eqs. (4.13) and (4.14) are repeated to obtain corrected solution to the stages $Y_i^{[M]},\ i = 1, 2, ..., s$, and $Y_s^{[M]}$ is equal to the output y_{n+1}.

Error estimation and Stepsize control: For variable step size implementation, changing the step size is used to control the error for efficient solution. The local truncation error can be defined as

$$E_n = C_{p+1}h^{p+1}y^{(p+1)}(x_n) + O(h^{p+2}), \qquad p \geq 1, \tag{4.17}$$

where C_{p+1} is the error constant of the specified method. Ignoring the $O(h^{p+2})$ term of (4.17), then

$$E_n \approx C_{p+1}h^{p+1}y^{(p+1)}(x_n),$$

and for

$$h^{p+1}y^{(p+1)}(x_n) \approx h\left(d_1 f(Y_1) + d_2 f(Y_2) + \cdots + d_s f(Y_s)\right),$$

where $d_1, d_2, ..., d_s$ are coefficients found by expanding $f(Y_i)$ by Taylor's series about x_n, the local truncation error can be expressed as

$$E_n \approx C_{p+1}\left[d_1 hf(Y_1) + d_2 hf(Y_2) + \cdots + d_s hf(Y_s)\right]. \tag{4.18}$$

Theorem 4.4. [31] *For $p \geq 1$ and $s = p+1$, the coefficients $d_i,\ i = 1, 2, ..., s$ in (4.18) satisfy the system of equations*

$$\sum_{i=1}^{s} \frac{1}{(k-1)!} c_i^{k-1} d_i = 0, \quad k = 1, 2, ..., p-1,$$
$$\sum_{i=1}^{s} \frac{1}{(k-1)!} c_i^{k-1} d_i = 1, \quad k = p.$$

The following example is the application of this theorem.
Example: For an order $p = 2$ MiGLM with abscissae values $c_1 = \frac{1}{3}, c_2 = \frac{2}{3}$ and $c_3 = 1$.

$$\begin{aligned} h^3 y^{(3)}(x_n) &= d_1 hf(Y_1) + d_2 hf(Y_2) + d_3 hf(Y_3) \\ &= d_1 hy'(x_n + \frac{1}{3}h) + d_2 hy'(x_n + \frac{2}{3}h) + d_3 hy'(x_n + h), \end{aligned}$$

by theorem 4.4, the coefficients $d_i,\ i = 1, 2, 3$ satisfy the system of equations

$$d_1 + d_2 + d_3 = 0, \quad \frac{d_1}{3} + \frac{2d_2}{3} + d_3 = 0, \quad \frac{1}{18}\left(d_1 + 4d_2 + 9d_3\right) = 1, \tag{4.19}$$

solving (4.19) gives $d_1 = 9, \quad d_2 = -18, \quad d_3 = 9.$ For the case I MiGLM (2.1), the error constant for the MiGLM (3.13) of order $p = 2$ is $C_3 = -\frac{1}{3}$, thus, the local truncation error here is given as

$$E_n \approx -3hf(Y_1) + 6hf(Y_2) - 3hf(Y_3).$$

Here, the step size controller used has the form

$$h_{n+1} = \theta_n h_n,$$

where h_n is the stepsize for the step n, h_{n+1} is the stepsize expected in the following step $n+1$ and θ_n is a coefficient computed using

$$\theta_n = min\left(2, max\left(\hat{\theta_n}, \frac{1}{2}\right)\right); \qquad \hat{\theta_n} = \gamma\left(\frac{TOL}{\parallel E_n \parallel}\right)^{\frac{1}{p+1}},$$

where γ is the safety factor chosen to be between 0 and 1, Here, we choose $\gamma = 0.9$, and TOL is the specified tolerance.

