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# FUNDAMENTAL TONE OF COMPLETE WEAKLY STABLE CONSTANT MEAN CURVATURE HYPERSURFACES IN HYPERBOLIC SPACE

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Abstract. In this paper, we give an upper bound for the fundamental tone of stable constant mean curvature hypersurfaces in hyperbolic space. Let $M$ be an $n$-dimensional complete non-compact constant mean curvature hypersurface with finite $L^2$-norm of the traceless second fundamental form. If $M$ is weakly stable, then $\lambda_1(M)$ is bounded above by $n^2 + O(n^{2+s})$ for arbitrary $s > 0$.

## 1. Introduction

Let $M$ be a complete non-compact Riemannian manifold. The *fundamental tone* $\lambda_1(M)$ of $M$ is defined as

$$\lambda_1(M) = \inf\{\lambda_1(\Omega) :\ \Omega \subset M,\ \Omega \text{ is compact}\}.$$

It can be characterized variationally as

$$\lambda_1(M) = \inf\left\{\frac{\int_M |\nabla f|^2}{\int_M f^2} :\ 0 \neq f \in W_0^{1,2}(M)\right\}. \tag{1.1}$$

To find $\lambda_1(M)$ or to estimate $\lambda_1(M)$ is a very important and interesting problem in differential geometry. McKean [12] showed the following famous theorem.

Theorem (McKean [12]). Let $M$ be a complete simply connected Riemannian manifold with sectional curvature bounded above by a constant $-\kappa^2 < 0$. Then $\lambda_1(M) \geq \frac{(n-1)^2\kappa^2}{4}$.

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Let $\mathbb{H}^m$ be an $m$-dimensional hyperbolic space with constant curvature $-1$. For a complete submanifold in hyperbolic space, Cheung and Leung [7] obtained the following theorem.

Theorem (Cheung and Leung [7]). Let $M$ be an $n$-dimensional complete non-compact submanifold in $\mathbb{H}^m$ with the mean curvature vector $\vec{H}$. If $|\vec{H}| \leq \alpha < n-1$, then

$$\lambda_1(M) \geq \frac{(n-1-\alpha)^2}{4}.$$

There are also upper bound estimates for the fundamental tone of a complete submanifold in hyperbolic space.

Theorem (Candel [5]). Let $M$ be a stable simply connected minimal surface in $\mathbb{H}^3$. Then

$$\frac{1}{4} \leq \lambda_1(M) \leq \frac{4}{3}.$$

Theorem (Seo [13]). Let $M$ be a complete stable minimal hypersurface in $\mathbb{H}^{n+1}$ with $\int_M |A|^2 < \infty$. Then

$$\frac{(n-1)^2}{4} \leq \lambda_1(M) \leq n^2.$$

Seo [14] also generalized his result to a complete minimal hypersurface in $\mathbb{H}^{n+1}$ with finite index. For a cmc-$H$ submanifold in hyperbolic space, Fu and Tao [11] showed the following.

Theorem (Fu and Tao [11]). Let $M$ be an $n$-dimensional complete non-compact orientable submanifold with parallel mean curvature vector in $\mathbb{H}^{n+p}$. If $\int_M |\Phi|^q < \infty$ for $q \geq n$, then

$$\lambda_1(M) \leq \frac{(n-1)^2(1-|H|^2)}{4},$$

where $\Phi$ is the traceless second fundamental form of $M$.

In particular, if $M$ is an $n(\leq 5)$-dimensional complete non-compact weakly stable cmc-$H$ hypersurface in $\mathbb{H}^{n+1}$ with $\int_M |\Phi|^d < \infty$ for $d = 1, 2, 3$, then $\lambda_1(M) \leq \frac{(n-1)^2(1-|H|^2)}{4}$.

