J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math.
http://dx.doi.org/10.7468/jksmeb.2021.28.4.315
Volume 28, Number 4 (November 2021), Pages 315–328

ISSN(Print) 1226-0657
ISSN(Online) 2287-6081

FRAME OPERATORS AND SEMI-FRAME OPERATORS OF FINITE GABOR FRAMES

N. M. Madhavan Namboothiri [a], T. C. Easwaran Nambudiri [b] and Jineesh Thomas [c,*]

Abstract. A characterization of frame operators of finite Gabor frames is presented here. Regularity aspects of Gabor frames in $l^2(\mathbb{Z}_N)$ are discussed by introducing associated semi-frame operators. Gabor type frames in finite dimensional Hilbert spaces are also introduced and discussed.

1. Introduction

Theory of frames is occupying its own space in both pure and applied mathematics because of its enormous applications. This mathematical theory, first initiated by Gabor [10] in his *Theory of Communication*, formulated a fundamental approach to signal decomposition in terms of elementary signals. His approach has become the paradigm for the spectral analysis associated with time-frequency methods.

The concept of frames in Hilbert spaces was introduced in 1952 by Duffin and Schaeffler [7] in their study of non harmonic Fourier series. The pivotal works of Janssen [12] made it an independent topic of mathematical investigation in 1980s. The importance of the theory of frames in modern signal processing and time frequency analysis is now entrenched (see [11], for example). Various generalizations of this concept have been proposed; frame of subspaces [1, 2], pseudo-frames [14], oblique frames [5] and so on, where in all, the mathematical theory of Gabor frames (also known as Weyl-Heisenberg frames) plays the key role. These frames are very specific as they are being generated by translations and modulations of a single vector in the space concerned.

Received by the editors March 16, 2021. Revised September 15, 2021. Accepted Sep. 25, 2021.
2010 *Mathematics Subject Classification.* 42C15, 47B90, 94A12.
Key words and phrases. Gabor frame operator, semi-frame operator, regular frames.
*Corresponding author.

Gabor analysis took a new turn with the fundamental works of Daubechies, Grossmann and Meyer in 1986 [6] and put forth the idea of combining Gabor analysis with frame theory. Systematic utilisation of time shifts (translations) and frequency shifts (modulations) lie at the heart of modern time-frequency analysis. Gabor analysis aims at representing functions (signals) $f \in L^2(\mathbb{R})$ as superpositions of translated and modulated versions of a fixed function $g \in L^2(\mathbb{R})$.

The most significant tool in frame theory (both in theoretical and applicational aspects) is the frame operator, associated with a given frame. Observables corresponding to certain physical process can be regarded as operators on an appropriate Hilbert space. Each observable has its own parameters which in turn determine the operator. Thus frames corresponding to a given *nice* operator have substantial practical importance. In particular, Gabor frame operators, which are very special in their construction, are receiving significant research attention and are of interest in this paper too.

In $l^2(\mathbb{Z}_N)$, a Gabor frame is the set of time-frequency translates of a single vector in it [16]. The concept of generalised Weyl-Heisenberg frames, frame operators in $L^2(\mathbb{R})$ and its characterization are discussed in [8, 9]. We present this paper in the following sequel. We begin with some basic definitions and results which are essential for the present work (see Section 2). The concept of generalised Weyl-Heisenberg frames in $l^2(\mathbb{Z}_N)$, frame operators and its characterization are discussed in Section 3. Section 4 mainly focused on regular Gabor frames and interplay between semi frame operators in connection with Weyl-Heisenberg frame operators. In Section 5, we deal with the construction and characterization of Gabor type frames in finite dimensional Hilbert spaces. Our basic references for both abstract Hilbert frame theory and the theory of Weyl-Heisenberg frames are [4, 11]. In this discussion, $\mathcal{H}$ will denote a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$.