4.3. **Test problems for the numerical experiments.** Experiments are carried out on an index 1 and 2 DAEs by implementing the MiGLM with IRK stability (3.13), (3.14), (3.22) and (3.23) using fixed step size and variable step size. MATLAB ode15s and the third order RADAU IIA [1]

$$\left[\begin{array}{c} y_{n+\frac{1}{3}} \\ y_{n+1} \\ \hline y_{n+1} \end{array}\right] = \left[\begin{array}{cc|c} \frac{5}{12} & -\frac{1}{12} & 1 \\ \frac{3}{4} & \frac{1}{4} & 1 \\ \hline \frac{3}{4} & \frac{1}{4} & 1 \end{array}\right] \left[\begin{array}{c} hy'_{n+\frac{1}{3}} \\ hy'_{n+1} \\ \hline y_n \end{array}\right], \tag{4.20}$$

were also implemented on the following test problems.
Problem 1. [32] The index 1 problem

$$\begin{aligned} y'(x) &= -(2+\epsilon^{-1})y(x) + \epsilon^{-1}z(x)^2, & y(0) &= 1; & y(x) &= e^{-2x} \\ 0 &= y(x) - z(x)(1+z(x)) + e^{-x}, & z(0) &= 1; & z(x) &= e^{-x} \\ & & & & \epsilon &= 10^{-1}, 10^{-2}. \end{aligned}$$

Problem 2. [32] The index 2 problem

$$\begin{aligned} y_1'(x) &= -(2+\epsilon^{-1})y_1(x) + \epsilon^{-1}y_2(x)^2, & y_1(0) &= 1; & y_1(x) &= e^{-2x} \\ y_2'(x) &= -e^{1-z(x)^2}, & y_2(0) &= 1; & y_2(x) &= e^{-x} \\ 0 &= y_1(x) - y_2(x)\left(1+y_2(x)\right) + \frac{y_1(x)}{y_2(x)}, & z(0) &= 1; & z(x) &= \sqrt{1+x} \\ & & & & \epsilon &= 10^{-1}, 10^{-2}. \end{aligned}$$

4.4. **Results and discussion.** Using fixed step size implementation, we verify that the numerical order of the methods match the theoretical order with the plot shown in Figs. 1 and 2. In variable step size implementation, we compare the number of function evaluations (nfe) and global errors e_h of the MiGLMs, MATLAB ode15s and RADAU IIA (order $p = 3$) defined in (4.20) with tolerances $TOL = 10^{-j},\ j = 2, 4, 6, 8, 10, 12$. The errors are measured in the $\|\cdot\|_2$ norm. The plots of the global error e_h versus nfe for the MiGLMs (3.13, 3.14, 3.24, 3.25), MATLAB ode15s and RADAU IIA (4.20) with $\epsilon = 10^{-1}$ are shown in Figs. 3 and 5 for problem 1 and Figs. 7 and 9 for problem 2. While the plots of the global error e_h versus nfe for the MiGLMs (3.13, 3.14, 3.24, 3.25), MATLAB ode15s and RADAU IIA (4.20) with $\epsilon = 10^{-2}$ are shown in Figs. 4 and 6 for problem 1 and Figs. 8 and 10 for problem 2

