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A NOTE ON THE INTEGRAL REPRESENTATIONS OF GENERALIZED RELATIVE ORDER (α, β) AND GENERALIZED RELATIVE TYPE (α, β) OF ENTIRE AND MEROMORPHIC FUNCTIONS WITH RESPECT TO AN ENTIRE FUNCTION

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Abstract. In this paper we wish to establish the integral representations of generalized relative order (α, β) and generalized relative type (α, β) of entire and meromorphic functions where α and β are continuous non-negative functions defined on $(-\infty, +\infty)$. We also investigate their equivalence relation under some certain condition.

1. Introduction

For any entire function $f = \sum\limits_{n=0}^{\infty} a_n z^n$ on $|z| = r$, $M_f(r)$, a function of r is defined as follows:

$$M_f(r) = \max_{|z|=r} |f(z)|.$$

If an entire function f is non-constant then $M_f(r)$ is strictly increasing and continuous and its inverse $M_f{}^{-1} : (|f(0)|, \infty) \to (0, \infty)$ exists and is such that $\lim\limits_{s \to +\infty} M_f^{-1}(s) = \infty$.

Whenever f is meromorphic, one can define another function $T_f(r)$ (see [13, p.4]) known as Nevanlinna's characteristic function of f plays the same role as $M_f(r)$. Moreover, if f is non-constant entire then $T_f(r)$ is also strictly increasing and continuous function of r. Therefore its inverse $T_f^{-1} : (T_f(0), \infty) \to (0, \infty)$ exists and is such that $\lim\limits_{s \to +\infty} T_f^{-1}(s) = \infty$.

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The order and lower order of entire or meromorphic functions which are generally used in computational purpose are classical in complex analysis. Mainly the growth investigation of entire and meromorphic functions has usually been done through their maximum moduli or Nevanlinna's characteristic function in comparison with those of exponential function. But if one is paying attention to evaluate the growth rates of any entire and meromorphic function with respect to a new entire function, the notions of relative growth indicators (see e.g. [1, 2, 15]) will come. On the other hand, Sheremeta [17] introduced the concept of generalized order of entire functions considering two continuous non-negative functions defined on $(-\infty, +\infty)$. During the past decades, several authors made close investigations on the properties of entire functions related to generalized order as introduced by Sheremeta [17] in some different direction. For the purpose of further applications, Biswas et al. (see e.g. [4] to [11]) rewrite the definition of the generalized order of entire and meromorphic function after giving a minor modification to the original definition introduced by Sheremeta (e.g. see, [17]). Further, in order to make some progress in the study of relative order, Biswas et al. (see e.g. [3, 8]) used the definitions of generalized relative order (α, β) and generalized relative lower order (α, β) of an entire and meromorphic function where α and β are continuous non-negative functions defined on $(-\infty, +\infty)$.

In order to refine the above growth scale, one may use the definitions of other growth indicators, such as generalized relative type (α, β) and generalized relative weak type (α, β) of entire and meromorphic function with respect to an entire function. For details one may see [3, 8].

Here in this paper, we wish to establish the integral representations of the definitions of generalized relative order (α, β), generalized relative lower order (α, β), generalized relative type (α, β) and generalized relative weak type (α, β) of entire and meromorphic function with respect to an entire function which considerably extend earlier results of [12, p.4]. We also investigate their equivalence relations under certain conditions. We do not explain the standard definitions and notations in the theory of entire and meromorphic functions (see, e.g., [13], [14], [16], [18] to [20]).

2. Preliminary Remarks and Definitions

Let L be a class of continuous non-negative on $(-\infty, +\infty)$ functions α such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ with $\alpha(x) \uparrow +\infty$ as $x \to +\infty$ and $\alpha((1 + o(1))x) =$

$(1+o(1))\alpha(x)$ as $x \to +\infty$. We say that $\alpha \in L^0$, if $\alpha \in L$ and $\alpha(cx) = (1+o(1))\alpha(x)$ as $x_0 \leq x \to +\infty$ for each $c \in (0,+\infty)$, i.e., α is slowly increasing function. Clearly $L^0 \subset L$. Further we assume that throughout the present paper α, $\beta \in L$ unless otherwise specifically stated. Further suppose that $\mu(x)$ and $\gamma(x)$ are any two positive continuous increasing to $+\infty$ on $[x_0,+\infty)$ functions such that $\mu^{-1}(x)$ exist.

Taking this into account, let us give the following definitions:

Definition 2.1. Let $\mu(x)$ and $\gamma(x)$ be any two positive continuous increasing to $+\infty$ on $[x_0,+\infty)$ functions. The generalized relative order (α,β) of $\gamma(x)$ with respect to $\mu(x)$ is defined by

$$\rho_{(\alpha,\beta)}[\gamma]_\mu = \limsup_{r\to+\infty} \frac{\alpha(\mu^{-1}(\gamma(r)))}{\beta(r)}.$$

Definition 2.2. Let $\mu(x)$ and $\gamma(x)$ be any two positive continuous increasing to $+\infty$ on $[x_0,+\infty)$ functions. The *growth indicator* $\rho_{(\alpha,\beta)}[\gamma]_\mu$ is alternatively defined as: *The integral* $\int\limits_{r_0}^{\infty} \frac{\exp(\alpha(\mu^{-1}(\gamma(r))))}{(\exp\beta(r))^{k+1}}dr$ $(r_0>0)$ *converges for* $k > \rho_{(\alpha,\beta)}[\gamma]_\mu$ *and diverges for* $k < \rho_{(\alpha,\beta)}[\gamma]_\mu$.

Definition 2.3. Let $\mu(x)$ and $\gamma(x)$ be any two positive continuous increasing to $+\infty$ on $[x_0,+\infty)$ functions. The generalized relative lower order (α,β) of $\gamma(x)$ with respect to $\mu(x)$ is defined as*:*

$$\lambda_{(\alpha,\beta)}[\gamma]_\mu = \liminf_{r\to+\infty} \frac{\alpha(\mu^{-1}(\gamma(r)))}{\beta(r)}.$$

Definition 2.4. Let $\mu(x)$ and $\gamma(x)$ be any two positive continuous increasing to $+\infty$ on $[x_0,+\infty)$ functions. The *growth indicator* $\lambda_{(\alpha,\beta)}[\gamma]_\mu$ is alternatively defined as: *The integral* $\int\limits_{r_0}^{\infty} \frac{\exp(\alpha(\mu^{-1}(\gamma(r))))}{(\exp\beta(r))^{k+1}}dr$ $(r_0>0)$ *converges for* $k > \lambda_{(\alpha,\beta)}[\gamma]_\mu$ *and diverges for* $k < \lambda_{(\alpha,\beta)}[\gamma]_\mu$.

If we consider $\mu(x) = M_g(x)$, $\gamma(x) = M_f(x)$ and α, $\beta \in L^0$ where f and g are any two entire functions, then the above definitions reduce to the definitions of generalized relative order (α,β) and generalized relative lower order (α,β) of an entire function f with respect to another entire function g respectively as introduced by Biswas et al. [3]. Similarly if we take $\mu(x) = T_g(x)$, $\gamma(x) = T_f(x)$ and α, $\beta \in L^0$ where f is a meromorphic function and g be any entire function, then the above definitions reduce to the definitions of generalized relative order (α,β) and generalized relative lower order (α,β) of a meromorphic function f with respect to an entire function g respectively. For details, one may see [8] and [3].

In order to refine the above growth scale, one may introduce the definition of an another growth indicator, called generalized relative type (α,β), as follows:

Definition 2.5. Let $\mu(x)$ and $\gamma(x)$ be any two positive continuous increasing to $+\infty$ on $[x_0,+\infty)$ functions. The generalized relative type (α,β) denoted by $\sigma_{(\alpha,\beta)}[\gamma]_\mu$ of $\gamma(x)$ with respect to $\mu(x)$ having finite positive generalized relative order (α,β), $\rho_{(\alpha,\beta)}[\gamma]_\mu$ $(0<\rho_{(\alpha,\beta)}[\gamma]_\mu<+\infty)$ is defined by

$$\sigma_{(\alpha,\beta)}[\gamma]_\mu=\limsup_{r\to+\infty}\frac{\exp(\alpha(\mu^{-1}(\gamma(r))))}{(\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[\gamma]_\mu}}.$$

The above definition can alternatively be given in the following manner:

Definition 2.6. Let $\mu(x)$ and $\gamma(x)$ be any two positive continuous increasing to $+\infty$ on $[x_0,+\infty)$ functions having finite positive generalized relative order (α,β), $\rho_{(\alpha,\beta)}[\gamma]_\mu$ $(0<\rho_{(\alpha,\beta)}[\gamma]_\mu<+\infty)$. Then the generalized relative type (α,β), $\sigma_{(\alpha,\beta)}[\gamma]_\mu$ of $\gamma(x)$ with respect to $\mu(x)$ is defined as: The integral $\int_{r_0}^{\infty}\frac{\exp(\exp(\alpha(\mu^{-1}(\gamma(r)))))}{[\exp((\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[\gamma]_\mu})]^{k+1}}dr$ $(r_0>0)$ converges for $k>\sigma_{(\alpha,\beta)}[\gamma]_\mu$ and diverges for $k<\sigma_{(\alpha,\beta)}[\gamma]_\mu$.