Meanwhile, Barbosa and do Carmo [2] proved that any compact cmc-$H$, $H \neq 0$, hypersurface in $\mathbb{R}^{n+1}$ is weakly stable if and only if it is a round sphere. This result was extended by Barbosa, do Carmo, and Eschenburg [3] to a compact cmc-$H$ hypersurface in space forms. Da Silveira [15] studied complete non-compact weakly stable cmc-$H$ surfaces in $\mathbb{R}^3$ and $\mathbb{H}^3$. In $\mathbb{R}^3$, he generalized do Carmo and Peng [6], Fischer-Colbrie and Schoen [9] as follows: Any complete non-compact cmc-$H$ surface is weakly stable if and only if it is totally geodesic. In $\mathbb{H}^3$, the situation turns out differently: If $|H| \geq 1$, then any complete non-compact weakly stable cmc-$H$ surface in $\mathbb{H}^3$ is a horosphere. However, there exists at least one one-parameter family of weakly stable non-umbilic cmc-$H$ embeddings if $|H| < 1$. Later, Cheung and Zhou [8] proved that a complete non-compact weakly stable cmc-$H$ hypersurface in $\mathbb{H}^{n+1}$, $n = 3, 4, 5$, with $|H| > 1$ is a compact geodesic sphere if the $L^2$-norm of the traceless second fundamental form is bounded. Not much is known about complete non-compact weakly stable cmc-$H$ hypersurfaces for higher dimensions.

In this paper, we obtain an upper bound for the fundamental tone of a complete non-compact weakly stable cmc-$H$ hypersurface in $\mathbb{H}^{n+1}$ with finite $L^2$-norm of the traceless second fundamental form.

THEOREM (Theorem 3.2). Let $M$ be an $n$-dimensional complete non-compact orientable cmc-$H$ hypersurface in $\mathbb{H}^{n+1}$ with $\int_M |\Phi|^2 < \infty$. Assume that $M$ is not a totally umbilical cmc-$H$ hypersurface. Let $s > 0$. If $M$ is weakly stable, then

$$\lambda_1(M) \leq n^2 + C_4,$$

where $C_4$ is a constant with $C_4 = O(n^{2+s})$. In particular, if $n = 2$, then $\lambda_1(M^2) \leq n^2 = 4$.

Note that there is no dimension restriction on $M$ in the above theorem.

## 2. Preliminaries

Let $M$ be an $n$-dimensional immersed orientable hypersurface in an $(n+1)$-dimensional Riemannian manifold $N$. Denote by $\overline{\nabla}$ and $\nabla$ the Levi-Civita connections of $N$ and $M$, respectively. The second fundamental form of $M$ is defined by, for all tangent vector fields $X, Y$,

$$\langle AX, Y\rangle = \langle \overline{\nabla}_X Y, \nu\rangle,$$

where $\nu$ is the unit normal vector field of $M$. The (normalized) mean curvature of $M$ is defined as

$$H = \frac{1}{n}\mathrm{tr}A.$$

An immersed hypersurface $M$ in $N$ is said to be *a constant mean curvature hypersurface* if $H$ is constant on $M$. Simply, we call $M$ a cmc-$H$

hypersurface. In particular, $M$ is said to be *a minimal hypersurface* if $H = 0$.

REMARK 2.1. If $M$ is a cmc-$H$ hypersurface with nonzero $H$, then $M$ is orientable. We may assume that $H > 0$ by choosing the suitable orientation.

DEFINITION 2.2. An $n$-dimensional cmc-$H$ hypersurface $M$ in an $(n+1)$-dimensional Riemannian manifold $N$ is called *strongly stable* if for all $f \in W_0^{1,2}(M)$,

$$\int_M \left\{ |\nabla f|^2 - \left( \overline{\mathrm{Ric}}(\nu,\nu) + |A|^2 \right) f^2 \right\} \geq 0, \tag{2.1}$$

where $\overline{\mathrm{Ric}}$ is the Ricci curvature of $N$ and $|A|^2$ is the squared norm of the second fundamental form of $M$ in $N$.

$M$ is said to be *weakly stable* if (2.1) holds for all $f \in W_0^{1,2}(M)$ satisfying

$$\int_M f = 0.$$

A minimal hypersurface $M$ is *stable* if it is strongly stable.

Remark that, for a cmc-$H$ hypersurface, weak stability is more natural than other stability conditions because a cmc-$H$ hypersurface can be viewed as a critical point of area-functional for volume-preserving variations (see [4]). From the definition, a strongly stable cmc-$H$ hypersurface is weakly stable. However, the converse does not hold: For example, a totally geodesic $\mathbb{S}^2$ isometrically immersed in $\mathbb{S}^3$ is weakly stable, but is not strongly stable.