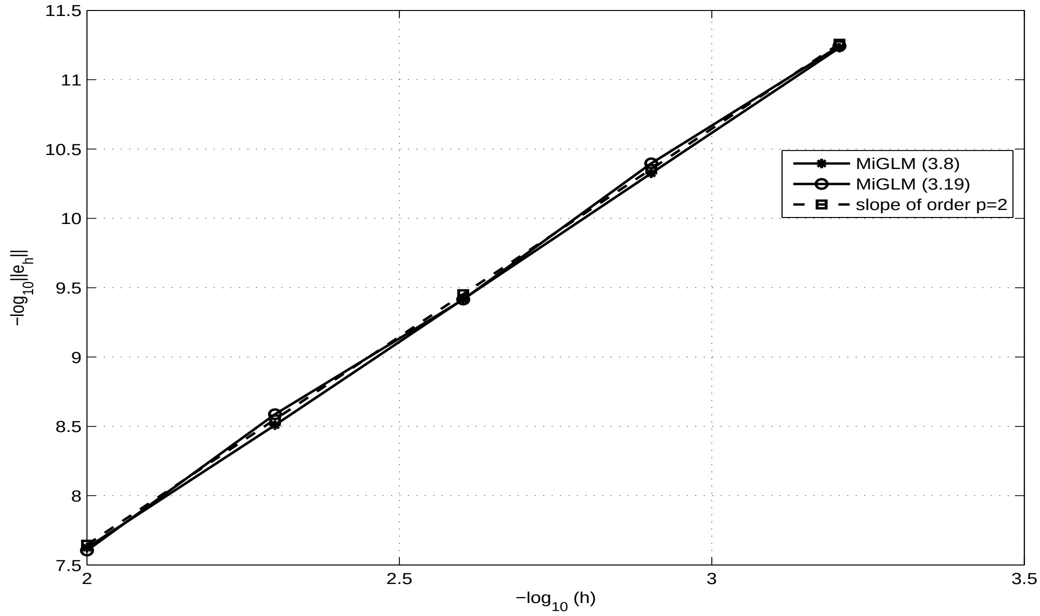

FIGURE 1. Numerical results for the MiGLM (3.13)and (3.24) on problem 1 at $x = 1$ with $\epsilon = 10^{-1}$.

The plots of the global error e_h versus the step size h for the MiGLMs (3.13) and (3.22) of order $p = 2$, MiGLMs (3.14) and (3.23) of order $p = 3$ applied to problem 1 with $x \in [0, 1]$ and $\epsilon = 10^{-1}$ is shown in Fig. 1. It is observed that the orders of the MiGLMs match the theoretical orders, reporting that the MiGLMs do not suffer from order reduction.

MiGLMs (3.13), (3.22) and MATLAB ode15s were implemented on problem 1 with $x \in [0, 1]$ and $\epsilon = 10^{-1}, 10^{2}$. For both cases of $\epsilon = 10^{-1}$ and $\epsilon = 10^{-2}$, the MATLAB ode15s has lesser number of function evaluations for all tolerances $TOL = 10^{-j},\ j = 2(2)12$. However, the MiGLMs reports better accuracy than the MATLAB ode15s as shown in Figs. 3 and 4. In Figs. 7 and 8, the MiGLMs (3.13) and (3.22) also show better accuracy in terms of global error than the MATLAB ode15s when applied on problem 2 with $x \in [0, 1]$ and $\epsilon = 10^{-1}, 10^{-2}$. The nfe and e_h of the MiGLMs (3.13) is equal to the nfe and e_h of the MiGLMs (3.22) for tolerances $TOL = 10^{-2}$ and $TOL = 10^{-4}$, thus the reason for only five nodes in Figs. 3,4,7 and 8.

The plots of nfe versus e_h for the MiGLMs (3.14), (3.23) and RADAU IIA when applied to problem 1 with $x \in [0, 1]$ and $\epsilon = 10^{-1}, 10^{-2}$ is shown in Figs. 5 and 6. It is observed that the MiGLMs (3.14) and (3.23) has improved accuracy and fewer functional evaluations than the RADAU IIA. In the case of problem 2, the MiGLMs (3.14) and (3.23) shows better accuracy in

