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EVALUATIONS OF THE ROGERS-RAMANUJAN CONTINUED FRACTION BY THETA-FUNCTION IDENTITIES

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Abstract. In this paper, we use theta-function identities involving parameters $l_{5,n}$, $l'_{5,n}$, and $l'_{5,4n}$ to evaluate the Rogers-Ramanujan continued fractions $R(e^{-2\pi\sqrt{n/20}})$ and $S(e^{-\pi\sqrt{n/5}})$ for some positive rational numbers n.

1. Introduction

The Rogers-Ramanujan continued fractions $R(q)$ and $S(q)$, for $|q| < 1$, are defined by

$$R(q) = \frac{q^{1/5}}{1} + \frac{q}{1} + \frac{q^2}{1} + \frac{q^3}{1} + \cdots$$

and

$$S(q) = -R(-q).$$

In his first two letters to G. H. Hardy, S. Ramanujan asserted that

$$R(e^{-2\pi}) = \sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{5}+1}{2},$$

$$S(e^{-\pi}) = \sqrt{\frac{5-\sqrt{5}}{2}} - \frac{\sqrt{5}-1}{2},$$

and

$$R(e^{-2\pi\sqrt{5}}) = \frac{\sqrt{5}}{1 + \left(5^{3/4}\left(\frac{\sqrt{5}-1}{2}\right)^{5/2} - 1\right)^{1/5}} - \frac{1+\sqrt{5}}{2}.$$

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See [4, 5] for more details about the proofs and historical remarks of these evaluations. Further explicit evaluations of $R(e^{-2\pi\sqrt{n}})$ and $S(e^{-\pi\sqrt{n}})$ for some positive rational numbers n were given in [4, 5, 12, 13]. Ramanathan [8, 9, 10] evaluated $R(e^{-2\pi\sqrt{n}})$ for $n = 1, 2, 5, \frac{2}{5}, \frac{17}{5}$ and $S(e^{-\pi\sqrt{n}})$ for $n = 1, 5, \frac{3}{5}, \frac{3}{7}, \frac{23}{5}$. Ramanathan [11] also evaluated $R(e^{-2\pi\sqrt{n}})$ for $n = \frac{2}{5}, \frac{17}{5}$ and $S(e^{-\pi\sqrt{n}})$ for $n = \frac{7}{5}, \frac{23}{5}$ by using Kronecker's limit formula. Berndt and Chan [4] established values of $R(e^{-2\pi\sqrt{n}})$ for $n = 4, 9, 16, 64$ and $S(e^{-\pi\sqrt{n}})$ for $n = \frac{3}{5}, \frac{7}{5}, \frac{1}{15}, \frac{1}{35}$ by employing an eta-function identity. Meanwhile, Chan [6] evaluated $S(e^{-\pi\sqrt{3}})$ by using a modular equation. Berndt, Chan, and Zhang [5] derived formulas for the explicit evaluations of $R(e^{-2\pi\sqrt{n}})$ and $S(e^{-\pi\sqrt{n}})$ for positive rational numbers n in terms of Ramanujan-Weber class invariants. In particular, they determined the values of $R(e^{-6\pi})$ and $S(e^{-\pi\sqrt{n}})$ for $n = 9$, $\frac{1}{5}, \frac{3}{5}, \frac{9}{5}, \frac{11}{5}, \frac{13}{5}, \frac{29}{5}, \frac{41}{5}, \frac{53}{5}, \frac{89}{5}, \frac{101}{5}$.