To work with a cmc-$H$ hypersurface $M \subset N$, the traceless second fundamental form is more useful than the second fundamental form. The traceless second fundamental form, denoted by $\Phi$, is defined by

$$\Phi = A - H \cdot g_M,$$

where $g_M$ is the metric on $M$. By a simple computation, we have

$$|A|^2 = |\Phi|^2 + nH^2,$$

and hence, for cmc-$H$ hypersurface, (2.1) becomes

$$\int_M \left\{ |\nabla f|^2 - \left( \overline{\mathrm{Ric}}(\nu,\nu) + |\Phi|^2 + nH^2 \right) f^2 \right\} \geq 0$$

For later use, we recall the famous Simons' inequality for a cmc-$H$ hypersurface in a space form.

THEOREM 2.3 (Simons' inequality [1, 8]). *Let $M$ be a cmc-$H$ hypersurface in a space form $N^{n+1}(c)$ with constant curvature $c$. If $H \geq 0$, then*

$$|\Phi|\triangle|\Phi| \geq \frac{2}{n}|\nabla|\Phi||^2 - |\Phi|^4 - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi|^3 + n(H^2+c)|\Phi|^2. \tag{2.2}$$

## 3. Fundamental tone

Let $M$ be a weakly stable cmc-$H$ hypersurface in $\mathbb{H}^{n+1}$. In $\mathbb{H}^{n+1}$, $\overline{\mathrm{Ric}}(\nu,\nu) = -n$, thus we write (2.1) as follows.

$$\int_M \left\{|\nabla f|^2 - \left(|\Phi|^2 + nH^2 - n\right) f^2\right\} \geq 0. \tag{3.1}$$

Fix a point $p \in M$. Let $r(x) = \mathrm{dist}(p,x)$ and $B(p,r) = \{x \in M \mid r(x) < r\}$ be a distance function from $p$ to $x$ in $M$ and a geodesic ball of radius $r$ centered at $p$, respectively. For any $R > 0$, define a function $\varphi_R(x) \in [-1,1]$ on $M$ as follows.

$$\varphi_R(x) = \begin{cases} 1 & \text{on } B(p,R); \\ 2 - \frac{r(x)}{R} & \text{on } B(p,3R) \setminus B(p,R); \\ -1 & \text{on } B(p,kR) \setminus B(p,3R); \\ -(k+1) + \frac{r(x)}{R} & \text{on } B(p,(k+1)R) \setminus B(p,kR); \\ 0 & \text{on } M \setminus B(p,(k+1)R). \end{cases}$$

Here, we can choose an integer $k > 0$ to make $\int_M \varphi_R < 0$ since $\varphi_R(x) > 0$ if and only if $r(x) < 2R$, and the volume of $M$ is infinite (see [10]). For $0 \leq t \leq R$, define a one-parameter family of functions $\varphi_{R,t}(x)$ to be

$$\varphi_{R,t}(x) = \begin{cases} 1 & \text{on } B(p,R); \\ 2 - \frac{r(x)}{R} & \text{on } B(p,2R+t) \setminus B(p,R); \\ -\frac{t}{R} & \text{on } B(p,(k+1)R-t) \setminus B(p,2R+t); \\ -(k+1) + \frac{r(x)}{R} & \text{on } B(p,(k+1)R) \setminus B(p,(k+1)R-t); \\ 0 & \text{on } M \setminus B(p,(k+1)R). \end{cases}$$

Since $\int_M \varphi_{R,0} > 0$, there exists $t_0 \in (0,R)$ such that $\int_M \varphi_{R,t_0} = 0$. We take $\varphi_{R,t_0}(x) \in [-1,1]$ as a cut-off function on $M$. For the sake of convenience, we simply write it as $\varphi(x)$. The following lemma is originally proved in [8]. Here, we analyze the order of constants.

Lemma 3.1. *Let $M$ be an $n(\geq 3)$-dimensional complete non-compact orientable cmc-$H$ hypersurface in $\mathbb{H}^{n+1}$ with $\int_M |\Phi|^2 < \infty$. Let $s > 0$. If $M$ is weakly stable, then there exist a constant $C_3 = O(n^{1+s})$ independent of $R$ and a constant $R_3 > 0$ such that*

$$\int_M |\Phi|^3\varphi^2 < C_3 \int_M \varphi^2|\Phi|^2,$$

*for all $R > R_3$.*

*Proof.* Multiplying $\varphi^2$ on both sides of (2.2), and integrating on $M$, we have