2. Preliminaries

A frame in a Hilbert space $\mathcal{H}$ is a sequence $\{f_k\}_{k=1}^{\infty}$ of elements in $\mathcal{H}$ such that

$$A\|f\|^2 \leq \sum_{k=1}^{\infty} |\langle f, f_k \rangle|^2 \leq B\|f\|^2, \quad \forall\, f \in \mathcal{H}$$

for some constants $A, B > 0$. Here A and B are called *frame bounds*. If $A = B$, then the frame $\{f_k\}_{k=1}^{\infty}$ is called a *tight frame*. A tight frame is called a *Parseval frame or normalized tight frame* if $A = B = 1$. Whenever a sequence $\{f_k\}_{k=1}^{\infty}$ of elements in $\mathcal{H}$ satisfies

$$\sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 \;\leq\; B\|f\|^2, \;\; \forall\, f \in \mathcal{H}$$

for some constant $B > 0$, then $\{f_k\}_{k=1}^{\infty}$ is said to be a *Bessel sequence* or a *semi frame sequence* and is a *frame sequence* if it is a frame for $\overline{span}\{f_k\}_{k=1}^{\infty}$.

A frame $\{f_k\}_{k=1}^{\infty}$ in $\mathcal{H}$ with upper frame bound β spans a dense subspace of $\mathcal{H}$ and $\|f_k\| \leq \sqrt{\beta}$ for all k. In particular $\|f_k\| \leq 1$ when $\{f_k\}_{k=1}^{\infty}$ is a normalized tight frame. For a frame $\{f_k\}_{k=1}^{\infty}$ in a Hilbert space $\mathcal{H}$, the map S defined by

$$Sf = \sum_{k=1}^{\infty}\langle f, f_k\rangle f_k, \; f \in \mathcal{H}$$

is a bounded linear operator on $\mathcal{H}$, called the *frame operator* of the frame $\{f_k\}_{k=1}^{\infty}$. The frame operator of a tight frame is a scalar multiple of the identity operator and that of a Parseval frame is the identity operator.

Remark 2.1. Let $\{f_k\}_{k=1}^{\infty}$ be a frame with frame operator S and frame bounds A, B in a Hilbert space $\mathcal{H}$. Then,

(i) S is bounded, invertible, self-adjoint and positive. In fact $AI \leq S \leq BI$.

(ii) $\{S^{-1}f_k\}_{k=1}^{\infty}$ is a frame with frame bounds B^{-1}, A^{-1} and $\{S^{-1/2}f_k\}_{k=1}^{\infty}$ is a normalized tight frame.

(iii) If A and B are the optimal frame bounds for $\{f_k\}_{k=1}^{\infty}$, then the bounds B^{-1}, A^{-1} are the optimal frame bounds for $\{S^{-1}f_k\}_{k=1}^{\infty}$. The frame operator for $\{S^{-1}f_k\}_{k=1}^{\infty}$ is S^{-1}. Further $B^{-1}I \leq S^{-1} \leq A^{-1}I$.

Operator with the same definition of S exists for a Bessel sequence $\{f_k\}_{k=1}^{\infty}$ too, which is not invertible, but still is known as a frame operator in literature. For a frame $\{f_k\}_{k=1}^{\infty}$ in $\mathcal{H}$ with frame operator S, the frame $\{\tilde{f}_k\}_{k\in\mathbb{N}} = \{S^{-1}f_k\}_{k\in\mathbb{N}}$ is known as the *(canonical) dual frame* of $\{f_k\}_{k=1}^{\infty}$. A frame $\{f_k\}_{k=1}^{\infty}$ together with its dual frame $\{\tilde{f}_k\}_{k\in\mathbb{N}}$ yields the following *frame decomposition* by which every element in $\mathcal{H}$ has a representation as a superposition of the frame elements.

Theorem 2.2. *Let $\{f_k\}_{k=1}^{\infty}$ be a frame in a Hilbert space $\mathcal{H}$ with frame operator S. Then for all $f \in \mathcal{H}$, $f = \sum_{k=1}^{\infty}\langle f, S^{-1}f_k\rangle f_k$ and $f = \sum_{k=1}^{\infty}\langle f, f_k\rangle S^{-1}f_k$.*

Here both series converge unconditionally for all $f \in \mathcal{H}$.