Yi [13] established some formulas and found values of $R(e^{-2\pi\sqrt{n}})$ and $S(e^{-\pi\sqrt{n}})$ by using modular equations of degrees 5 or 25. In addition, Yi [13] employed modular equations of degree 1, p, 5, and $5p$ for any positive integer p so that she was able to compute $R(e^{-2\pi\sqrt{n}})$ for $n = 1, 2, 3, 4, 9, 16, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{7}{5}, \frac{8}{5}, \frac{9}{5}, \frac{1}{9}, \frac{1}{10}, \frac{1}{15}, \frac{1}{16}, \frac{1}{20}, \frac{1}{25}, \frac{1}{35}, \frac{1}{40}, \frac{1}{45}$ and $S(e^{-\pi\sqrt{n}})$ for $n = 1, 3, 9, 27, \frac{1}{3}, \frac{1}{5}, \frac{3}{5}, \frac{7}{5}, \frac{9}{5}, \frac{1}{9}, \frac{1}{15}, \frac{1}{25}, \frac{1}{27}, \frac{1}{35}, \frac{1}{45}$. Recently, Paek and Yi [7] evaluated $R(e^{-2\pi\sqrt{n}})$ and $S(e^{-\pi\sqrt{n}})$ for $n = \frac{4}{5}$, $\frac{16}{5}$ by using modular equations of degree 5. Saikia [12] evaluated explicit values of $R(e^{-2\pi\sqrt{n}})$ for $n = 1, 4, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{9}{5}, \frac{1}{10}$ and $S(e^{-\pi\sqrt{n}})$ for $n = 1, \frac{1}{5}, \frac{3}{5}, \frac{9}{5}$ by using parametrization of Ramanujan's theta-functions.

In this paper, we use theta-function identities involving parameters $l_{5,n}$, $l'_{5,n}$, and $l'_{5,4n}$ to show how to evaluate $R(e^{-2\pi\sqrt{n}})$ for $n = \frac{3}{5\cdot 4^m}, \frac{9}{5\cdot 4^m}, \frac{1}{15\cdot 4^m}, \frac{1}{45\cdot 4^m}$ and $S(e^{-\pi\sqrt{n}})$ for $n = \frac{3}{5\cdot 4^{m-1}}, \frac{9}{5\cdot 4^{m-1}}, \frac{1}{15\cdot 4^{m-1}}, \frac{1}{45\cdot 4^{m-1}}$, where m is any positive integer. Furthermore, we establish explicit evaluations of these for $m = 1$, 2, or 3.

Ramanujan's theta-function ψ is defined by

$$\psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2},$$

where $|q| < 1$.

Recall the parameters $l_{k,n}$ and $l'_{k,n}$ for the theta-function ψ introduced in [14]. For any positive real numbers k and n, define $l_{k,n}$ and $l'_{k,n}$ by

$$l_{k,n} = \frac{\psi(-q)}{k^{1/4}q^{(k-1)/8}\psi(-q^k)} \tag{1.1}$$

and

$$l'_{k,n} = \frac{\psi(q)}{k^{1/4}q^{(k-1)/8}\psi(q^k)}, \tag{1.2}$$

where $q = e^{-\pi\sqrt{n/k}}$.

We employ the following formulas in [12, Theorem 3.1] to evaluate $R(e^{-2\pi\sqrt{n/20}})$ and $S(e^{-\pi\sqrt{n/5}})$ in terms of $l'_{5,n}$ and $l_{5,n}$, respectively.

$$\frac{1}{R^5(e^{-2\pi\sqrt{n/20}})} - 11 - R^5(e^{-2\pi\sqrt{n/20}}) = \sqrt{5}\, l'^2_{5,n} \left(\frac{\sqrt{5}\, l'^2_{5,n} - 5}{\sqrt{5}\, l'^2_{5,n} - 1}\right)^2 \tag{1.3}$$

and

$$\frac{1}{S^5(e^{-\pi\sqrt{n/5}})} + 11 - S^5(e^{-\pi\sqrt{n/5}}) = \sqrt{5}\, l^2_{5,n} \left(\frac{\sqrt{5}\, l^2_{5,n} + 5}{\sqrt{5}\, l^2_{5,n} + 1}\right)^2 . \tag{1.4}$$

From (1.3) and (1.4), we have

$$R^5(e^{-2\pi\sqrt{n/20}}) = \sqrt{a^2+1} - a, \tag{1.5}$$

where

$$2a = 11 + \sqrt{5}\, l'^2_{5,n} \left(\frac{\sqrt{5}\, l'^2_{5,n} - 5}{\sqrt{5}\, l'^2_{5,n} - 1}\right)^2$$

and

$$S^5(e^{-\pi\sqrt{n/5}}) = \sqrt{b^2+1} - b, \tag{1.6}$$

where

$$2b = -11 + \sqrt{5}\, l^2_{5,n} \left(\frac{\sqrt{5}\, l^2_{5,n} + 5}{\sqrt{5}\, l^2_{5,n} + 1}\right)^2 .$$