$$\begin{aligned}&\int_M |\Phi|\triangle|\Phi|\varphi^2 + \int_M |\Phi|^4\varphi^2 + aH\int_M |\Phi|^3\varphi^2\\ &\geq \frac{2}{n}\int_M |\nabla|\Phi||^2\varphi^2 + \int_M (nH^2 - n)|\Phi|^2\varphi^2,\end{aligned}$$

where $a = \frac{n(n-2)}{\sqrt{n(n-1)}}$. The divergence theorem can be applied such that

$$\begin{aligned}&-\int_M |\nabla|\Phi||^2\varphi^2 - 2\int_M |\Phi|\varphi\langle\nabla|\Phi|, \nabla\varphi\rangle + \int_M |\Phi|^4\varphi^2 + aH\int_M |\Phi|^3\varphi^2\\ &\geq \frac{2}{n}\int_M |\nabla|\Phi||^2\varphi^2 + \int_M (nH^2 - n)|\Phi|^2\varphi^2.\end{aligned} \tag{3.2}$$

Since $M$ is stable, (3.1) becomes

$$\begin{aligned}&\int_M |\Phi|^4\varphi^2 + \int_M \left(nH^2 - n\right)|\Phi|^2\varphi^2\\ &\leq \int_M |\nabla(|\Phi|\varphi)|^2\\ &= \int_M |\nabla|\Phi||^2\varphi^2 + \int_M |\Phi|^2|\nabla\varphi|^2 + 2\int_M |\Phi|\varphi\langle\nabla|\Phi|, \nabla\varphi\rangle.\end{aligned} \tag{3.3}$$

Applying Cauchy-Schwarz inequality,

$$\begin{aligned}&\int_M |\Phi|^4\varphi^2 + \int_M \left(nH^2 - n\right)|\Phi|^2\varphi^2\\ &\leq 2\int_M |\nabla|\Phi||^2\varphi^2 + 2\int_M |\Phi|^2|\nabla\varphi|^2.\end{aligned} \tag{3.4}$$

Combining (3.2) and (3.3),

$$
\begin{aligned}
&aH\int_M |\Phi|^3\varphi^2 + \int_M |\Phi|^2|\nabla\varphi|^2 \\
&\geq \frac{2}{n}\int_M |\nabla|\Phi||^2\varphi^2 + 2\int_M (nH^2-n)|\Phi|^2\varphi^2. \qquad (3.5)
\end{aligned}
$$

Multiplying $\frac{1}{n}$ to (3.4), and then combining with (3.5), we have

$$
\begin{aligned}
&aH\int_M |\Phi|^3\varphi^2 + \int_M |\Phi|^2|\nabla\varphi|^2 \\
&\geq \frac{1}{n}\int_M |\Phi|^4\varphi^2 + (2n+1)(H^2-1)\int_M |\Phi|^2\varphi^2 - \frac{2}{n}\int_M |\Phi|^2|\nabla\varphi|^2. \qquad (3.6)
\end{aligned}
$$

Note that $a \neq 0$ if $n \geq 3$. From the Young's inequality, $xy \leq \frac{\epsilon x^2}{2} + \frac{y^2}{2\epsilon}$, we have the following estimate:

$$
\int_M |\Phi|^3\varphi^2 \leq \frac{\epsilon_1}{2}\int_M |\Phi|^4\varphi^2 + \frac{1}{2\epsilon_1}\int_M |\Phi|^2\varphi^2, \qquad (3.7)
$$

where the constant $\epsilon_1 > 0$ will be chosen later. From (3.6) and (3.7), we get

$$
\begin{aligned}
&\left(\frac{1}{n} - \frac{aH\epsilon_1}{2}\right)\int_M |\Phi|^4\varphi^2 \\
&\leq \left(\frac{aH}{2\epsilon_1} - (2n+1)(H^2-1)\right)\int_M |\Phi|^2\varphi^2 + \left(1+\frac{2}{n}\right)\int_M |\Phi|^2|\nabla\varphi|^2.
\end{aligned}
$$

Let $A = \frac{1}{n} - \frac{aH\epsilon_1}{2}$, $B = \frac{aH}{2\epsilon_1} - (2n+1)(H^2-1)$, and $C = 1 + \frac{2}{n}$. We can choose $\epsilon_1$ sufficiently small such that $A, B, C > 0$. Moreover, if we let $\epsilon_1 = \theta n^{-2s}$ for some $\theta > 0$, then constants $C_1, C_2$ can be obtained by choosing sufficiently small $\theta$ such that $\frac{B}{A} < C_1 = O(n^{2+2s})$ and $\frac{C}{A} < C_2 = O(n)$. Therefore

$$
\int_M |\Phi|^4\varphi^2 \leq C_1\int_M |\Phi|^2\varphi^2 + C_2\int_M |\Phi|^2|\nabla\varphi|^2.
$$