Analogously, to determine the relative growth of two increasing functions having same non-zero finite generalized relative lower order (α,β), one can introduce the definition of generalized relative weak type (α,β) in the following way.

Definition 2.7. Let $\mu(x)$ and $\gamma(x)$ be any two positive continuous increasing to $+\infty$ on $[x_0,+\infty)$ functions having finite positive generalized relative lower order (α,β), $\lambda_{(\alpha,\beta)}[\gamma]_\mu$ $(0<\lambda_{(\alpha,\beta)}[\gamma]_\mu<+\infty)$. Then the generalized relative weak type (α,β) denoted by $\overline{\tau}_{(\alpha,\beta)}[\gamma]_\mu$ of $\gamma(x)$ with respect to $\mu(x)$ is defined as

$$\overline{\tau}_{(\alpha,\beta)}[\gamma]_\mu=\liminf_{r\to+\infty}\frac{\exp(\alpha(\mu^{-1}(\gamma(r))))}{(\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[\gamma]_\mu}}.$$

The above definition can also alternatively be given

Definition 2.8. Let $\mu(x)$ and $\gamma(x)$ be any two positive continuous increasing to $+\infty$ on $[x_0,+\infty)$ functions having finite positive generalized relative lower order (α,β), $\lambda_{(\alpha,\beta)}[\gamma]_\mu$ $(0<\lambda_{(\alpha,\beta)}[\gamma]_\mu<+\infty)$. Then the generalized relative weak type

(α,β), $\overline{\tau}_{(\alpha,\beta)}[\gamma]_\mu$ of $\gamma(x)$ with respect to $\mu(x)$ is defined as

$$\int_{r_0}^{\infty} \frac{\exp(\exp(\alpha(\mu^{-1}(\gamma(r)))))}{[\exp((\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[\gamma]_\mu})]^{k+1}} dr \ (r_0 > 0)$$

converges for $k > \overline{\tau}_{(\alpha,\beta)}[\gamma]_\mu$ and diverges for $k < \overline{\tau}_{(\alpha,\beta)}[\gamma]_\mu$.

Now a question may arise about the equivalence of the definitions of generalized relative order (α,β), generalized relative lower order (α,β), generalized relative type (α,β) and generalized relative weak type (α,β) with their integral representations. In this paper we would like to establish such equivalence of Definition 2.1, Definition 2.3, Definition 2.5 and Definition 2.7 with Definition 2.2, Definition 2.4, Definition 2.6 and Definition 2.8 respectively and also investigate some growth properties related to generalized relative type (α,β) and generalized relative weak type (α,β) of $\gamma(x)$ with respect to $\mu(x)$.

3. Main Results

In this section we state the main results of this paper. First of all we prove the following lemma which will be needed in the sequel.

Lemma 3.1. *Let the integral* $\int_{R_0}^{\infty} \frac{\exp(\alpha(\mu^{-1}(\gamma(r))))}{(\exp\beta(r))^{k+1}} dr$ $(r_0 > 0)$ *converges for* $0 < k < +\infty$. *Then*

$$\lim_{r\to+\infty} \frac{\exp(\alpha(\mu^{-1}(\gamma(r))))}{(\exp\beta(r))^{k}} = 0.$$

Proof. Since the integral $\int_{r_0}^{\infty} \frac{\exp(\alpha(\mu^{-1}(\gamma(r))))}{(\exp\beta(r))^{k+1}} dr$ is convergent for $0 < k < +\infty$, given ε (> 0) there exists a number $\Re = \Re(\varepsilon)$ such that

$$\int_{r_0}^{\infty} \frac{\exp(\alpha(\mu^{-1}(\gamma(r))))}{(\exp\beta(r))^{k+1}} dr < \varepsilon \text{ for } r_0 > \Re.$$

i.e., for $r_0 > \Re$,

$$\int_{r_0}^{r_0+r} \frac{\exp(\alpha(\mu^{-1}(\gamma(r))))}{(\exp\beta(r))^{k+1}} dr < \varepsilon.$$

Since $\exp(\alpha(\mu^{-1}(\gamma(r))))$ is an increasing function of r, so

$$\int_{r_0}^{r_0+\exp\beta(r_0)} \frac{\exp(\alpha(\mu^{-1}(\gamma(r))))}{(\exp\beta(r))^{k+1}}dr \geq \frac{\exp(\alpha(\mu^{-1}(\gamma(r_0))))}{(\exp\beta(r_0))^{k+1}} \cdot (\exp\beta(r_0))$$

$$\text{i.e.,} \quad \int_{r_0}^{r_0+\exp\beta(r_0)} \frac{\exp(\alpha(\mu^{-1}(\gamma(r))))}{(\exp\beta(r))^{k+1}}dr \geq \frac{\exp(\alpha(\mu^{-1}(\gamma(r_0))))}{(\exp\beta(r_0))^{k}} \text{ for } r_0 > \Re,$$

$$i.e., \ \frac{\exp(\alpha(\mu^{-1}(\gamma(r_0))))}{(\exp\beta(r_0))^{k}} < \varepsilon \text{ for } r_0 > \Re,$$

from which it follows that

$$\lim_{r\to+\infty} \frac{\exp(\alpha(\mu^{-1}(\gamma(r))))}{(\exp\beta(r))^{k}} = 0.$$

This proves the lemma. □

Theorem 3.2. *Let $\mu(x)$ and $\gamma(x)$ be any two positive continuous increasing to $+\infty$ on $[x_0, +\infty)$ functions. Then Definition 2.1 and Definition 2.2 are equivalent.*

Proof. CASE 1. $\rho_{(\alpha,\beta)}[\gamma]_\mu = +\infty$.

DEFINITION 2.1 ⇒ DEFINITION 2.2.

As $\rho_{(\alpha,\beta)}[\gamma]_\mu = \infty$, from Definition 2.1 we have for arbitrary positive G and for a sequence of values of r tending to infinity that

$$\alpha(\mu^{-1}(\gamma(r))) > G\beta(r)$$

$$i.e, \exp(\alpha(\mu^{-1}(\gamma(r)))) > (\exp\beta(r))^{G}. \tag{3.1}$$

If possible let the integral $\int_{r_0}^{\infty} \frac{\exp(\alpha(\mu^{-1}(\gamma(r))))}{(\exp\beta(r))^{G+1}}dr$ $(r_0 > 0)$ be converge. Then by Lemma 3.1,

$$\limsup_{r\to+\infty} \frac{\exp(\alpha(\mu^{-1}(\gamma(r))))}{(\exp\beta(r))^{G}} = 0.$$

So for all sufficiently large values of r,

$$\exp(\alpha(\mu^{-1}(\gamma(r)))) < (\exp\beta(r))^{G}. \tag{3.2}$$

Now from (3.1) and (3.2) we arrive at a contradiction.

Hence $\int_{r_0}^{\infty} \frac{\exp(\alpha(\mu^{-1}(\gamma(r))))}{(\exp\beta(r))^{G+1}}dr$ $(r_0 > 0)$ diverges whenever G is finite, which is Definition 2.2.

DEFINITION 2.2 ⇒ DEFINITION 2.1.

Suppose G be any positive number. Since $\rho_{(\alpha,\beta)}[\gamma]_\mu = +\infty$, from Definition 2.2 the divergence of the integral $\int\limits_{r_0}^{\infty} \frac{\exp(\alpha(\mu^{-1}(\gamma(r))))}{(\exp\beta(r))^{G+1}}dr$ $(r_0 > 0)$ gives for arbitrary positive ε and for a sequence of values of r tending to infinity,

$$\exp(\alpha(\mu^{-1}(\gamma(r)))) > (\exp\beta(r))^{G-\varepsilon}$$

$$i.e,\ \alpha(\mu^{-1}(\gamma(r))) > (G-\varepsilon)\beta(r).$$

This gives that

$$\limsup_{r\to+\infty}\frac{\alpha(\mu^{-1}(\gamma(r)))}{\beta(r)} \geq (G-\varepsilon).$$

Since $G > 0$ is arbitrary, it follows that

$$\limsup_{r\to+\infty}\frac{\alpha(\mu^{-1}(\gamma(r)))}{\beta(r)} = \infty.$$

Thus Definition 2.1 follows.