Consequently, in order to compute $R(e^{-2\pi\sqrt{n/20}})$ and $S(e^{-\pi\sqrt{n/5}})$, it suffices to evaluate $l'^2_{5,n}$ and $l^2_{5,n}$, respectively. Thus, in this paper, we employ theta-function identities involving $l_{5,n}$, $l'_{5,n}$, and $l'_{5,4n}$ to find some new explicit values of the Rogers-Ramanujan continued fraction.

2. Evaluations of $l_{5,n}$ and $l'_{5,n}$

In this section, we compute $l'^2_{5,n}$ and $l^2_{5,n}$ for some positive rational numbers n to evaluate $R(e^{-2\pi\sqrt{n/20}})$ and $S(e^{-\pi\sqrt{n/5}})$. We begin by recalling the known values of $l_{5,3}$, $l_{5,1/3}$, $l_{5,9}$, and $l_{5,1/9}$ in [14] which will play key roles in evaluating the Rogers-Ramanujan continued fraction later on.

Lemma 2.1 ([14, Theorem 4.8]). *We have*

(i) $l_{5,3} = \sqrt{2+\sqrt{5}}\,,$

(ii) $l_{5,1/3} = \sqrt{-2+\sqrt{5}}\,,$

(iii) $l_{5,9} = \dfrac{1+\sqrt{3}}{\sqrt{5}-\sqrt{3}}\,,$

(iv) $l_{5,1/9} = \dfrac{\sqrt{5}-\sqrt{3}}{1+\sqrt{3}}$.

Note that the values of (i) and (ii) in Lemma 2.1 were given in [14] as follow:

$$l_{5,3} = (38+17\sqrt{5})^{1/6} \quad \text{and} \quad l_{5,1/3} = (-38+17\sqrt{5})^{1/6}.$$

We now need a couple of theta-function identities: one shows a relation between $l'_{5,n}$ and $l'_{5,4n}$ and the other shows a relation between $l'_{5,n}$ and $l_{5,n}$ for any positive rational number n.

Lemma 2.2 ([7, Corollary 3.8]). *For every positive real number n, we have*

$$\sqrt{5}\left(l'^2_{5,n} + \frac{1}{l'^2_{5,n}}\right) = \left(\frac{l'_{5,n}}{l'_{5,4n}}\right)^2 + \left(\frac{l'_{5,4n}}{l'_{5,n}}\right)^2 + 4 \tag{2.1}$$

Identity (2.1) follows from a modular equation of degree 5 in [7] such as $P^2 + \dfrac{5}{P^2} = \left(\dfrac{P}{Q}\right)^2 + \left(\dfrac{Q}{P}\right)^2 + 4$, where $P = \frac{\psi(q)}{q^{1/2}\psi(q^5)}$ and $Q = \frac{\psi(q^2)}{q\psi(q^{10})}$.

Lemma 2.3 ([7, Corollary 3.12]). *For every positive real number n, we have*

$$\left(l'^2_{5,n} - l^2_{5,n} - \sqrt{5}\right)\left(\frac{1}{l'^2_{5,n}} - \frac{1}{l^2_{5,n}} - \sqrt{5}\right) = 1 \tag{2.2}$$

Identity (2.2) follows from a modular equation of degree 5 in [7] such as

$$\left(P^2 - Q^2 - 5\right)\left(\frac{1}{P^2} - \frac{1}{Q^2} - 1\right) = 1,$$

where $P = \frac{\psi(q)}{q^{1/2}\psi(q^5)}$ and $Q = \frac{\psi(-q)}{q^{1/2}\psi(-q^5)}$.

We are in position to evaluate $l'^2_{5,n}$ for $n = 3, \frac{1}{3}, \frac{3}{4}, \frac{1}{12}, \frac{3}{16}$, and $\frac{1}{48}$.