Remark 2.3. Let $\{f_k\}_{k=1}^{\infty}$ be a frame in $\mathcal{H}$ and $A : \mathcal{H} \longrightarrow \mathcal{H}$ be a bounded linear surjective operator on $\mathcal{H}$. Then $\{Af_k\}_{k=1}^{\infty}$ is a frame in $\mathcal{H}$ (see [4]).

In particular, if A is an invertible bounded linear operator on $\mathcal{H}$, then $\{Af_k\}_{k=1}^{\infty}$ is also a frame for $\mathcal{H}$. If $\{f_k\}_{k=1}^{\infty}$ is exact, so is $\{Af_k\}_{k=1}^{\infty}$. If A is unitary, then $\{Af_k\}_{k=1}^{\infty}$ has the same frame bounds as those of $\{f_k\}_{k=1}^{\infty}$.

3. Gabor Frame Operators on $l^2(\mathbb{Z}_N)$

Gabor analysis is mainly based on the interplay between the Fourier transform and the translation and modulation operators. In our discussion, the finite dimensional space $\mathbb{C}^N$ is identified with the space $l^2(\mathbb{Z}_N)$ of complex functions on $\mathbb{Z}_N$ equipped with the standard inner product. For each $k, l \in \mathbb{Z}_N$ and $g \in l^2(\mathbb{Z}_N)$, the *Translation operator* $T_k : l^2(\mathbb{Z}_N) \to l^2(\mathbb{Z}_N)$ is defined by

$$(T_k g)(j)) = g(j-k) \quad \text{for} \quad j = 0, 1, 2, ..., N-1.$$

and the *Modulation operator* $M_l : l^2(\mathbb{Z}_N) \to l^2(\mathbb{Z}_N)$ is defined by

$$(M_l g)(j) = e^{2\pi i l j/N} g(j) \quad \text{for} \quad j = 0, 1, 2, ..., N-1.$$

The Fourier transform $\mathcal{F}$ on $l^2(\mathbb{Z}_N)$ is the linear transformation defined by

$$F(f)(k) = \sum_{j=0}^{N-1} f(j) e^{\frac{-2\pi k j}{N}}$$

for all $f \in l^2(\mathbb{Z}_N)$ and $k = 0, 1, 2, ..., N-1$.

Definition. For $g \in l^2(\mathbb{Z}_N) - \{0\}$ and $\Lambda \subseteq \mathbb{Z}_N \times \mathbb{Z}_N$, the set $\{M_l T_k g;\ (k, l) \in \Lambda\}$ is called a *Gabor System* generated by the *window function g*. A *Gabor frame* (also known as a *Weyl-Heisenberg frame*) in $l^2(\mathbb{Z}_N)$ is a Gabor system which spans $l^2(\mathbb{Z}_N)$.

In the literature, $\pi(k, l)$ denotes the time-frequency shift operator in $l^2(\mathbb{Z}_N)$ ([3]), defined by $\pi(k, l)g = M_l T_k g$ for all $g \in l^2(\mathbb{Z}_N)$. Following lemma appeared in [15], brings out a couple of significant properties of the Gabor frame operators on $l^2(\mathbb{Z}_N)$.

Lemma 3.1. *The frame operator of a Gabor frame* $\{M_l T_k g : (k, l) \in \Lambda\}$ *in* $l^2(\mathbb{Z}_N)$*, where* $\Lambda = \Lambda_1 \times \Lambda_2$ *and* Λ_1*,* Λ_2 *are subgroups of* $\mathbb{Z}_N$ *commutes with* $\pi(k, l)$ *for all* $(k, l) \in \Lambda_1 \times \Lambda_2$*.*

In particular, the frame operator S commutes with all translations T_k, $k \in \Lambda_1$ and all modulations M_l, $l \in \Lambda_2$.