CASE 2. $0 \leq \rho_{(\alpha,\beta)}[\gamma]_\mu < +\infty$.

DEFINITION 2.1 $\Rightarrow$ DEFINITION 2.2.

SUBCASE (I). $0 < \rho_{(\alpha,\beta)}[\gamma]_\mu < +\infty$.

If $0 < \rho_{(\alpha,\beta)}[\gamma]_\mu < \infty$, then for arbitrary $\varepsilon(> 0)$ and for all sufficiently large values of r,

$$\frac{\alpha(\mu^{-1}(\gamma(r)))}{\beta(r)} < \rho_{(\alpha,\beta)}[\gamma]_\mu + \varepsilon$$

$$i.e, \exp(\alpha(\mu^{-1}(\gamma(r)))) < (\exp\beta(r))^{(\rho_{(\alpha,\beta)}[\gamma]_\mu+\varepsilon)}$$

$$i.e, \frac{\exp(\alpha(\mu^{-1}(\gamma(r))))}{(\exp\beta(r))^k} < \frac{(\exp\beta(r))^{(\rho_{(\alpha,\beta)}[\gamma]_\mu+\varepsilon)}}{(\exp\beta(r))^k}$$

$$i.e, \frac{\exp(\alpha(\mu^{-1}(\gamma(r))))}{(\exp\beta(r))^k} < \frac{1}{(\exp\beta(r))^{k-(\rho_{(\alpha,\beta)}[\gamma]_\mu+\varepsilon)}}.$$

Therefore $\int\limits_{r_0}^{\infty} \frac{\exp(\alpha(\mu^{-1}(\gamma(r))))}{(\exp\beta(r))^{k+1}}dr$ $(r_0 > 0)$ converges if $k > \rho_{(\alpha,\beta)}[\gamma]_\mu$ and diverges if $k < \rho_{(\alpha,\beta)}[\gamma]_\mu$.

SUBCASE (II).

When $\rho_{(\alpha,\beta)}[\gamma]_\mu = 0$, Definition 2.1 gives for all sufficiently large values of r that

$$\frac{\alpha(\mu^{-1}(\gamma(r)))}{\beta(r)} \leq \varepsilon.$$

Then as before we obtain that $\int\limits_{R_0}^{\infty} \frac{\exp(\alpha(\mu^{-1}(\gamma(r))))}{(\exp\beta(r))^{k+1}}dr$ $(r_0 > 0)$ converges for $k > 0$ and diverges for $k < 0$.

Thus from Subcase (I) and Subcase (II) Definition 2.2 follows.

DEFINITION 2.2 $\Rightarrow$ DEFINITION 2.1.

By Definition 2.2, for arbitrary ε (>0) the integral $\int\limits_{r_0}^{\infty} \frac{\exp(\alpha(\mu^{-1}(\gamma(r))))}{(\exp\beta(r))^{\rho_{(\alpha,\beta)}[\gamma]_\mu+\varepsilon+1}}dr$ converges. Then by Lemma 3.1 we have

$$\limsup_{r\to+\infty}\frac{\exp(\alpha(\mu^{-1}(\gamma(r))))}{(\exp\beta(R))^{\rho_{(\alpha,\beta)}[\gamma]_\mu+\varepsilon}}=0$$

i.e, for all sufficiently large values of r,

$$\frac{\exp(\alpha(\mu^{-1}(\gamma(r))))}{(\exp\beta(r))^{\rho_{(\alpha,\beta)}[\gamma]_\mu+\varepsilon}}<\varepsilon_0$$

$$i.e,\ \exp(\alpha(\mu^{-1}(\gamma(r))))<\varepsilon_0\cdot(\exp\beta(r))^{\rho_{(\alpha,\beta)}[\gamma]_\mu+\varepsilon}$$

$$i.e,\ \alpha(\mu^{-1}(\gamma(r)))<\log\varepsilon_0+(\rho_{(\alpha,\beta)}[\gamma]_\mu+\varepsilon)\beta(r)$$

$$i.e,\ \frac{\alpha(\mu^{-1}(\gamma(r)))}{\beta(r)}\le\frac{\log\varepsilon_0}{\beta(r)}+(\rho_{(\alpha,\beta)}[\gamma]_\mu+\varepsilon)$$

$$i.e,\ \limsup_{r\to+\infty}\frac{\alpha(\mu^{-1}(\gamma(r)))}{\beta(r)}\le\rho_{(\alpha,\beta)}[\gamma]_\mu+\varepsilon.$$

Since ε (>0) is arbitrary, it follows from above that

$$\limsup_{r\to+\infty}\frac{\alpha(\mu^{-1}(\gamma(r)))}{\beta(r)}\le\rho_{(\alpha,\beta)}[\gamma]_\mu. \tag{3.3}$$

Again by Definition 2.2 the divergence of the integral $\int\limits_{r_0}^{\infty} \frac{\exp(\alpha(\mu^{-1}(\gamma(r))))}{(\exp\beta(r))^{\rho_{(\alpha,\beta)}[\gamma]_\mu-\varepsilon+1}}dr$ implies that there exists a sequence of values of r tending to infinity such that

$$\frac{\exp(\alpha(\mu^{-1}(\gamma(r))))}{(\exp\beta(r))^{\rho_{(\alpha,\beta)}[\gamma]_\mu-\varepsilon+1}}>\frac{1}{(\exp\beta(r))^{1+\varepsilon}}$$

$$i.e, \exp(\alpha(\mu^{-1}(\gamma(r))))>(\exp\beta(r))^{\rho_{(\alpha,\beta)}[\gamma]_\mu-2\varepsilon}$$

$$i.e,\ \alpha(\mu^{-1}(\gamma(r)))>(\rho_{(\alpha,\beta)}[\gamma]_\mu-2\varepsilon)\beta(r)$$

$$i.e,\ \frac{\alpha(\mu^{-1}(\gamma(r)))}{\beta(r)}>(\rho_{(\alpha,\beta)}[\gamma]_\mu-2\varepsilon).$$

As ε (>0) is arbitrary, we get that

$$\limsup_{r\to+\infty}\frac{\alpha(\mu^{-1}(\gamma(r)))}{\beta(r)}\ge\rho_{(\alpha,\beta)}[\gamma]_\mu. \tag{3.4}$$

Thus from (3.3) and (3.4) it follows that

$$\limsup_{r\to+\infty}\frac{\alpha(\mu^{-1}(\gamma(r)))}{\beta(r)}=\rho_{(\alpha,\beta)}[\gamma]_\mu.$$

Thus we obtain Definition 2.1.

Now combining Case 1 and Case 2, the theorem follows. □

In the line of Theorem 3.2 we may now state the following theorem without proof.

Theorem 3.3. *Let $\mu(x)$ and $\gamma(x)$ be any two positive continuous increasing to $+\infty$ on $[x_0, +\infty)$ functions. Then Definition 2.3 and Definition 2.4 are equivalent.*

Next we prove the equivalence of the definitions of generalized relative type (α, β) and generalized relative weak type (α, β) with their integral representations. For this purpose we need the following lemma.

Lemma 3.4. *Let the integral $\int\limits_{r_0}^{\infty} \frac{\exp(\exp(\alpha(\mu^{-1}(\gamma(r)))))}{[\exp((\exp(\beta(r)))^A)]^{k+1}} dr$ $(r_0 > 0)$ converge where $0 < A < +\infty$. Then*

$$\lim_{r \to +\infty} \frac{\exp(\exp(\alpha(\mu^{-1}(\gamma(r)))))}{[\exp((\exp(\beta(r)))^A)]^{k}} = 0.$$

Proof. Since the integral $\int\limits_{r_0}^{\infty} \frac{\exp(\exp(\alpha(\mu^{-1}(\gamma(r)))))}{[\exp((\exp(\beta(r)))^A)]^{k+1}} dr$ $(r_0 > 0)$ converges, then there exist $R(\varepsilon) > 0$ such that

$$\int\limits_{r_0}^{\infty} \frac{\exp(\exp(\alpha(\mu^{-1}(\gamma(r)))))}{[\exp((\exp(\beta(r)))^A)]^{k+1}} dr < \varepsilon, \text{ if } r_0 > R(\varepsilon).$$

Therefore,

$$\int\limits_{r_0}^{\exp((\exp(\beta(r_0)))^A)+r_0} \frac{\exp(\exp(\alpha(\mu^{-1}(\gamma(r)))))}{[\exp((\exp(\beta(r)))^A)]^{k+1}} dr < \varepsilon.$$