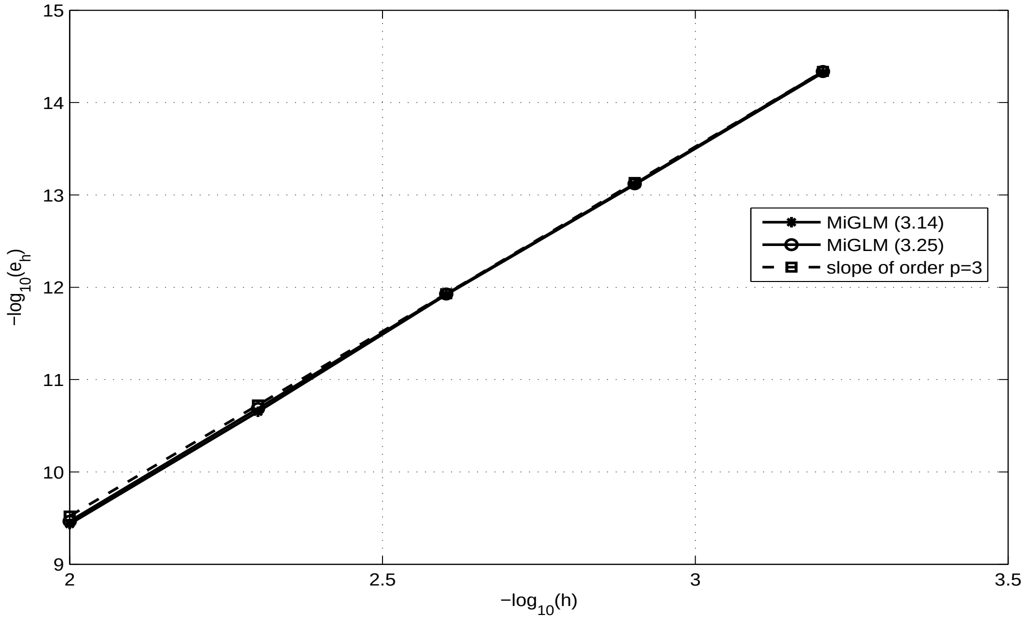

FIGURE 2. Numerical results for the MiGLM (3.14) and (3.25) on problem 1 at $x = 1$ with $\epsilon = 10^{-1}$.

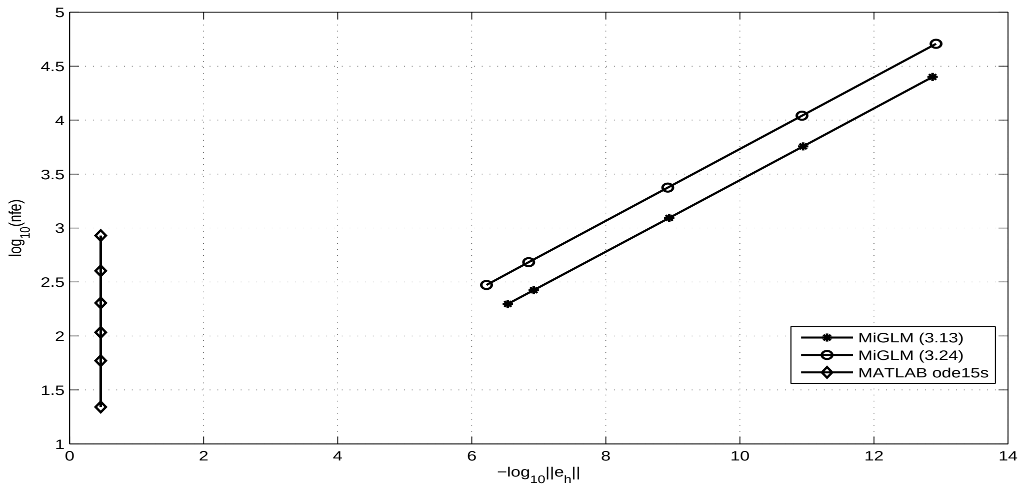

FIGURE 3. nfe versus $\| e_h \|$ at $x = 1$ with $\epsilon = 10^{-1}$ for problem 1.

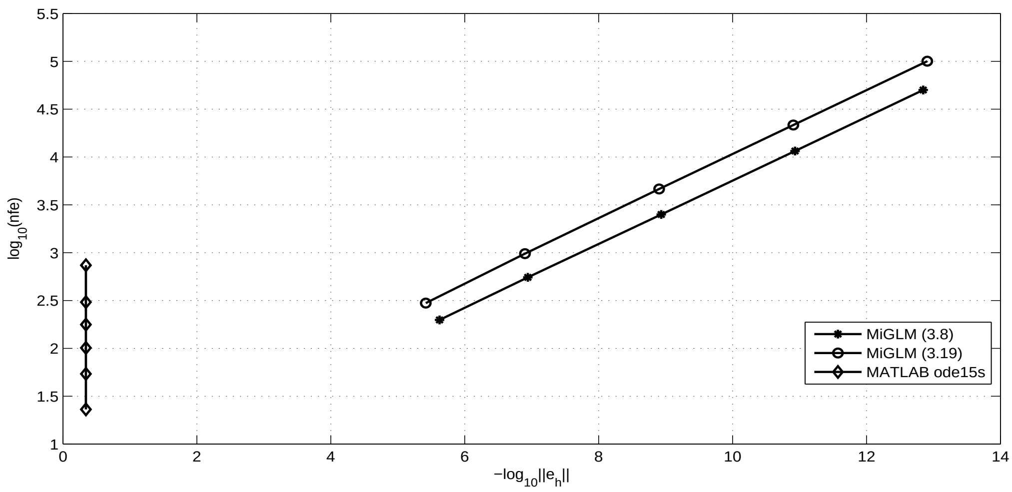

FIGURE 4. nfe versus $\| e_h \|$ at $x = 1$ with $\epsilon = 10^{-2}$ for problem 1.