Remark 3.2. The existence of a Gabor frame in $l^2(\mathbb{Z}_N)$ has been established by Jim Laurence [13] for primes and by Romanos-Digenes Malikiosis [15] for any $N \geq 4$. Consequently, for any positive integer N, there is a Gabor frame in $l^2(\mathbb{Z}_N)$ of the form $\{M_l T_k g : (k, l) \in \Lambda_1 \times \Lambda_2\}$ where $|\Lambda_1 \times \Lambda_2| \geq N$ and Λ_1, Λ_2 are subgroups of $\mathbb{Z}_N$. This yield the following observation which will be used in sequel.

Proposition 3.3. *For any two subgroups* Λ_1 *and* Λ_2 *of* $\mathbb{Z}_N$ *with* $|\Lambda_1 \times \Lambda_2| \geq N$, *there is a tight Gabor frame in* $l^2(\mathbb{Z}_N)$ *with the identity operator as its frame operator.*

Proof. As observed in Remark 3.2, for any two subgroups Λ_1 and Λ_2 of $\mathbb{Z}_N$ with $|\Lambda_1 \times \Lambda_2| \geq N$, there is always a Gabor frame $\{M_l T_k g : (k,l) \in \Lambda_1 \times \Lambda_2\}$ in $l^2(\mathbb{Z}_N)$. By Lemma 3.1, the frame operator S of this frame is (trivially bounded), invertible, positive and commutes with the translation $T_p, p \in \Lambda_1$ and the modulation $M_q, q \in \Lambda_2$. Hence $S^{-1/2}$ also possesses all the properties. By Remark 2.3, $\{S^{-1/2} M_l T_k g = M_l T_k S^{-1/2} g : (k,l) \in \Lambda_1 \times \Lambda_2\}$ is a Gabor frame on $l^2(\mathbb{Z}_N)$. For $x \in l^2(\mathbb{Z}_N)$,

$$\begin{aligned} x &= S^{-1/2} S(S^{-1/2}(x)) \\ &= S^{-1/2}\Big(\sum_{(k,l)\in\Lambda} \langle S^{-1/2}(x), M_l T_k g\rangle M_l T_k g \\ &= \sum_{(k,l)\in\Lambda} \langle x, S^{-1/2} M_l T_k g\rangle M_l T_k S^{-1/2} g \\ &= \sum_{(k,l)\in\Lambda} \langle x, M_l T_k S^{-1/2} g\rangle M_l T_k S^{-1/2} g \end{aligned}$$

Hence the frame operator of the Gabor frame $\{M_l T_k S^{-1/2} g : (k,l) \in \Lambda_1 \times \Lambda_2\}$ is the identity operator. □

Definition. A Gabor frame in $l^2(\mathbb{Z}_N)$ of the form $\{M_l T_k g : (k,l) \in \Lambda_1 \times \Lambda_2\}$ where Λ_1 and Λ_2 are subgroups of $\mathbb{Z}_N$ is called a *regular Gabor frame* in $l^2(\mathbb{Z}_N)$.

Construction of frames with a given operator as their frame operator is an interesting theme having practical relevance in the context of frames. As shown in [8], positive and invertible operators can become frame operators on separable Hilbert spaces. However, in view of Lemma 3.1, those operators on $l^2(\mathbb{Z}_N)$ which are positive and invertible but fail to commute with certain translation and modulation operators can not correspond to any regular Gabor frame in $l^2(\mathbb{Z}_N)$. Here, we present a complete characterization of frame operators of regular Gabor frames in $l^2(\mathbb{Z}_N)$.

Proposition 3.4. *A bounded linear operator on* $l^2(\mathbb{Z}_N)$ *can be realized as a frame operator of a regular Gabor frame in* $l^2(\mathbb{Z}_N)$ *if and only if it is positive, invertible and commutes with translation operator* T_p, $p \in \Lambda_1$ *and modulation operator* M_q, $q \in \Lambda_2$, *for some subgroups* Λ_1 *and* Λ_2 *of* $\mathbb{Z}_N$ *with* $|\Lambda_1 \times \Lambda_2| \geq N$.