Theorem 2.4. *We have*

(i) $l'^2_{5,3} = \dfrac{1}{2}(2+\sqrt{3})(1+\sqrt{5})$,

(ii) $l'^2_{5,1/3} = \dfrac{1}{2}(2+\sqrt{3})(-1+\sqrt{5})$,

(iii) $l'^2_{5,3/4} = \dfrac{4+\sqrt{2}+\sqrt{30}}{2-\sqrt{3}+\sqrt{15}}$,

(iv) $l'^2_{5,1/12} = \dfrac{4-\sqrt{2}+\sqrt{30}}{-2+\sqrt{3}+\sqrt{15}}$.

(v) $l'^2_{5,3/16} = \dfrac{4(2+\sqrt{10}) + 2\sqrt{12+33\sqrt{10}+\sqrt{270(13-4\sqrt{10})}}}{17+17\sqrt{2}+9\sqrt{3}-7\sqrt{5}+9\sqrt{6}-5\sqrt{10}-3\sqrt{15}-3\sqrt{30}}$,

$$\text{(vi)}\ \ l'^2_{5,1/48}=\frac{4(2+\sqrt{10}\,)+2\sqrt{12+33\sqrt{10}-\sqrt{270(13-4\sqrt{10})}}}{-17+17\sqrt{2}-9\sqrt{3}-7\sqrt{5}+9\sqrt{6}+5\sqrt{10}-3\sqrt{15}+3\sqrt{30}}\,.$$

Proof. For (i), let $n=3$ in (2.2). Putting $l'^2_{5,3}=x$ and $l_{5,3}=\sqrt{2+\sqrt{5}}$ in Lemma 2.1(i), we find that

$$2x^2-4(1+\sqrt{5}\,)x+3+\sqrt{5}=0\,.$$

Solving the equation for x and using $x>1$, we complete the proof.

For (iii), let $n=\frac{3}{4}$ in (2.1). Putting $l'^2_{5,3/4}=x$ and $l'^2_{5,3}=\dfrac{1}{2}(2+\sqrt{3}\,)(1+\sqrt{5}\,)$ obtained in (i), we deduce that

$$(2-\sqrt{3}+\sqrt{15}\,)x^2-8x-2-\sqrt{3}-\sqrt{15}=0.$$

Solving the last equation for x and using $x>1$, we complete the proof.

The proofs of (ii), (iv), (v), and (vi) are similar to those of (i) or (iii). □

Remark 1. Letting $n=\frac{3}{64}$ and $\frac{1}{192}$ in (2.1) and using the value $l'^2_{5,3/16}$ and $l'^2_{5,1/48}$ in Theorem 2.4(v) and (vi), respectively, we can also evaluate $l'^2_{5,3/64}$ and $l'^2_{5,1/192}$. Hence, by the same argument, we can evaluate $l'^2_{5,3/4^m}$ and $l'^2_{5,1/3\cdot 4^m}$ for every nonnegative integer m.

We now evaluate $l^2_{5,3/4}$ and $l^2_{5,1/12}$.

Theorem 2.5. *We have*

$$\text{(i)}\ \ l^2_{5,3/4}=\frac{4+\sqrt{2}-5\sqrt{3}-2\sqrt{5}+\sqrt{15}+\sqrt{30}}{4-2\sqrt{3}+2\sqrt{15}} +\frac{\sqrt{26-84\sqrt{2}+18\sqrt{3}+63\sqrt{5}-47\sqrt{6}-\sqrt{10}+38\sqrt{15}}}{(2+\sqrt{3})\sqrt{2+6\sqrt{5}+4\sqrt{15}}}\,,$$

$$\text{(ii)}\ \ l^2_{5,1/12}=\frac{4-\sqrt{2}-5\sqrt{3}+2\sqrt{5}-\sqrt{15}+\sqrt{30}}{-4+2\sqrt{3}+2\sqrt{15}} +\frac{\sqrt{-26-84\sqrt{2}-18\sqrt{3}+63\sqrt{5}-47\sqrt{6}+\sqrt{10}+38\sqrt{15}}}{(2+\sqrt{3})\sqrt{-2+6\sqrt{5}+4\sqrt{15}}}\,.$$

Proof. The results follow directly from (2.2) and Theorem 2.4(ii) and (iv) with the help of *Mathematica*. □

Remark 2. Letting $n=\frac{3}{16}$ and $\frac{1}{48}$ in (2.2) and using the value $l'^2_{5,3/16}$ and $l'^2_{5,1/48}$ in Theorem 2.4(v) and (vi), respectively, we can also evaluate $l^2_{5,3/16}$ and $l^2_{5,1/48}$. By the same argument as before, we are able to evaluate $l^2_{5,3/4^m}$ and $l^2_{5,1/3\cdot 4^m}$ for every positive integer m.