Since here $\exp(\exp(\alpha(\mu^{-1}(\gamma(r)))))$ increases with r, so

$$\int\limits_{r_0}^{\exp((\exp(\beta(r_0)))^A)+r_0} \frac{\exp(\exp(\alpha(\mu^{-1}(\gamma(r)))))}{[\exp((\exp(\beta(r)))^A)]^{k+1}} dr \geq .$$

$$\frac{\exp(\exp(\alpha(\mu^{-1}(\gamma(r_0)))))}{[\exp((\exp(\beta(r_0)))^A)]^{k+1}} \cdot [\exp((\exp(\beta(r_0)))^A)].$$

Therefore for all sufficiently large values of r,

$$\int\limits_{r_0}^{\exp((\exp(\beta(r_0)))^A)+r_0} \frac{\exp(\exp(\alpha(\mu^{-1}(\gamma(r)))))}{[\exp((\exp(\beta(r)))^A)]^{k+1}} dr \geq \frac{\exp(\exp(\alpha(\mu^{-1}(\gamma(r_0)))))}{[\exp((\exp(\beta(r_0)))^A)]^{k}},$$

so that

$$\frac{\exp(\exp(\alpha(\mu^{-1}(\gamma(r_0)))))}{[\exp((\exp(\beta(r_0)))^A)]^k} < \varepsilon \text{ if } r_0 > R(\varepsilon).$$

$$i.e., \lim_{r\to+\infty} \frac{\exp(\exp(\alpha(\mu^{-1}(\gamma(r_0)))))}{[\exp((\exp(\beta(r_0)))^A)]^k} = 0.$$

This proves the lemma. □

Theorem 3.5. *Let $\mu(x)$ and $\gamma(x)$ be any two positive continuous increasing to $+\infty$ on $[x_0,+\infty)$ functions having finite positive generalized relative order (α,β), $\rho_{(\alpha,\beta)}[\gamma]_\mu$ $(0<\rho_{(\alpha,\beta)}[\gamma]_\mu<+\infty)$ and generalized relative type (α,β), $\sigma_{(\alpha,\beta)}[\gamma]_\mu$. Then Definition 2.5 and Definition 2.6 are equivalent.*

Proof. Let us consider $\mu(x)$ and $\gamma(x)$ be any two positive continuous increasing to $+\infty$ on $[x_0,+\infty)$ functions such that $\rho_{(\alpha,\beta)}[\gamma]_\mu$ $(0<\rho_{(\alpha,\beta)}[\gamma]_\mu<+\infty)$ exists.

CASE I. $\sigma_{(\alpha,\beta)}[\gamma]_\mu=+\infty$.

DEFINITION 2.5 ⇒ DEFINITION 2.6.

As $\sigma_{(\alpha,\beta)}[\gamma]_\mu=+\infty$, from Definition 2.5 we have for arbitrary positive G and for a sequence of values of r tending to infinity that

$$\exp(\alpha(\mu^{-1}(\gamma(r)))) > G\cdot(\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[\gamma]_\mu}$$

$$i.e., \exp(\exp(\alpha(\mu^{-1}(\gamma(r))))) > [\exp((\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[\gamma]_\mu})]^G. \tag{3.5}$$

If possible let the integral $\int_{r_0}^{\infty}\frac{\exp(\exp(\alpha(\mu^{-1}(\gamma(r)))))}{[\exp((\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[\gamma]_\mu})]^{G+1}}dr$ $(r_0>0)$ be convergent. Then by Lemma 3.4,

$$\limsup_{r\to+\infty}\frac{\exp(\exp(\alpha(\mu^{-1}(\gamma(r)))))}{[\exp((\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[\gamma]_\mu})]^G}=0.$$

So for all sufficiently large values of r,

$$\exp(\exp(\alpha(\mu^{-1}(\gamma(r))))) < [\exp((\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[\gamma]_\mu})]^G. \tag{3.6}$$

Therefore from (3.5) and (3.6) we arrive at a contradiction.

Hence $\int_{r_0}^{\infty}\frac{\exp(\exp(\alpha(\mu^{-1}(\gamma(r)))))}{[\exp((\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[\gamma]_\mu})]^{G+1}}dr$ $(r_0>0)$ diverges whenever G is finite, which is the Definition 2.6.

DEFINITION 2.6 ⇒ DEFINITION 2.5.

Let G be any positive number. Since $\sigma_{(\alpha,\beta)}[\gamma]_\mu=+\infty$, from Definition 2.6, the divergence of the integral $\int_{r_0}^{\infty}\frac{\exp(\exp(\alpha(\mu^{-1}(\gamma(r)))))}{[\exp((\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[\gamma]_\mu})]^{G+1}}dr$ $(r_0>0)$ gives for arbitrary positive ε and for a sequence of values of r tending to infinity

$$\exp(\exp(\alpha(\mu^{-1}(\gamma(r))))) > [\exp((\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[\gamma]_\mu})]^{G-\varepsilon}$$
$$i.e.,\ \exp(\alpha(\mu^{-1}(\gamma(r)))) > (G-\varepsilon)(\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[\gamma]_\mu},$$

which implies that

$$\limsup_{r\to+\infty}\frac{\exp(\alpha(\mu^{-1}(\gamma(r))))}{(\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[\gamma]_\mu}} \geq G-\varepsilon.$$

Since $G>0$ is arbitrary, it follows that

$$\limsup_{r\to+\infty}\frac{\exp(\alpha(\mu^{-1}(\gamma(r))))}{(\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[\gamma]_\mu}} = \infty.$$

Thus Definition 2.5 follows.

Case II. $0 \leq \sigma_{(\alpha,\beta)}[\gamma]_\mu < +\infty$.

Definition 2.5 $\Rightarrow$ Definition 2.6.

Subcase (A). $0 < \sigma_{(\alpha,\beta)}[\gamma]_\mu < +\infty$.

Let $\mu(x)$ and $\gamma(x)$ be any two positive continuous increasing to $+\infty$ on $[x_0,+\infty)$ functions such that $0<\sigma_{(\alpha,\beta)}[\gamma]_\mu<+\infty$ exists. Then according to the Definition 2.5, for arbitrary positive ε and for all sufficiently large values of r, we obtain

$$\exp(\alpha(\mu^{-1}(\gamma(r)))) < (\sigma_{(\alpha,\beta)}[\gamma]_\mu+\varepsilon)(\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[\gamma]_\mu}$$
$$i.e.,\ \exp(\exp(\alpha(\mu^{-1}(\gamma(r))))) < [\exp((\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[\gamma]_\mu})]^{\sigma_{(\alpha,\beta)}[\gamma]_\mu+\varepsilon}$$
$$i.e.,\ \frac{\exp(\exp(\alpha(\mu^{-1}(\gamma(r)))))}{[\exp((\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[\gamma]_\mu})]^k} < \frac{[\exp((\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[\gamma]_\mu})]^{\sigma_{(\alpha,\beta)}[\gamma]_\mu+\varepsilon}}{[\exp((\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[\gamma]_\mu})]^k}$$
$$i.e.,\ \frac{\exp(\exp(\alpha(\mu^{-1}(\gamma(r)))))}{[\exp((\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[\gamma]_\mu})]^k} < \frac{1}{[\exp((\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[\gamma]_\mu})]^{k-(\sigma_{(\alpha,\beta)}[\gamma]_\mu+\varepsilon)}}$$

Therefore $\int\limits_{r_0}^{\infty}\frac{\exp(\exp(\alpha(\mu^{-1}(\gamma(r)))))}{[\exp((\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[\gamma]_\mu})]^{k+1}}dr$ $(r_0>0)$ converges for $k>\sigma_{(\alpha,\beta)}[\gamma]_\mu$.

Again by Definition 2.5, we obtain for a sequence values of r tending to infinity that

$$\exp(\alpha(\mu^{-1}(\gamma(r)))) > (\sigma_{(\alpha,\beta)}[\gamma]_\mu-\varepsilon)(\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[\gamma]_\mu}$$
$$i.e.,\ \exp(\exp(\alpha(\mu^{-1}(\gamma(r))))) > [\exp((\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[\gamma]_\mu})]^{\sigma_{(\alpha,\beta)}[\gamma]_\mu-\varepsilon}. \tag{3.7}$$

So for $k<\sigma_{(\alpha,\beta)}[\gamma]_\mu$, we get from (3.7) that

$$\frac{\exp(\exp(\alpha(\mu^{-1}(\gamma(r)))))}{[\exp((\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[\gamma]_\mu})]^k} > \frac{1}{[\exp((\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[\gamma]_\mu})]^{k-(\sigma_{(\alpha,\beta)}[\gamma]_\mu-\varepsilon)}}.$$

Therefore $\int\limits_{r_0}^{\infty} \frac{\exp(\exp(\alpha(\mu^{-1}(\gamma(r)))))}{[\exp((\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[\gamma]_\mu})]^{k+1}} dr$ $(r_0 > 0)$ diverges for $k < \sigma_{(\alpha,\beta)}[\gamma]_\mu$.