Proof. Let $\{M_l T_k g : (k,l) \in \Lambda_1 \times \Lambda_2\}$ be a regular Gabor frame in $l^2(\mathbb{Z}_N)$ with S as its frame operator. By Remark 2.1 and Lemma 3.1, S is positive, invertible and commutes with translation operator $T_p,\ p \in \Lambda_1$ and modulation operator $M_q,\ q \in \Lambda_2$.

Conversely if S is a linear positive and invertible operator on $l^2(\mathbb{Z}_N)$ that commutes with the translation operator $T_p,\ p \in \Lambda_1$ and modulation operator $M_q,\ q \in \Lambda_2$, for some subgroups Λ_1 and Λ_2 of $\mathbb{Z}_N$ with $|\Lambda_1 \times \Lambda_2| \geq N$, then so is its positive square root $S^{1/2}$. By Proposition 3.3, there is always a tight regular Gabor frame $\{M_l T_k g : (k,l) \in \Lambda_1 \times \Lambda_2\}$ in $l^2(\mathbb{Z}_N)$ with identity operator I as its frame operator. The invertibility and commutativity of $S^{1/2}$ ensure that the image of this frame under $S^{1/2}$ is a frame.
This frame is a regular Gabor frame in $l^2(\mathbb{Z}_N)$, since

$$\begin{aligned} S^{1/2}(\{M_l T_k g\}_{(k,l)\in\Lambda_1\times\Lambda_2}) &= \{S^{1/2} M_l T_k g\}_{(k,l)\in\Lambda_1\times\Lambda_2} \\ &= \{M_l T_k S^{1/2} g\}_{(k,l)\in\Lambda_1\times\Lambda_2}. \end{aligned}$$

Now, for all $x \in l^2(\mathbb{Z}_N)$,

$$\begin{aligned} \sum_{(k,l)\in\Lambda} \langle x, M_l T_k S^{1/2} g\rangle M_l T_k S^{1/2} g &= S^{1/2} \sum_{(k,l)\in\Lambda} \langle S^{1/2} x, M_l T_k g\rangle M_l T_k g \\ &= S^{1/2} I S^{1/2}(x) = S(x). \end{aligned}$$

Thus S is the frame operator of the regular Gabor frame $\{M_l T_k S^{1/2} g\}_{(k,l)\in\Lambda_1\times\Lambda_2}$. □

4. Gabor Semi-frame Operators and Regularity

Any finite sequence of elements of $l^2(\mathbb{Z}_N)$ can be considered as a Bessel sequence in $l^2(\mathbb{Z}_N)$. Let $\{u_k\}_{k\in\Lambda}$, $|\Lambda| < \infty$ is such a sequence. Then there is a bounded linear positive operator S on $l^2(\mathbb{Z}_N)$ defined by $S(x) = \sum\limits_{k\in\Lambda} \langle x, u_k\rangle u_k$ for all $x \in l^2(\mathbb{Z}_N)$. We call this operator as the *semi-frame operator* associated to $\{u_k\}_{k\in\Lambda}$.

Analogously, the family $\mathcal{G}(g,\Lambda) = \{M_l T_k g : (k,l) \in \Lambda\}$ where $g \in l^2(\mathbb{Z}_N)$ and $\Lambda \subseteq \mathbb{Z}_N \times \mathbb{Z}_N$, is a Bessel sequence in $l^2(\mathbb{Z}_N)$. Hence there is an associated semi-frame operator to this, called the *Gabor semi-frame operator* associated with the generating set Λ and generating function g.

Proposition 4.1. *Let $\Lambda = \Lambda_1 \times \Lambda_2 \subseteq \mathbb{Z}_N \times \mathbb{Z}_N$ be such that $\Lambda_1' = \Lambda_1 - r$ and $\Lambda_2' = \Lambda_2 - t$ are subgroups of $\mathbb{Z}_N$ for some $(r,t) \in \mathbb{Z}_N \times \mathbb{Z}_N$ with $|\ \Lambda\ | \geq N$. If S is a Gabor semi-frame operator on $l^2(\mathbb{Z}_N)$ associated with $\mathcal{G}(g,\Lambda)$ then there are Gabor semi-frame operators S_r and S_t on $\mathbb{C}^N$ such that $ST_r = T_r S_r$ and $SM_t = M_t S_t$. Moreover $S_r T_h = T_h S_r$ for all $h \in \Lambda_1'$ and $S_t M_p = M_p S_t$ for all $p \in \Lambda_2'$.*