We now evaluate $l'^2_{5,n}$ for $n=9$, $\frac{9}{4}$, $\frac{1}{9}$, and $\frac{1}{36}$.

Theorem 2.6. *We have*

(i) $l'^2_{5,9} = (2+\sqrt{3}\,)\left(2+\sqrt{5}+2\sqrt{2+\sqrt{5}}\right)$,

(ii) $l'^2_{5,9/4} = \dfrac{(2+\sqrt{3}\,)\left(5+2\sqrt{2}+\sqrt{5}+\sqrt{6+6\sqrt{5}}\right)}{\sqrt{2}\left(-2+\sqrt{5}+\sqrt{15}+2\sqrt{2+\sqrt{5}}\right)}$,

(iii) $l'^2_{5,1/9} = (2-\sqrt{3}\,)\left(2+\sqrt{5}+2\sqrt{2+\sqrt{5}}\right)$,

(iv) $l'^2_{5,1/36} = \dfrac{(2-\sqrt{3}\,)\left(5+2\sqrt{2}+\sqrt{5}-\sqrt{6+6\sqrt{5}}\right)}{\sqrt{2}\left(-2+\sqrt{5}-\sqrt{15}+2\sqrt{2+\sqrt{5}}\right)}$.

Proof. For (i), let $n = 3$ in (2.2). Putting $l'^2_{5,9} = x$ and $l_{5,9} = \dfrac{1+\sqrt{3}}{\sqrt{5}-\sqrt{3}}$ in Lemma 2.1(iii), we find that

$$x^2 - 2(2+\sqrt{5}\,)\,x + 2 + \sqrt{3} = 0\,.$$

Solving the equation for x and using $x > 1$, we complete the proof.

For (ii), let $n = \frac{9}{4}$ in (2.1). Putting $l'^2_{5,9/4} = x$ and the value $l'^2_{5,9}$ obtained in (i), we deduce that

$$\frac{-2+\sqrt{5}+\sqrt{15}+2\sqrt{2+\sqrt{5}}}{2+\sqrt{3}}\,x^2 - 4x - \frac{2-\sqrt{5}+\sqrt{15}+2\sqrt{2+\sqrt{5}}}{2-\sqrt{3}} = 0\,.$$

Solve the last equation for x with the help of *Mathematica* and use $x > 1$ to complete the proof.

The proofs of (iii) and (iv) are similar to those of (i) and (ii), respectively. □

Remark 3. Letting $n = \frac{9}{16}$ and $\frac{1}{144}$ in (2.1) and using the value of $l'^2_{5,9/4}$ and $l'^2_{5,1/36}$ in Theorem 2.6(ii) and (iv), respectively, we can also evaluate $l'^2_{5,9/16}$ and $l'^2_{5,1/144}$. Hence we can evaluate $l'^2_{5,9/4^m}$ and $l'^2_{5,1/9\cdot 4^m}$ for every nonnegative integer m.

We end this section by evaluating $l^2_{5,9/4}$ and $l^2_{5,1/36}$.