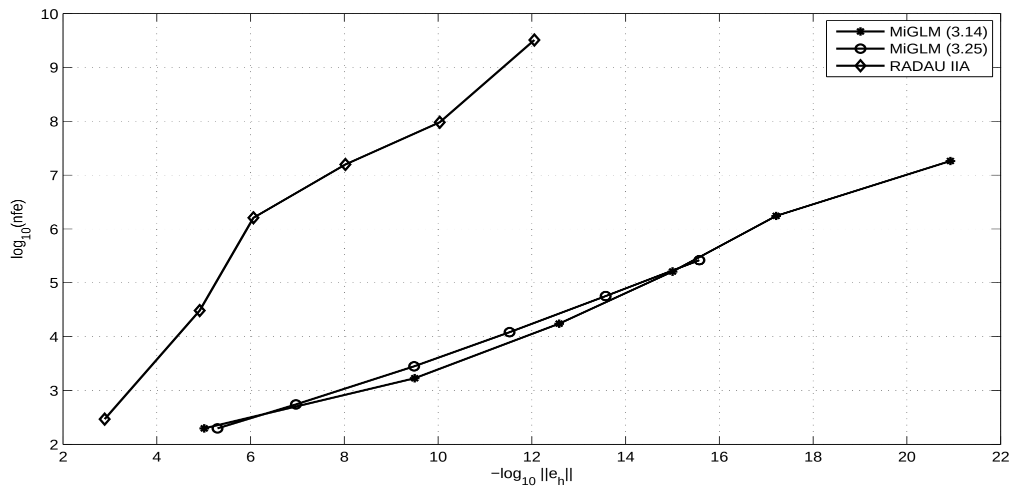

FIGURE 5. nfe versus $\| e_h \|$ at $x = 1$ with $\epsilon = 10^{-1}$ for problem 1.

terms of global error than the RADAU IIA. However, for $\epsilon = 10^{-2}$ and $TOL = 10^{-10}, 10^{-12}$, the MiGLMs (3.23) has more function evaluations with good accuracy. The nfe and global error e_h of MiGLMs (3.14) when $TOL = 10^{-2}$ is equal to that of 10^{-4} for problem 2 with $\epsilon = 10^{-2}$.

The results of the MiGLMs for problems 1 and 2 show that the the methods do not amplify

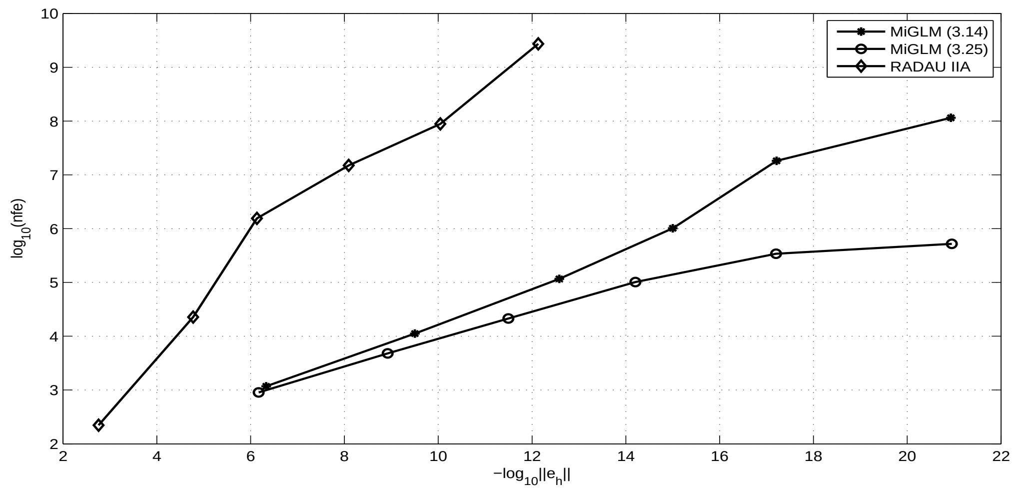

FIGURE 6. nfe versus $\| e_h \|$ at $x = 1$ with $\epsilon = 10^{-2}$ for problem 1.