Proof. Let S be the semi-frame operator of $\mathcal{G}(g,\Lambda)$ as given in the statement. Then for $x \in l^2(\mathbb{Z}_N)$, $S(x) = \sum_{(k,l)\in\Lambda_1\times\Lambda_2} \langle x, M_lT_kg\rangle M_lT_kg$.

$$\begin{aligned} ST_r(x) &= \sum_{(k,l)\in\Lambda_1\times\Lambda_2} \langle T_r(x), M_lT_kg\rangle M_lT_kg \\ &= \sum_{(k,l)\in\Lambda_1\times\Lambda_2} \langle x, e^{2\pi ilr/N} M_lT_{k-r}g\rangle M_lT_kg \\ &= T_r \sum_{(k',l)\in\Lambda_1'\times\Lambda_2} \langle x, M_lT_{k'}g\rangle M_lT_{k'}g \\ &= T_rS_r(x), \end{aligned}$$

where $S_r(x) = \sum_{(k',l)\in\Lambda_1'\times\Lambda_2} \langle x, M_lT_{k'}g\rangle M_lT_{k'}g$. Also,

$$\begin{aligned} SM_t(x) &= \sum_{(k,l)\in\Lambda_1\times\Lambda_2} \langle M_t(x), M_lT_kg\rangle M_lT_kg \\ &= \sum_{(k,l)\in\Lambda_1\times\Lambda_2} \langle x, M_{l-t}T_kg\rangle M_lT_kg \\ &= \sum_{(k,l')\in\Lambda_1\times\Lambda_2'} \langle x, M_{l'}T_kg\rangle M_{l'+t}T_kg \\ &= M_t \sum_{(k,l')\in\Lambda_1\times\Lambda_2'} \langle x, M_{l'}T_kg\rangle M_{l'}T_kg \\ &= M_tS_t(x), \end{aligned}$$

where $S_t(x) = \sum_{(k,l')\in\Lambda_{\prime}\times\Lambda_2'} \langle x, M_{l'}T_kg\rangle M_{l'}T_kg$.

Hence there are Gabor semi-frame operators S_r and S_t on $l^2(\mathbb{Z}_N)$ such that $ST_r = T_rS_r$ and $SM_t = M_tS_t$. Now for $h \in \Lambda_1'$,

$$\begin{aligned} S_rT_h(x) &= \sum_{(k',l)\in\Lambda_1'\times\Lambda_2} \langle T_h(x), M_lT_{k'}g\rangle M_lT_{k'}g \\ &= \sum_{(k',l)\in\Lambda_1'\times\Lambda_2} \langle x, T_{-h}M_lT_{k'}g\rangle M_lT_{k'}g \\ &= \sum_{(k',l)\in\Lambda_1'\times\Lambda_2} \langle x, e^{2\pi ilh/N} M_lT_{k'-h}g\rangle M_lT_{k'}g. \end{aligned}$$

Taking $k' - h = k''$, we have

$$\begin{aligned} S_rT_h(x) &= \sum_{(k'',l)\in\Lambda_1'\times\Lambda_2} e^{-2\pi ilh/N}\langle x, M_lT_{k''}g\rangle M_lT_{k''+h}g \\ &= T_h \sum_{(k'',l)\in\Lambda_1'\times\Lambda_2} \langle x, M_lT_{k''}g\rangle M_lT_{k''}g \\ &= T_hS_r(x). \end{aligned}$$

Similarly, for each $p \in \Lambda_2'$, $S_tM_p = M_pS_t$. Thus S_r commutes with all translations T_h for every $h \in \Lambda_1'$ and S_t commutes with all modulations M_p for every $p \in \Lambda_2'$. □

Remark 4.2. It can be noted that if S is a Gabor semi-frame operator as in Proposition 4.1, then the invertibility of S, S_r and S_t are equivalent.