Theorem 2.7. *We have*

(i) $l^2_{5,9/4} = \dfrac{(c-4)\sqrt{c}+\sqrt{(c-4)(c^2+4)}}{2\sqrt{5c}}$, *where*

$$c = -1 + \frac{\sqrt{5}\,(2+\sqrt{3}\,)\left(5+2\sqrt{2}+\sqrt{5}+2\sqrt{6+6\sqrt{5}}\right)}{\sqrt{2}\left(-2+\sqrt{5}+\sqrt{15}+2\sqrt{2+\sqrt{5}}\right)},$$

(ii) $l^2_{5,1/36} = \dfrac{(c-4)\sqrt{c}+\sqrt{(c-4)(c^2+4)}}{2\sqrt{5c}}$, *where*

$$c = -1 + \frac{\sqrt{5}\,(2-\sqrt{3}\,)\left(5+2\sqrt{2}+\sqrt{5}-2\sqrt{6+6\sqrt{5}}\right)}{\sqrt{2}\left(-2+\sqrt{5}-\sqrt{15}+2\sqrt{2+\sqrt{5}}\right)}.$$

Proof. The results follow directly from (2.2) and Theorem 2.6(ii) and (iv) with the help of *Mathematica*. □

Remark 4. As mentioned in Remark 3, if we evaluate $l'^2_{5,9/16}$ and $l'^2_{5,1/144}$, then we have $l^2_{5,9/16}$ and $l^2_{5,1/144}$. Thus we are able to compute $l^2_{5,9/4^m}$ and $l^2_{5,1/9\cdot 4^m}$ for every positive integer m.

3. Evaluations of $R(q)$ and $S(q)$

In view of Remark 1, we can evaluate $R(e^{-2\pi\sqrt{n}})$ for $n = \frac{3}{5\cdot 4^m}$ and $\frac{1}{15\cdot 4^m}$, where m is any positive integer. We evaluate the cases for $m = 1$, 2, and 3.

Theorem 3.1. *We have*

(i) $R^5(e^{-2\pi\sqrt{3/20}})$
$$= \frac{57 - 50\sqrt{3} + 16\sqrt{5} - 36\sqrt{15} + \sqrt{15(2105 - 760\sqrt{3} + 842\sqrt{5} - 380\sqrt{15})}}{1 + 3\sqrt{5} + 2\sqrt{15}},$$

(ii) $R^5(e^{-2\pi\sqrt{1/60}})$
$$= \frac{57 - 50\sqrt{3} - 16\sqrt{5} + 36\sqrt{15} - \sqrt{15(2105 - 760\sqrt{3} - 842\sqrt{5} + 380\sqrt{15})}}{1 - 3\sqrt{5} - 2\sqrt{15}},$$

(iii) $R^5(e^{-2\pi\sqrt{3/80}}) = \sqrt{a^2+1} - a$, *where*
$$a = \frac{11}{2} + \frac{5\sqrt{5}}{2}\left(\frac{4+\sqrt{2}+\sqrt{30}}{2-\sqrt{3}+\sqrt{15}}\right)\left(\frac{4+\sqrt{2}-5\sqrt{3}-2\sqrt{5}+\sqrt{15}+\sqrt{30}}{2-\sqrt{3}-4\sqrt{5}-5\sqrt{6}-\sqrt{10}+\sqrt{15}}\right)^2,$$

(iv) $R^5(e^{-2\pi\sqrt{1/240}}) = \sqrt{a^2+1} - a$, *where*
$$a = \frac{11}{2} - \frac{5\sqrt{5}}{2}\left(\frac{4-\sqrt{2}+\sqrt{30}}{2-\sqrt{3}-\sqrt{15}}\right)\left(\frac{4-\sqrt{2}-5\sqrt{3}+2\sqrt{5}-\sqrt{15}+\sqrt{30}}{2-\sqrt{3}+4\sqrt{5}+5\sqrt{6}-\sqrt{10}-\sqrt{15}}\right)^2.$$

(v) $R^5(e^{-2\pi\sqrt{3/320}}) = \dfrac{-2 + (a^2+2a+2)(\sqrt{a^2+1} - a)}{a^2}$, *where*
$$a = -\frac{1}{2} + \frac{10\sqrt{2} + 4\sqrt{5} + \sqrt{5}\sqrt{12 + 33\sqrt{10} + \sqrt{270(13 - 4\sqrt{10})}}}{17 + 17\sqrt{2} + 9\sqrt{3} - 7\sqrt{5} + 9\sqrt{6} - 5\sqrt{10} - 3\sqrt{15} - 3\sqrt{30}},$$