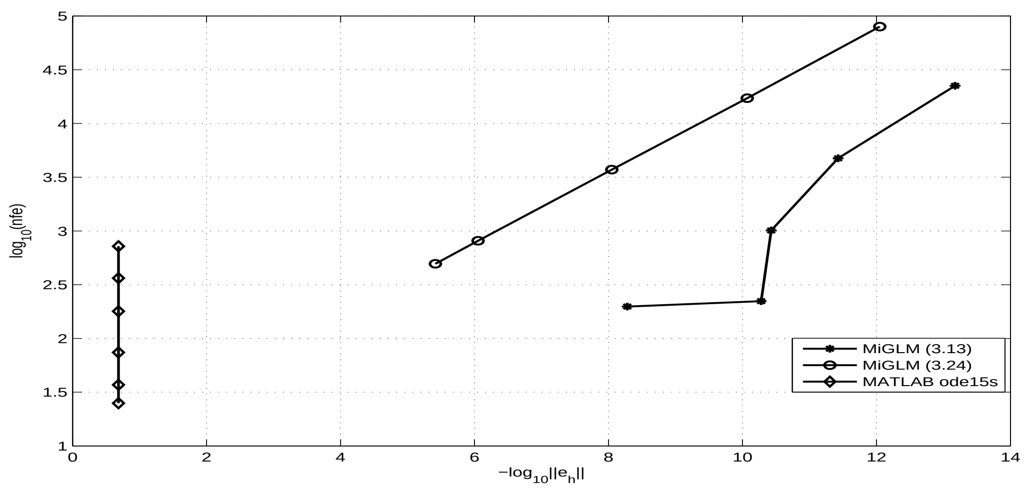

FIGURE 7. nfe versus $\| e_h \|$ at $x = 1$ with $\epsilon = 10^{-1}$ for problem 2.

errors (as shown in Figs. 3-10). This attest that the proposed methods are highly stable and suitable for DAEs. Hence, satisfying its L-stability properties.

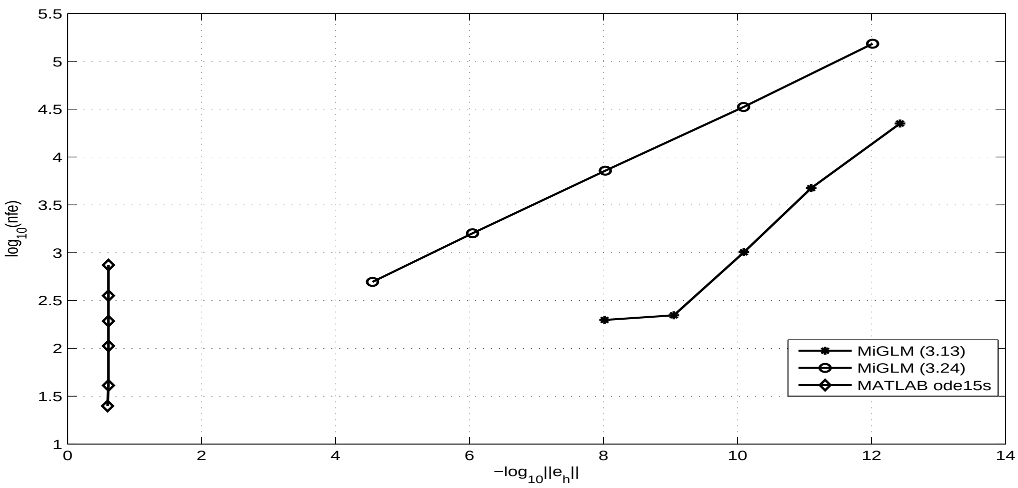

FIGURE 8. nfe versus $\| e_h \|$ at $x = 1$ with $\epsilon = 10^{-2}$ for problem 2.