Hence $\int\limits_{r_0}^{\infty} \frac{\exp(\exp(\alpha(\mu^{-1}(\gamma(r)))))}{[\exp((\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[\gamma]_\mu})]^{k+1}} dr$ $(r_0 > 0)$ converges for $k > \sigma_{(\alpha,\beta)}[\gamma]_\mu$ and diverges for $k < \sigma_{(\alpha,\beta)}[\gamma]_\mu$.

SUBCASE (B). $\sigma_{(\alpha,\beta)}[\gamma]_\mu = 0$.

When $\sigma_{(\alpha,\beta)}[\gamma]_\mu = 0$, Definition 2.5 gives for all sufficiently large values of r that

$$\frac{\exp(\alpha(\mu^{-1}(\gamma(r))))}{(\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[\gamma]_\mu}} < \varepsilon.$$

Then as before we obtain that $\int\limits_{r_0}^{\infty} \frac{\exp(\exp(\alpha(\mu^{-1}(\gamma(r)))))}{[\exp((\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[\gamma]_\mu})]^{k+1}} dr$ $(r_0 > 0)$ converges for $k > 0$ and diverges for $k < 0$.

Thus combining Subcase (A) and Subcase (B), Definition 2.6 follows.

DEFINITION 2.6 $\Rightarrow$ DEFINITION 2.5.

From Definition 2.6 and for arbitrary positive ε the integral

$$\int\limits_{r_0}^{\infty} \frac{\exp(\exp(\alpha(\mu^{-1}(\gamma(r)))))}{[\exp((\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[\gamma]_\mu})]^{\sigma_{(\alpha,\beta)}[\gamma]_\mu+\varepsilon+1}} dr \ (r_0 > 0)$$

converges. Then by Lemma 3.4, we get

$$\limsup_{r\to+\infty} \frac{\exp(\exp(\alpha(\mu^{-1}(\gamma(r)))))}{[\exp((\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[\gamma]_\mu})]^{\sigma_{(\alpha,\beta)}[\gamma]_\mu+\varepsilon}} = 0.$$

So we obtain all sufficiently large values of r that

$$\frac{\exp(\exp(\alpha(\mu^{-1}(\gamma(r)))))}{[\exp((\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[\gamma]_\mu})]^{\sigma_{(\alpha,\beta)}[\gamma]_\mu+\varepsilon}} < \varepsilon$$

$$i.e., \ \exp(\exp(\alpha(\mu^{-1}(\gamma(r))))) < \varepsilon \cdot [\exp((\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[\gamma]_\mu})]^{\sigma_{(\alpha,\beta)}[\gamma]_\mu+\varepsilon}$$

$$i.e., \ \exp(\alpha(\mu^{-1}(\gamma(r)))) < \log\varepsilon + (\sigma_{(\alpha,\beta)}[\gamma]_\mu + \varepsilon)(\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[\gamma]_\mu}$$

$$i.e., \ \limsup_{r\to+\infty} \frac{\exp(\alpha(\mu^{-1}(\gamma(r))))}{(\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[\gamma]_\mu}} \le \sigma_{(\alpha,\beta)}[\gamma]_\mu + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows from above that

$$\limsup_{r\to+\infty} \frac{\exp(\alpha(\mu^{-1}(\gamma(r))))}{(\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[\gamma]_\mu}} \le \sigma_{(\alpha,\beta)}[\gamma]_\mu. \tag{3.8}$$

On the other hand the divergence of the integral $\int\limits_{r_0}^{\infty} \frac{\exp(\exp(\alpha(\mu^{-1}(\gamma(r)))))}{[\exp((\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[\gamma]_\mu})]^{\sigma_{(\alpha,\beta)}[\gamma]_\mu-\varepsilon+1}} dr$ $(r_0 > 0)$ implies that there exists a sequence of values of r tending to infinity such that

$$\frac{\exp(\exp(\alpha(\mu^{-1}(\gamma(r)))))}{[\exp((\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[\gamma]_\mu})]^{\sigma_{(\alpha,\beta)}[\gamma]_\mu-\varepsilon+1}} > \frac{1}{[\exp((\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[\gamma]_\mu})]^{1+\varepsilon}}$$

$$i.e.,\ \exp(\exp(\alpha(\mu^{-1}(\gamma(r))))) > [\exp((\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[\gamma]_\mu})]^{\sigma_{(\alpha,\beta)}[\gamma]_\mu-2\varepsilon}$$

$$i.e.,\ \exp(\alpha(\mu^{-1}(\gamma(r)))) > (\sigma_{(\alpha,\beta)}[\gamma]_\mu - 2\varepsilon)((\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[\gamma]_\mu})$$

$$i.e.,\ \frac{\exp(\alpha(\mu^{-1}(\gamma(r))))}{(\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[\gamma]_\mu}} > (\sigma_{(\alpha,\beta)}[\gamma]_\mu - 2\varepsilon).$$

As $\varepsilon > 0$ is arbitrary, it follows from above that

$$\limsup_{r\to+\infty} \frac{\exp(\alpha(\mu^{-1}(\gamma(r))))}{(\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[\gamma]_\mu}} \geq \sigma_{(\alpha,\beta)}[\gamma]_\mu. \tag{3.9}$$

So from (3.8) and (3.9) , we obtain that

$$\limsup_{r\to+\infty} \frac{\exp(\alpha(\mu^{-1}(\gamma(r))))}{(\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[\gamma]_\mu}} = \sigma_{(\alpha,\beta)}[\gamma]_\mu.$$

This proves the theorem. □

Theorem 3.6. *Let $\mu(x)$ and $\gamma(x)$ be any two positive continuous increasing to $+\infty$ on $[x_0, +\infty)$ functions having finite positive generalized relative lower order (α, β), $\lambda_{(\alpha,\beta)}[\gamma]_\mu$ $(0 < \lambda_{(\alpha,\beta)}[\gamma]_\mu < +\infty)$ and generalized relative weak type (α, β), $\overline{\tau}_{(\alpha,\beta)}[\gamma]_\mu$. Then Definition 2.7 and Definition 2.8 are equivalent.*

Proof. Let us consider $\mu(x)$ and $\gamma(x)$ be any two positive continuous increasing to $+\infty$ on $[x_0, +\infty)$ functions such that $\lambda_{(\alpha,\beta)}[\gamma]_\mu$ $(0 < \lambda_{(\alpha,\beta)}[\gamma]_\mu < +\infty)$ exists.

Case I. $\overline{\tau}_{(\alpha,\beta)}[\gamma]_\mu = +\infty$.

Definition 2.7 ⇒ Definition 2.8.

As $\overline{\tau}_{(\alpha,\beta)}[\gamma]_\mu = +\infty$, from Definition 2.7 we obtain for arbitrary positive G and for all sufficiently large values of r that

$$\exp(\alpha(\mu^{-1}(\gamma(r)))) > G \cdot (\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[\gamma]_\mu}$$

$$i.e.,\ \exp(\exp(\alpha(\mu^{-1}(\gamma(r))))) > [\exp((\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[\gamma]_\mu})]^G. \tag{3.10}$$

Now if possible let the integral $\int_{r_0}^{\infty} \frac{\exp(\exp(\alpha(\mu^{-1}(\gamma(r)))))}{[\exp((\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[\gamma]_\mu})]^{G+1}} dr$ $(r_0 > 0)$ be converge. Then by Lemma 3.4,

$$\liminf_{r\to+\infty} \frac{\exp(\exp(\alpha(\mu^{-1}(\gamma(r)))))}{[\exp((\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[\gamma]_\mu})]^G} = 0.$$

So for a sequence of values of r tending to infinity we get

$$\exp(\exp(\alpha(\mu^{-1}(\gamma(r))))) < [\exp((\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[\gamma]_\mu})]^G. \tag{3.11}$$

Therefore from (3.10) and (3.11), we arrive at a contradiction.

Hence $\int\limits_{r_0}^{\infty} \frac{\exp(\exp(\alpha(\mu^{-1}(\gamma(r)))))}{[\exp((\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[\gamma]_\mu})]^{G+1}} dr$ $(r_0 > 0)$ diverges whenever G is finite, which is the Definition 2.8.