Note that $C_1$ and $C_2$ are independent of $R$. By using the Cauchy-Schwarz inequality,

$$
\begin{aligned}
&\int_M |\Phi|^3\varphi^2 \leq \left(\int_M |\Phi|^2\varphi^2\right)^{\frac{1}{2}} \cdot \left(\int_M |\Phi|^4\varphi^2\right)^{\frac{1}{2}} \\
&\leq \left(\int_M |\Phi|^2\varphi^2\right)^{\frac{1}{2}} \cdot \left(C_1\int_M |\Phi|^2\varphi^2 + C_2\int_M |\Phi|^2|\nabla\varphi|^2\right)^{\frac{1}{2}}.
\end{aligned}
$$

Note that $\varphi$ is a function of $R$. For every $\epsilon > 0$, there is $R_1 > 0$ such that $\int_M |\Phi|^2|\nabla\varphi|^2 < \epsilon$ if $R > R_1$. As $\epsilon$ goes to 0, $\int_M |\Phi|^2\varphi^2$ converges to $\int_M |\Phi|^2$, which is positive unless $M$ is totally umbilical. For every positive $\epsilon < \frac{1}{2}\int_M |\Phi|^2$, there is $R_2 > 0$ such that $-\epsilon+\int_M |\Phi|^2 < \int_M |\Phi|^2\varphi^2$ if $R > R_2$. Put $R_3 = \max\{R_1, R_2\}$. Then, for all $R > R_3$,

$$\int_M |\Phi|^3\varphi^2 \leq C_3 \int_M |\Phi|^2\varphi^2,$$

where $C_3$ is a constant such that $(C_1 + C_2)^{\frac{1}{2}} \leq C_3 = O(n^{1+s})$. □

Now we give an upper bound for the fundamental tone of a complete non-compact weakly stable cmc-$H$ hypersurface in $\mathbb{H}^{n+1}$.

THEOREM 3.2. *Let $M$ be an $n$-dimensional complete non-compact orientable cmc-$H$ hypersurface in $\mathbb{H}^{n+1}$ with $\int_M |\Phi|^2 < \infty$. Assume that $M$ is not a totally umbilical cmc-$H$ hypersurface. Let $s > 0$. If $M$ is weakly stable, then*

$$\lambda_1(M) \leq n^2 + C_4,$$

*where $C_4$ is a constant with $C_4 = O(n^{2+s})$. In particular, if $n = 2$, then $\lambda_1(M^2) \leq n^2 = 4$.*

*Proof.* Putting $f = |\Phi|\varphi$ in (1.1), we have

$$\begin{aligned}
&\lambda_1(M) \int_M |\Phi|^2\varphi^2 \\
&\leq \int_M |\nabla(|\Phi|\varphi)|^2 \\
&= \int_M |\nabla|\Phi||^2\varphi^2 + \int_M |\Phi|^2|\nabla\varphi|^2 + 2\int_M |\Phi|\varphi\langle\nabla|\Phi|, \nabla\varphi\rangle \\
&= \left(1 + \frac{1}{\epsilon_2}\right) \int_M |\nabla|\Phi||^2\varphi^2 + (1 + \epsilon_2) \int_M |\Phi|^2|\nabla\varphi|^2.
\end{aligned}$$

In the last equality, we use the Cauchy-Schwarz and the Young's inequality. The constant $\epsilon_2 > 0$ will be determined later. By remark 2.1, we may assume that $H \geq 0$. The inequality (3.5) still holds, and thus we have

$$\begin{aligned}
&\frac{2}{n}\int_M |\nabla|\Phi||^2\varphi^2 \\
&\leq aH \int_M |\Phi|^3\varphi^2 + \int_M |\Phi|^2|\nabla\varphi|^2 + 2n\int_M |\Phi|^2\varphi^2. \qquad (3.8)
\end{aligned}$$