Theorem 4.3. *If the Gabor semi-frame operator S as in Proposition 4.1 is invertible, then there are regular Gabor frame operators S' and S'' on $l^2(\mathbb{Z}_N)$ such that $T_rM_tS' = ST_rM_t$ and $M_tT_rS'' = SM_tT_r$.*

Proof. By Proposition 4.1, there is a Gabor semi-frame operator S_r on $l^2(\mathbb{Z}_N)$ such that $ST_r = T_rS_r$, where S_r commutes with all translations T_h, $h \in \Lambda_1' = \Lambda_1 - r$. Also, by the same proposition there is a Gabor semi-frame operator $(S_r)_t$ on $l^2(\mathbb{Z}_N)$ such that $S_rM_t = M_t(S_r)_t$, and $(S_r)_t$ commutes with all modulations M_p, $p \in \Lambda_2'$ and $\Lambda_2' = \Lambda_2 - t$. Now

$$(S_r)_t = \sum_{(k,l)\in\Lambda_1'\times\Lambda_2'} \langle x, M_lT_kg\rangle M_lT_kg$$

and $(S_r)_t$ commutes with all translations T_h, $h \in \Lambda_1'$ since Λ_1' is a subgroup of $\mathbb{Z}_N$. Also, $ST_rM_t = T_rS_rM_t = T_rM_t(S_r)_t$ so that $(S_r)_t = (T_rM_t)^*S(T_rM_t)$.
Since S is invertible and positive, so is $(S_r)_t$. Thus $(S_r)_t$ is an invertible Gabor semi-frame operator which commutes with all translations T_h, $h \in \Lambda_1'$ and modulations M_t, $t \in \Lambda_2'$ where $\Lambda_1' \times \Lambda_2'$ is the generating set for the corresponding semi-frame. Hence by Proposition 3.4, $S' = (S_r)_t$ is a regular Gabor frame operator. By taking S_t instead of S_r, it follows that there exists S'' on $l^2(\mathbb{Z}_N)$ such that $M_tT_rS'' = SM_tT_r$. □

Example 4.4. In the space $\mathbb{C}^{12}$ with $g = (1,0,1,0,1,0,1,0,1,0,1,0) \in \mathbb{C}^{12}$, $\Lambda_1 = \{1,4,7,10\}$ and $\Lambda_2 = \{1,3,5,7,9,11\}$, the family $P = \{M_lT_kg : l \in \Lambda_2,\ k \in \{1,4\}\}$ is linearly independent since g identifies the $span(\{M_lT_k : l \in \Lambda_2,\ k \in \{1,4\}\})$ (see

[13]). Hence P is a basis for $\mathbb{C}^{12}$. Thus $\{M_lT_kg : (k,l) \in \Lambda_1 \times \Lambda_2\}$ is a Gabor frame in $\mathbb{C}^{12}$, but non regular.

However, by Theorem 4.3, when $\Lambda_1' = \Lambda_1 + 2$ as well as $\Lambda_2' = \Lambda_1 + 1$, the family $\{M_lT_kg; (k,l) \in \Lambda_1' \times \Lambda_2'\}$ becomes a regular Gabor frame in $\mathbb{C}^{12}$.

Let Λ_1 and Λ_2 be any two subsets of $\mathbb{Z}_N$. Define $r = min\Lambda_1$ and $s = min\Lambda_2$. Also let Λ_1' and Λ_2' are the subgroups of $\mathbb{Z}_N$ generated by the subsets $\Lambda_1 - r$ and $\Lambda_2 - s$ respectively. Then $\tilde{\Lambda_1} = \Lambda_1' + r$ and $\tilde{\Lambda_2} = \Lambda_2' + s$ are subsets of $\mathbb{Z}_N$ such that $\Lambda_1 \subseteq \tilde{\Lambda_1}$ and $\Lambda_2 \subseteq \tilde{\Lambda_2}$.