DEFINITION 2.8 ⇒ DEFINITION 2.7.

Let G be any positive number. Since $\overline{\tau}_{(\alpha,\beta)}[\gamma]_\mu = +\infty$, from Definition 2.8, the divergence of the integral $\int\limits_{r_0}^{\infty} \frac{\exp(\exp(\alpha(\mu^{-1}(\gamma(r)))))}{[\exp((\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[\gamma]_\mu})]^{G+1}} dr$ $(r_0 > 0)$ gives for arbitrary positive ε and for all sufficiently large values of r that

$$\exp(\exp(\alpha(\mu^{-1}(\gamma(r))))) > [\exp((\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[\gamma]_\mu})]^{G-\varepsilon}$$

$$i.e.,\ \exp(\alpha(\mu^{-1}(\gamma(r)))) > (G-\varepsilon)(\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[\gamma]_\mu},$$

which implies that

$$\liminf_{r\to+\infty} \frac{\exp(\alpha(\mu^{-1}(\gamma(r))))}{(\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[\gamma]_\mu}} \geq G-\varepsilon.$$

Since $G > 0$ is arbitrary, it follows that

$$\liminf_{r\to+\infty} \frac{\exp(\alpha(\mu^{-1}(\gamma(r))))}{(\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[\gamma]_\mu}} = \infty.$$

Thus Definition 2.7 follows.

CASE II. $0 \leq \overline{\tau}_{(\alpha,\beta)}[\gamma]_\mu < +\infty$.

DEFINITION 2.7 ⇒ DEFINITION 2.8.

SUBCASE (C). $0 < \overline{\tau}_{(\alpha,\beta)}[\gamma]_\mu < +\infty$.

Let $\mu(x)$ and $\gamma(x)$ be any two positive continuous increasing to $+\infty$ on $[x_0, +\infty)$ functions such that $0 < \overline{\tau}_{(\alpha,\beta)}[\gamma]_\mu < +\infty$ exists. Then according to the Definition 2.7, for a sequence of values of r tending to infinity we get

$$\exp(\alpha(\mu^{-1}(\gamma(r)))) < (\overline{\tau}_{(\alpha,\beta)}[\gamma]_\mu + \varepsilon)(\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[\gamma]_\mu}$$

$$i.e.,\ \exp(\exp(\alpha(\mu^{-1}(\gamma(r))))) < [\exp((\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[\gamma]_\mu})]^{\overline{\tau}_{(\alpha,\beta)}[\gamma]_\mu+\varepsilon}$$

$$i.e.,\ \frac{\exp(\exp(\alpha(\mu^{-1}(\gamma(r)))))}{[\exp((\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[\gamma]_\mu})]^k} < \frac{[\exp((\log^{[q-1]} r)^{\lambda_\alpha^{(p,q)}(\beta)})]^{\overline{\tau}_{(\alpha,\beta)}[\gamma]_\mu+\varepsilon}}{[\exp((\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[\gamma]_\mu})]^k}$$

$$i.e.,\ \frac{\exp(\exp(\alpha(\mu^{-1}(\gamma(r)))))}{[\exp((\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[\gamma]_\mu})]^k} < \frac{1}{[\exp((\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[\gamma]_\mu})]^{k-(\overline{\tau}_{(\alpha,\beta)}[\gamma]_\mu+\varepsilon)}}.$$

Therefore $\int\limits_{r_0}^{\infty} \frac{\exp(\exp(\alpha(\mu^{-1}(\gamma(r)))))}{[\exp((\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[\gamma]_\mu})]^{k+1}} dr$ $(r_0 > 0)$ converges for $k > \overline{\tau}_{(\alpha,\beta)}[\gamma]_\mu$.

Again by Definition 2.7, we obtain for all sufficiently large values of r that

$$\exp(\alpha(\mu^{-1}(\gamma(r)))) > (\overline{\tau}_{(\alpha,\beta)}[\gamma]_\mu - \varepsilon)(\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[\gamma]_\mu}$$

$$\text{(3.12)} \qquad i.e., \ \exp(\exp(\alpha(\mu^{-1}(\gamma(r))))) > [\exp((\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[\gamma]_\mu})]^{\overline{\tau}_{(\alpha,\beta)}[\gamma]_\mu - \varepsilon}.$$

So for $k < \overline{\tau}_{(\alpha,\beta)}[\gamma]_\mu$, we get from (3.12) that

$$\frac{\exp(\exp(\alpha(\mu^{-1}(\gamma(r)))))}{[\exp((\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[\gamma]_\mu})]^k} > \frac{1}{[\exp((\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[\gamma]_\mu})]^{k-(\overline{\tau}_{(\alpha,\beta)}[\gamma]_\mu - \varepsilon)}}.$$

Therefore $\int_{r_0}^{\infty} \frac{\exp(\exp(\alpha(\mu^{-1}(\gamma(r)))))}{[\exp((\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[\gamma]_\mu})]^{k+1}} dr$ $(r_0 > 0)$ diverges for $k < \overline{\tau}_{(\alpha,\beta)}[\gamma]_\mu$. Hence $\int_{r_0}^{\infty} \frac{\exp(\exp(\alpha(\mu^{-1}(\gamma(r)))))}{[\exp((\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[\gamma]_\mu})]^{k+1}} dr$ $(r_0 > 0)$ converges for $k > \overline{\tau}_{(\alpha,\beta)}[\gamma]_\mu$ and diverges for $k < \overline{\tau}_{(\alpha,\beta)}[\gamma]_\mu$.

Subcase (D). $\overline{\tau}_{(\alpha,\beta)}[\gamma]_\mu = 0$.

When $\overline{\tau}_{(\alpha,\beta)}[\gamma]_\mu = 0$, Definition 2.7 gives for a sequence of values of r tending to infinity that

$$\frac{\exp(\alpha(\mu^{-1}(\gamma(r))))}{(\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[\gamma]_\mu}} < \varepsilon.$$

Then as before we obtain that $\int_{r_0}^{\infty} \frac{\exp(\exp(\alpha(\mu^{-1}(\gamma(r)))))}{[\exp((\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[\gamma]_\mu})]^{k+1}} dr$ $(r_0 > 0)$ converges for $k > 0$ and diverges for $k < 0$.

Thus combining Subcase (C) and Subcase (D), Definition 2.8 follows.

Definition 2.8 ⇒ Definition 2.7.

From Definition 2.8 and for arbitrary positive ε the integral

$$\int_{r_0}^{\infty} \frac{\exp(\exp(\alpha(\mu^{-1}(\gamma(r)))))}{[\exp((\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[\gamma]_\mu})]^{\overline{\tau}_{(\alpha,\beta)}[\gamma]_\mu + \varepsilon + 1}} dr \ (r_0 > 0)$$

converges. Then by Lemma 3.4, we get

$$\liminf_{r\to+\infty} \frac{\exp(\exp(\alpha(\mu^{-1}(\gamma(r)))))}{[\exp((\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[\gamma]_\mu})]^{\overline{\tau}_{(\alpha,\beta)}[\gamma]_\mu + \varepsilon}} = 0.$$

So we get for a sequence of values of r tending to infinity that

$$\frac{\exp(\exp(\alpha(\mu^{-1}(\gamma(r)))))}{[\exp((\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[\gamma]_\mu})]^{\overline{\tau}_{(\alpha,\beta)}[\gamma]_\mu + \varepsilon}} < \varepsilon$$

$$i.e.,\ \exp(\exp(\alpha(\mu^{-1}(\gamma(r))))) < \varepsilon \cdot [\exp((\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[\gamma]_\mu})]^{\overline{\tau}_{(\alpha,\beta)}[\gamma]_\mu+\varepsilon}$$

$$i.e.,\ \exp(\alpha(\mu^{-1}(\gamma(r)))) < \log\varepsilon + (\overline{\tau}_{(\alpha,\beta)}[\gamma]_\mu + \varepsilon)(\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[\gamma]_\mu}$$

$$i.e.,\ \liminf_{r\to+\infty} \frac{\exp(\alpha(\mu^{-1}(\gamma(r))))}{(\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[\gamma]_\mu}} \leq \overline{\tau}_{(\alpha,\beta)}[\gamma]_\mu + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows from above that

$$\liminf_{r\to+\infty} \frac{\exp(\alpha(\mu^{-1}(\gamma(r))))}{(\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[\gamma]_\mu}} \leq \overline{\tau}_{(\alpha,\beta)}[\gamma]_\mu. \tag{3.13}$$