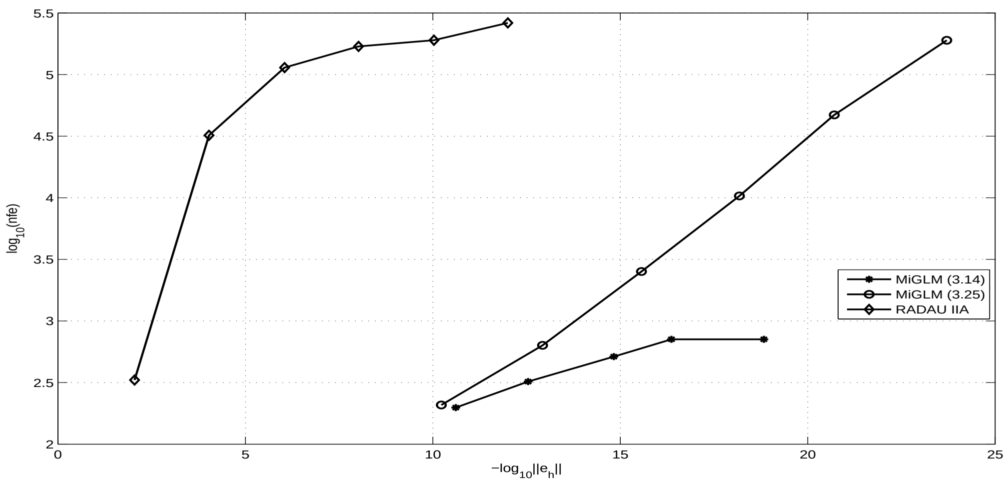

FIGURE 9. nfe versus $\| e_h \|$ at $x = 1$ with $\epsilon = 10^{-1}$ for problem 2.

5. CONCLUSION

Mono-implicit Nordsieck general linear methods (MiGLMs) (2.1) with inherent Runge-Kutta (IRK) stability property have been introduced herein. The methods have been implemented as first same as last (FSAL). Conditions necessary for the MiGLMs (2.1) to possess

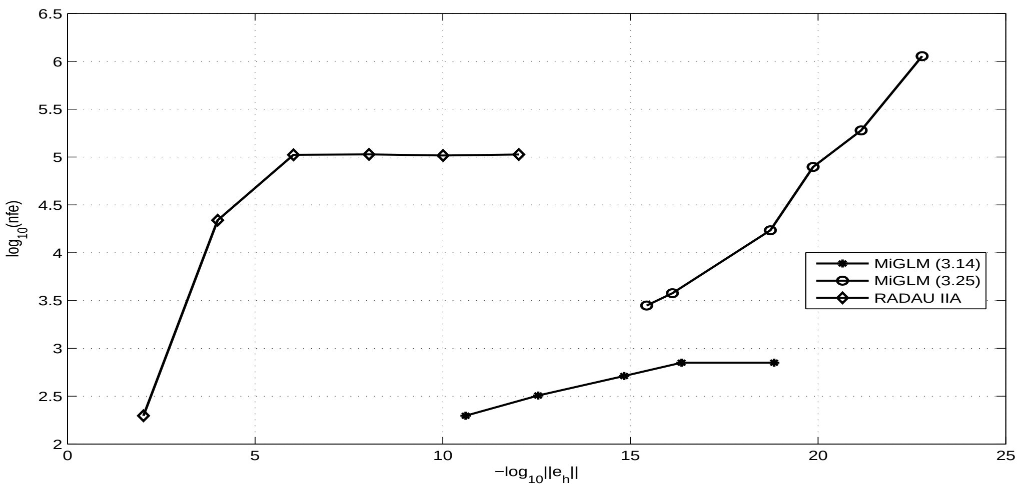

FIGURE 10. nfe versus $\| e_h \|$ at $x = 1$ with $\epsilon = 10^{-2}$ for problem 2.

IRK stability property have been discussed and procedures for constructing such are highlighted in section 3. Two examples of MiGLMs with IRK stability property are derived up to methods with internal stages $s = 5$. The case I MiGLMs (2.1) constructed are L-stable upto internal stages $s = 4$ and L-stable upto internal stages $s = 5$ for case II MiGLMs (2.1). Higher order methods can be derived following the approach discussed in section 3. The methods are also suitable for the solution of DAEs for their reason of L-stability properties. Implementation procedures by means of numerical test, suggest that the MiGLMs (2.1) do not suffer from order reduction and also have improved accuracy than MATLAB ode15s and RADAU IIA (4.20) on the problems considered. Future work will address issues pertaining to variable order variable step size implementation.

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