(vi) $R^5(e^{-2\pi\sqrt{1/960}}) = \dfrac{-2 + (a^2+2a+2)(\sqrt{a^2+1} - a)}{a^2}$, *where*
$$a = -\frac{1}{2} + \frac{10\sqrt{2} + 4\sqrt{5} + \sqrt{5}\sqrt{12 + 33\sqrt{10} - \sqrt{270(13 - 4\sqrt{10})}}}{-17 + 17\sqrt{2} - 9\sqrt{3} - 7\sqrt{5} + 9\sqrt{6} + 5\sqrt{10} - 3\sqrt{15} + 3\sqrt{30}}.$$

Proof. The results follow directly from (1.5) and Theorem 2.4 with the help of *Mathematica*. □

In view of Remark 2, we can evaluate $S(e^{-\pi\sqrt{n}})$ for $n = \frac{3}{5\cdot 4^{m-1}}$ and $\frac{1}{15\cdot 4^{m-1}}$, where m is any positive integer. We evaluate the cases for $m = 1$ and 2.

Theorem 3.2. *We have*

(i) $S^5(e^{-\pi\sqrt{3/5}}) = \dfrac{1}{4}\left(-3 - 5\sqrt{5} + \sqrt{30(5+\sqrt{5}\,)}\right)$,

(ii) $S^5(e^{-\pi\sqrt{1/15}}) = \dfrac{1}{4}\left(-3 + 5\sqrt{5} + \sqrt{30(5-\sqrt{5}\,)}\right)$,

(iii) $S^5(e^{-\pi\sqrt{3/20}}) = \dfrac{2 + (b^2 - 2b + 2)(\sqrt{b^2+1} - b)}{b^2}$, *where*

$$b = \frac{1}{2} + \frac{\sqrt{5}\,(4 + \sqrt{2} - 5\sqrt{3} - 2\sqrt{5} + \sqrt{15} + \sqrt{30})}{4(2 - \sqrt{3} + \sqrt{15})} + \frac{\sqrt{5(26 - 84\sqrt{2} + 18\sqrt{3} + 63\sqrt{5} - 47\sqrt{6} - \sqrt{10} + 38\sqrt{15})}}{(4 + 2\sqrt{3})\sqrt{2 + 6\sqrt{5} + 4\sqrt{15}}},$$

(iv) $S^5(e^{-\pi\sqrt{1/60}}) = \dfrac{2 + (b^2 - 2b + 2)(\sqrt{b^2+1} - b)}{b^2}$, *where*

$$b = \frac{1}{2} + \frac{\sqrt{5}\,(4 - \sqrt{2} - 5\sqrt{3} + 2\sqrt{5} - \sqrt{15} + \sqrt{30})}{4(-2 + \sqrt{3} + \sqrt{15})} + \frac{\sqrt{5(-26 - 84\sqrt{2} - 18\sqrt{3} + 63\sqrt{5} - 47\sqrt{6} + \sqrt{10} + 38\sqrt{15})}}{(4 + 2\sqrt{3})\sqrt{-2 + 6\sqrt{5} + 4\sqrt{15}}}.$$

Proof. The results are immediate consequences of (1.6), Lemma 2.1(i) and (ii), and Theorem 2.5 with the help of *Mathematica*. □

See [13, Corollary 4.12(i) and (ii)] for alternative proofs of Theorem 3.2(i) and (ii), respectively.

In view of Remark 3, we are able to evaluate $R(e^{-2\pi\sqrt{n}})$ for $n = \frac{9}{5\cdot 4^m}$ and $\frac{1}{45\cdot 4^m}$, where m is any positive integer. We evaluate the cases for $m = 1$ and 2.