On the other hand the divergence of the integral $\int_{r_0}^{\infty} \frac{\exp(\exp(\alpha(\mu^{-1}(\gamma(r)))))}{[\exp((\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[\gamma]_\mu})]^{\overline{\tau}_{(\alpha,\beta)}[\gamma]_\mu-\varepsilon+1}} dr$ ($r_0 > 0$) implies for all sufficiently large values of r that

$$\frac{\exp(\exp(\alpha(\mu^{-1}(\gamma(r)))))}{[\exp((\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[\gamma]_\mu})]^{\overline{\tau}_{(\alpha,\beta)}[\gamma]_\mu-\varepsilon+1}} > \frac{1}{[\exp((\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[\gamma]_\mu})]^{1+\varepsilon}}$$

$$i.e.,\ \exp(\exp(\alpha(\mu^{-1}(\gamma(r))))) > [\exp((\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[\gamma]_\mu})]^{\overline{\tau}_{(\alpha,\beta)}[\gamma]_\mu-2\varepsilon}$$

$$i.e.,\ \exp(\alpha(\mu^{-1}(\gamma(r)))) > (\overline{\tau}_{(\alpha,\beta)}[\gamma]_\mu - 2\varepsilon)((\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[\gamma]_\mu})$$

$$i.e.,\ \frac{\exp(\alpha(\mu^{-1}(\gamma(r))))}{(\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[\gamma]_\mu}} > (\overline{\tau}_{(\alpha,\beta)}[\gamma]_\mu - 2\varepsilon).$$

As $\varepsilon > 0$ is arbitrary, it follows from above that

$$\liminf_{r\to+\infty} \frac{\exp(\alpha(\mu^{-1}(\gamma(r))))}{(\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[\gamma]_\mu}} \geq \overline{\tau}_{(\alpha,\beta)}[\gamma]_\mu. \tag{3.14}$$

So from (3.13) and (3.14) we obtain

$$\liminf_{r\to+\infty} \frac{\exp(\alpha(\mu^{-1}(\gamma(r))))}{(\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[\gamma]_\mu}} = \overline{\tau}_{(\alpha,\beta)}[\gamma]_\mu.$$

This proves the theorem. □

Next we introduce the following two relative growth indicators which will also help our subsequent study.

Definition 3.7. Let $\mu(x)$ and $\gamma(x)$ be any two positive continuous increasing to $+\infty$ on $[x_0, +\infty)$ functions having finite positive generalized relative order (α, β), $(0 < \rho_{(\alpha,\beta)}[\gamma]_\mu < +\infty)$. Then the generalized relative lower type (α, β) of $\gamma(x)$ with respect to $\mu(x)$ is defined as:

$$\overline{\sigma}_{(\alpha,\beta)}[\gamma]_\mu = \liminf_{r\to+\infty} \frac{\exp(\alpha(\mu^{-1}(\gamma(r))))}{(\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[\gamma]_\mu}}.$$

The above definition can alternatively be defined in the following manner:

Definition 3.8. Let $\mu(x)$ and $\gamma(x)$ be any two positive continuous increasing to $+\infty$ on $[x_0, +\infty)$ functions having finite positive generalized relative order (α, β), $\rho_{(\alpha,\beta)}[\gamma]_\mu$ $(0 < \rho_{(\alpha,\beta)}[\gamma]_\mu < +\infty)$. Then the generalized relative lower type (α, β) denoted by $\overline{\sigma}_{(\alpha,\beta)}[\gamma]_\mu$ of $\gamma(x)$ with respect to $\mu(x)$ is defined as: The integral $\int_{r_0}^{\infty} \frac{\exp(\exp(\alpha(\mu^{-1}(\gamma(r)))))}{[\exp((\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[\gamma]_\mu})]^{k+1}} dr$ $(r_0 > 0)$ converges for $k > \overline{\sigma}_{(\alpha,\beta)}[\gamma]_\mu$ and diverges for $k < \overline{\sigma}_{(\alpha,\beta)}[\gamma]_\mu$.

Definition 3.9. Let $\mu(x)$ and $\gamma(x)$ be any two positive continuous increasing to $+\infty$ on $[x_0, +\infty)$ functions having finite positive generalized relative lower order (α, β) $(0 < \lambda_{(\alpha,\beta)}[\gamma]_\mu < +\infty)$. Then the generalized relative upper weak type (α, β) of $\gamma(x)$ with respect to $\mu(x)$ is defined as:

$$\tau_{(\alpha,\beta)}[\gamma]_\mu = \limsup_{r \to +\infty} \frac{\exp(\alpha(\mu^{-1}(\gamma(r))))}{(\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[\gamma]_\mu}}.$$

The above definition can also alternatively defined as:

Definition 3.10. Let $\mu(x)$ and $\gamma(x)$ be any two positive continuous increasing to $+\infty$ on $[x_0, +\infty)$ functions having finite positive generalized relative lower order (α, β) $(0 < \lambda_{(\alpha,\beta)}[\gamma]_\mu < +\infty)$. Then generalized relative upper weak type (α, β) denoted by $\tau_{(\alpha,\beta)}[\gamma]_\mu$ of $\gamma(x)$ with respect to $\mu(x)$ is defined as: The integral $\int_{r_0}^{\infty} \frac{\exp(\exp(\alpha(\mu^{-1}(\gamma(r)))))}{[\exp((\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[\gamma]_\mu})]^{k+1}} dr$ $(r_0 > 0)$ converges for $k > \tau_{(\alpha,\beta)}[\gamma]_\mu$ and diverges for $k < \tau_{(\alpha,\beta)}[\gamma]_\mu$.

Now we state the following two theorems without their proofs as those can easily be carried out with help of Lemma 3.4 and in the line of Theorem 3.5 and Theorem 3.6 respectively.

Theorem 3.11. *Let $\mu(x)$ and $\gamma(x)$ be any two positive continuous increasing to $+\infty$ on $[x_0, +\infty)$ functions having finite positive generalized relative order (α, β) $(0 < \rho_{(\alpha,\beta)}[\gamma]_\mu < +\infty)$ and generalized relative lower type (α, β) denoted by $\overline{\sigma}_{(\alpha,\beta)}[\gamma]_\mu$. Then Definition 3.7 and Definition 3.8 are equivalent.*

Theorem 3.12. *Let $\mu(x)$ and $\gamma(x)$ be any two positive continuous increasing to $+\infty$ on $[x_0, +\infty)$ functions having finite positive generalized relative lower order (α, β) $(0 < \lambda_{(\alpha,\beta)}[\gamma]_\mu < +\infty)$ and generalized relative upper weak type (α, β) denoted by $\tau_{(\alpha,\beta)}[\gamma]_\mu$. Then Definition 3.9 and Definition 3.10 are equivalent.*

Theorem 3.13. *$\mu(x)$ and $\gamma(x)$ be any two positive continuous increasing to $+\infty$ on $[x_0, +\infty)$ functions with $0 < \lambda_{(\alpha,\beta)}[\gamma]_\mu \leq \rho_{(\alpha,\beta)}[\gamma]_\mu < +\infty$. Then*

$$(i)\ \sigma_{(\alpha,\beta)}[\gamma]_\mu = \limsup_{r\to+\infty} \frac{\exp(\alpha(\mu^{-1}(r)))}{(\exp(\beta(\gamma^{-1}(r))))^{\rho_{(\alpha,\beta)}[\gamma]_\mu}},$$

$$(ii)\ \overline{\sigma}_{(\alpha,\beta)}[\gamma]_\mu = \liminf_{r\to+\infty} \frac{\exp(\alpha(\mu^{-1}(r)))}{(\exp(\beta(\gamma^{-1}(r))))^{\rho_{(\alpha,\beta)}[\gamma]_\mu}},$$

$$(iii)\ \overline{\tau}_{(\alpha,\beta)}[\gamma]_\mu = \liminf_{r\to+\infty} \frac{\exp(\alpha(\mu^{-1}(r)))}{(\exp(\beta(\gamma^{-1}(r))))^{\lambda_{(\alpha,\beta)}[\gamma]_\mu}}$$

and

$$(iv)\ \tau_{(\alpha,\beta)}[\gamma]_\mu = \limsup_{r\to+\infty} \frac{\exp(\alpha(\mu^{-1}(r)))}{(\exp(\beta(\gamma^{-1}(r))))^{\lambda_{(\alpha,\beta)}[\gamma]_\mu}}.$$

Proof. Taking $\gamma(r) = R$, theorem follows from the definitions of $\sigma_{(\alpha,\beta)}[\gamma]_\mu$, $\overline{\sigma}_{(\alpha,\beta)}[\gamma]_\mu$, $\overline{\tau}_{(\alpha,\beta)}[\gamma]_\mu$ and $\tau_{(\alpha,\beta)}[\gamma]_\mu$ respectively. □

In the following theorem we obtain a relationship between $\sigma_{(\alpha,\beta)}[\gamma]_\mu$, $\overline{\sigma}_{(\alpha,\beta)}[\gamma]_\mu$, $\tau_{(\alpha,\beta)}[\gamma]_\mu$ and $\overline{\tau}_{(\alpha,\beta)}[\gamma]_\mu$.