Note with $\lambda_1(M) > 0$ if $H < \alpha < \frac{n-1}{n}$ by Cheung and Leung [7]. However, it is not known whether it is usually positive. Here, what we want to get is an upper bound so that, without loss of generality, we may assume $\lambda_1(M) > 0$. Applying Lemma 3.1 to (3.8) for $n \geq 3$, if $R > R_3$, then

$$\left(\frac{2}{n} - \frac{(2n + C_3)(1 + \frac{1}{\epsilon_2})}{\lambda_1(M)}\right) \int_M |\nabla|\Phi||^2 \varphi^2$$

$$\leq \left(1 + \frac{(2n + C_3)(1 + \epsilon_2)}{\lambda_1(M)}\right) \int_M |\Phi|^2 |\nabla\varphi|^2. \tag{3.9}$$

For a sufficiently large $\epsilon_2 > 0$, the right hand side of (3.9) converges to zero as $R$ goes to infinity because a complete non-compact stable cmc-$H$ hypersurface in hyperbolic space has infinite volume. If $\frac{2}{n} > \frac{(2n+C_3)(1+\frac{1}{\epsilon_2})}{\lambda_1(M)}$, then $|\nabla|\Phi||^2 \equiv 0$ on $M$, and thus $M$ is a totally umbilical cmc-$H$ hypersurface. This is a contradiction. Therefore we get

$$\lambda_1(M) \leq n^2 + O(n^{2+s}).$$

If $n = 2$, then $a = 0$. Similarly, we get $\lambda_1(M^2) \leq n^2 = 4$. □

## References


[1] H. Alencar and M. do Carmo, *Hypersurfaces with constant mean curvature in spheres*, Proc. Amer. Math. Soc., **120** (1994), no. 4, 1223-1229.

[2] J. L. Barbosa and M. do Carmo, *Stability of hypersurfaces with constant mean curvature*, Math. Z., **185** (1984), no. 3, 339-353.

[3] J. L. Barbosa and M. do Carmo and J. Eschenburg, *Stability of hypersurfaces of constant mean curvature in Riemannian manifolds*, Math. Z., **197** (1988), no. 1, 123-138.

[4] J. L. Barbosa and P. Bérard, *Eigenvalue and "twisted" eigenvalue problems, applications to CMC surfaces*, J. Math. Pures Appl., (9), **79** (2009), no. 5, 427-450.

[5] A. Candel, *Eigenvalue estimates for minimal surfaces in hyperbolic space*, Trans. Amer. Math. Soc., **359** (2007), no. 8, 3567-3575.

[6] M. do Carmo. and C. K. Peng, *Stable complete minimal surfaces in* $\mathbf{R}^3$ *are planes*, Bull. Amer. Math. Soc., (N.S.), **1** (1979), no. 6, 903-906.

[7] L.-F. Cheung and P.-F. Leung, *Eigenvalue estimates for submanifolds with bounded mean curvature in the hyperbolic space*, Math. Z., **236** (2001), no. 3, 525-530.

[8] L.-F. Cheung and D. Zhou, *Stable constant mean curvature hypersurfaces in* $\mathbb{R}^{n+1}$ *and* $\mathbb{H}^{n+1}(-1)$, Bull. Braz. Math. Soc., (N.S.), **36** (2005), no. 1, 99-114.

[9] D. Fischer-Colbrie and R. Schoen, *The structure of complete stable minimal surfaces in* 3*-manifolds of nonnegative scalar curvature*, Comm. Pure Appl. Math., **33** (1980), no. 2, 199-211.

[10] K. R. Frensel, *Stable complete surfaces with constant mean curvature*, An. Acad. Brasil. Ciênc., **60** (1988), no. 2, 115-117.

[11] H. Fu and Y. Tao, *Eigenvalue estimates for complete submanifolds in the hyperbolic spaces*, J. Math. Res. Appl., **33** (2013), no. 5, 598-606.
[12] H. P. McKean, *An upper bound to the spectrum of $\Delta$ on a manifold of negative curvature*, J. Differ. Geom., **4** (1970), no. 3, 359-366.
[13] K. Seo, *Stable minimal hypersurfaces in the hyperbolic space*, J. Korean Math. Soc., **48** (2011), no. 2, 253-266.
[14] K. Seo, *Fundamental tone of minimal hypersurfaces with finite index in hyperbolic space*, J. Inequal. Appl., (2016), no. 127, 1-5.
[15] A. M. da Silveira, *Stability of complete noncompact surfaces with constant mean curvature*, Math. Ann., **277** (1987), no. 4, 629-638.

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