Theorem 3.3. *We have*

(i) $R^5(e^{-2\pi\sqrt{9/20}}) = \sqrt{a^2+1} - a$, *where*

$$a = \frac{11}{2} + \frac{5}{2}\left(\frac{5 + 14\sqrt{5} - 10\sqrt{2 + 5\sqrt{5}}}{54 - 25\sqrt{3} + 16\sqrt{5} - 12\sqrt{15}}\right),$$

(ii) $R^5(e^{-2\pi\sqrt{1/180}}) = \sqrt{a^2+1} - a$, *where*

$$a = \frac{11}{2} + \frac{5}{2}\left(\frac{5 + 14\sqrt{5} - 10\sqrt{2 + 5\sqrt{5}}}{54 + 25\sqrt{3} + 16\sqrt{5} + 12\sqrt{15}}\right),$$

(iii) $R^5(e^{-2\pi\sqrt{9/80}}) = \dfrac{-2 + (a^2 + 2a + 2)(\sqrt{a^2+1} - a)}{a^2}$, *where*

$$a=-\frac{1}{2}+\frac{\sqrt{5}\,(2+\sqrt{3}\,)\left(5+2\sqrt{2}+\sqrt{5}+\sqrt{6+6\sqrt{5}}\right)}{2\sqrt{2}\left(-2+\sqrt{5}+\sqrt{15}+2\sqrt{2+\sqrt{5}}\right)},$$

(iv) $R^5(e^{-2\pi\sqrt{1/720}})=\dfrac{-2+(a^2+2a+2)(\sqrt{a^2+1}-a)}{a^2}$, *where*

$$a=-\frac{1}{2}+\frac{\sqrt{5}\,(2-\sqrt{3}\,)\left(5+2\sqrt{2}+\sqrt{5}-\sqrt{6+6\sqrt{5}}\right)}{2\sqrt{2}\left(-2+\sqrt{5}-\sqrt{15}+2\sqrt{2+\sqrt{5}}\right)}.$$

Proof. The results follow directly from (1.5) and Theorem 2.4. We used *Mathematica* to verify (i)–(iv). □

In view of Remark 4, we can evaluate $S(e^{-\pi\sqrt{n}})$ for $n=\frac{9}{5\cdot 4^{m-1}}$ and $\frac{1}{45\cdot 4^{m-1}}$, where m is any positive integer. We evaluate the cases for $m=1$ and 2.

Theorem 3.4. *We have*

(i) $S^5(e^{-\pi\sqrt{9/5}})$

$$=\frac{22-65\sqrt{5}-32\sqrt{15}+2\sqrt{5(1850+1040\sqrt{3}-143\sqrt{5}-70\sqrt{15}\,)}}{4+\sqrt{15}},$$

(ii) $S^5(e^{-\pi\sqrt{1/45}})$

$$=\frac{22-65\sqrt{5}+32\sqrt{15}+2\sqrt{5(1850-1040\sqrt{3}-143\sqrt{5}+70\sqrt{15}\,)}}{4-\sqrt{15}},$$

(iii) $S^5(e^{-\pi\sqrt{9/20}})=\dfrac{2+(b^2-2b+2)(\sqrt{b^2+1}-b)}{b^2}$, *where*

$$b=-\frac{1}{2}+\frac{c\sqrt{c}+\sqrt{(c-4)(c^2+4)}}{4\sqrt{c}} \quad \textit{and}$$

$$c=-1+\frac{\sqrt{5}\,(2+\sqrt{3}\,)\left(5+2\sqrt{2}+\sqrt{5}+\sqrt{6+6\sqrt{5}}\right)}{\sqrt{2}\left(-2+\sqrt{5}+\sqrt{15}+2\sqrt{2+\sqrt{5}}\right)},$$

(iv) $S^5(e^{-\pi\sqrt{1/180}})=\dfrac{2+(b^2-2b+2)(\sqrt{b^2+1}-b)}{b^2}$, *where*

$$b=-\frac{1}{2}+\frac{c\sqrt{c}+\sqrt{(c-4)(c^2+4)}}{4\sqrt{c}} \quad \textit{and}$$

$$c=-1+\frac{\sqrt{5}\,(2-\sqrt{3}\,)\left(5+2\sqrt{2}+\sqrt{5}-\sqrt{6+6\sqrt{5}}\right)}{\sqrt{2}\left(-2+\sqrt{5}-\sqrt{15}+2\sqrt{2+\sqrt{5}}\right)}.$$

Proof. The results follow from (1.6), Lemma 2.1(iii) and (iv), and Theorem 2.7. We used *Mathematica* to verify (i)–(iv). □

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