Theorem 3.14. *Let $\mu(x)$ and $\gamma(x)$ be any two positive continuous increasing to $+\infty$ on $[x_0, +\infty)$ functions such $\rho_{(\alpha,\beta)}[\gamma]_\mu = \lambda_{(\alpha,\beta)}[\gamma]_\mu$ $(0 < \lambda_{(\alpha,\beta)}[\gamma]_\mu = \rho_{(\alpha,\beta)}[\gamma]_\mu < +\infty)$, then the following quantities*

$$(i)\ \sigma_{(\alpha,\beta)}[\gamma]_\mu,\ (ii)\ \overline{\tau}_{(\alpha,\beta)}[\gamma]_\mu,\ (iii)\ \overline{\sigma}_{(\alpha,\beta)}[\gamma]_\mu \text{ and } (iv)\ \tau_{(\alpha,\beta)}[\gamma]_\mu$$

are all equivalent.

Proof. From Definition 2.8, it follows that the integral $\int_{r_0}^{\infty} \frac{\exp(\exp(\alpha(\mu^{-1}(\gamma(r)))))}{[\exp((\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[\gamma]_\mu})]^{k+1}} dr$ $(r_0 > 0)$ converges for $k > \overline{\tau}_{(\alpha,\beta)}[\gamma]_\mu$ and diverges for $k < \overline{\tau}_{(\alpha,\beta)}[\gamma]_\mu$. On the other hand, Definition 2.6 implies that the integral $\int_{r_0}^{\infty} \frac{\exp(\exp(\alpha(\mu^{-1}(\gamma(r)))))}{[\exp((\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[\gamma]_\mu})]^{k+1}} dr$ $(r_0 > 0)$ converges for $k > \sigma_{(\alpha,\beta)}[\gamma]_\mu$ and diverges for $k < \sigma_{(\alpha,\beta)}[\gamma]_\mu$.

$(i) \Rightarrow (ii)$.

Now it is obvious that all the quantities in the expression

$$\left[\frac{\exp(\exp(\alpha(\mu^{-1}(\gamma(r)))))}{[\exp((\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[\gamma]_\mu})]^{k+1}} - \frac{\exp(\exp(\alpha(\mu^{-1}(\gamma(r)))))}{[\exp((\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[\gamma]_\mu})]^{k+1}}\right]$$

are of non negative type. So

$$\int_{r_0}^{\infty}\left[\frac{\exp(\exp(\alpha(\mu^{-1}(\gamma(r)))))}{[\exp((\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[\gamma]_\mu})]^{k+1}} - \frac{\exp(\exp(\alpha(\mu^{-1}(\gamma(r)))))}{[\exp((\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[\gamma]_\mu})]^{k+1}}\right]dr \geq 0 \text{ for } r_0 > 0$$

$$i.e., \int_{r_0}^{\infty}\frac{\exp(\exp(\alpha(\mu^{-1}(\gamma(r)))))}{[\exp((\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[\gamma]_\mu})]^{k+1}}dr \geq \int_{r_0}^{\infty}\frac{\exp(\exp(\alpha(\mu^{-1}(\gamma(r)))))}{[\exp((\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[\gamma]_\mu})]^{k+1}}dr \text{ for } r_0 > 0.$$

$$i.e., \ \overline{\tau}_{(\alpha,\beta)}[\gamma]_\mu \geq \sigma_{(\alpha,\beta)}[\gamma]_\mu. \tag{3.15}$$

Further as $\rho_{(\alpha,\beta)}[\gamma]_\mu = \lambda_{(\alpha,\beta)}[\gamma]_\mu$, therefore we get that

$$\sigma_{(\alpha,\beta)}[\gamma]_\mu = \limsup_{r\to+\infty}\frac{\exp(\alpha(\mu^{-1}(\gamma(r))))}{(\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[\gamma]_\mu}} \geq \liminf_{r\to+\infty}\frac{\exp(\alpha(\mu^{-1}(\gamma(r))))}{(\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[\gamma]_\mu}}$$

$$= \liminf_{r\to+\infty}\frac{\exp(\alpha(\mu^{-1}(\gamma(r))))}{(\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[\gamma]_\mu}} = \overline{\tau}_{(\alpha,\beta)}[\gamma]_\mu. \tag{3.16}$$

Hence from (3.15) and (3.16) we obtain

$$\sigma_{(\alpha,\beta)}[\gamma]_\mu = \overline{\tau}_{(\alpha,\beta)}[\gamma]_\mu. \tag{3.17}$$

$(ii) \Rightarrow (iii)$.

Since $\rho_{(\alpha,\beta)}[\gamma]_\mu = \lambda_{(\alpha,\beta)}[\gamma]_\mu$, we get

$$\overline{\tau}_{(\alpha,\beta)}[\gamma]_\mu = \liminf_{r\to+\infty}\frac{\exp(\alpha(\mu^{-1}(\gamma(r))))}{(\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[\gamma]_\mu}} = \liminf_{r\to+\infty}\frac{\exp(\alpha(\mu^{-1}(\gamma(r))))}{(\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[\gamma]_\mu}} = \overline{\sigma}_{(\alpha,\beta)}[\gamma]_\mu.$$

$(iii) \Rightarrow (iv)$.

In view of (3.17) and the condition $\rho_{(\alpha,\beta)}[\gamma]_\mu = \lambda_{(\alpha,\beta)}[\gamma]_\mu$, it follows that

$$\overline{\sigma}_{(\alpha,\beta)}[\gamma]_\mu = \liminf_{r\to+\infty}\frac{\exp(\alpha(\mu^{-1}(\gamma(r))))}{(\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[\gamma]_\mu}}$$

$$i.e., \ \overline{\sigma}_{(\alpha,\beta)}[\gamma]_\mu = \liminf_{r\to+\infty}\frac{\exp(\alpha(\mu^{-1}(\gamma(r))))}{(\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[\gamma]_\mu}}$$

$$i.e., \ \overline{\sigma}_{(\alpha,\beta)}[\gamma]_\mu = \overline{\tau}_{(\alpha,\beta)}[\gamma]_\mu$$

$$i.e., \ \overline{\sigma}_{(\alpha,\beta)}[\gamma]_\mu = \sigma_{(\alpha,\beta)}[\gamma]_\mu$$

$$i.e.,\ \overline{\sigma}_{(\alpha,\beta)}[\gamma]_\mu = \limsup_{r\to+\infty} \frac{\exp(\alpha(\mu^{-1}(\gamma(r))))}{(\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[\gamma]_\mu}}$$

$$i.e.,\ \overline{\sigma}_{(\alpha,\beta)}[\gamma]_\mu = \limsup_{r\to+\infty} \frac{\exp(\alpha(\mu^{-1}(\gamma(r))))}{(\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[\gamma]_\mu}}$$

$$i.e.,\ \overline{\sigma}_{(\alpha,\beta)}[\gamma]_\mu = \tau_{(\alpha,\beta)}[\gamma]_\mu.$$

$(iv) \Rightarrow (i)$.

As $\rho_{(\alpha,\beta)}[\gamma]_\mu = \lambda_{(\alpha,\beta)}[\gamma]_\mu$, we obtain that

$$\tau_{(\alpha,\beta)}[\gamma]_\mu = \limsup_{r\to+\infty} \frac{\exp(\alpha(\mu^{-1}(\gamma(r))))}{(\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[\gamma]_\mu}} = \limsup_{r\to+\infty} \frac{\exp(\alpha(\mu^{-1}(\gamma(r))))}{(\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[\gamma]_\mu}} = \sigma_{(\alpha,\beta)}[\gamma]_\mu.$$

Thus the theorem follows. □

Remark 3.15. If we consider $\mu(x) = M_g(x)$ and $\gamma(x) = M_f(x)$ where f and g are any two entire functions, then the above results reduces for the generalized relative growth indicators such as generalized relative order (α, β), generalized relative type (α, β) etc. of an entire function f with respect to another entire function g.

Remark 3.16. If we take $\mu(x) = T_g(x)$ and $\gamma(x) = T_f(x)$ where f be a meromorphic function and g be any entire function, then the above theorems reduces for generalized relative growth indicators such as generalized relative order (α, β), generalized relative type (α, β) etc. of a meromorphic function f with respect to an entire function g.

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