Remark 4.5. Let $\{M_lT_kg : (k,l) \in \Lambda_1 \times \Lambda_2\}$ be an irregular Gabor frame in $\mathbb{C}^N$. Then by above discussion there exists unique numbers r and s corresponding to Λ_1 and Λ_2 and there exists unique subsets $\tilde{\Lambda_1}$ and $\tilde{\Lambda_2}$ of $\mathbb{Z}_N$ with $\Lambda_1 \subseteq \tilde{\Lambda_1}$ and $\Lambda_2 \subseteq \tilde{\Lambda_2}$ such that $\Lambda_1' = \tilde{\Lambda_1} - r$ and $\Lambda_2' = \tilde{\Lambda_2} - s$ are subgroups of $\mathbb{Z}_N$. Also note that $\{M_lT_kg : (k,l) \in \tilde{\Lambda_1} \times \tilde{\Lambda_2}\}$ is again a Gabor frame in $\mathbb{C}^N$, actually it is a smallest possible extension of the given irregular Gabor frame to an almost regular Gabor frame in $\mathbb{C}^N$. Then from the proof of Proposition 4.1, $\{M_lT_kg : (k,l) \in \Lambda_1' \times \Lambda_2'\}$ is a regular Gabor frame.

5. Gabor Type Frames in Finite Dimensional Hilbert Spaces

Abstract frames in Hilbert spaces are countable subfamilies of the space, satisfying the required frame inequality. Where as Gabor frames are special types of frames having specific construction from a single vector by the action of a family of unitary operators, produced by certain translation and modulation operators, up on the generating vector. Such a construction is an interesting theme in the context of general finite dimensional Hilbert spaces. We discuss this in view of [8].

Let U be a bounded linear operator from a Hilbert space $\mathcal{H}$ to a Hilbert space $\mathcal{K}$ with its range set R_U is closed. Then there exists a bounded operator $U^\dagger$ from $\mathcal{K}$ to $\mathcal{H}$ such that $UU^\dagger f = f$ for all $f \in R_U$. Moreover $UU^\dagger$ is the orthogonal projection of $\mathcal{H}$ onto R_U (See [4], Lemma A.7.1).

Theorem 5.1. *Let $\{f_k\}_{k=1}^{\infty}$ be a frame in $\mathcal{K}$ with bounds A and B and let $U : \mathcal{K} \longrightarrow \mathcal{H}$ be a bounded linear operator with non trivial closed range. Then $\{Uf_k\}_{k=1}^{\infty}$ is a frame sequence with bounds $A \parallel U^\dagger \parallel^{-2}$ and $B \parallel U \parallel^2$.*

Proof. First observe that

$$\begin{aligned}\sum_{k=1}^{\infty} \mid \langle f, Uf_k\rangle \mid^2 &= \sum_{k=1}^{\infty} \mid \langle U^* f,\ f_k\rangle \mid^2 \\ &\leq B \parallel U^* f \parallel^2 \leq B \parallel U^* \parallel^2 \parallel f \parallel^2 = B \parallel U \parallel^2 \parallel f \parallel^2 .\end{aligned}$$

Thus $\{Uf_k\}_{k=1}^{\infty}$ is a Bessel sequence in $\mathcal{H}$ with upper frame bound $B \parallel U \parallel^2$.

For $h \in Span\{Uf_k\}_{k=1}^{\infty}$, there is $f \in Span\{f_k\}_{k=1}^{\infty}$ with $h = Uf$.
Since $UU^{\dagger}$ is the orthogonal projection onto R_U, it is self adjoint and hence,

$$h = Uf = (UU^{\dagger})(Uf) = (U^{\dagger})^* U^*(Uf).$$

Thus, $\parallel h \parallel^2 \leq \parallel (U^{\dagger})^* \parallel^2 \parallel U^* U f \parallel^2 \leq \frac{\|U^{\dagger}\|^2}{A} \sum_{k=1}^{\infty} \mid \langle U^* U f, f_k\rangle \mid^2$
$= \frac{\|U^{\dagger}\|^2}{A} \sum_{k=1}^{\infty} \mid \langle h, Uf_k\rangle \mid^2.$

Since $U : \mathcal{K} \longrightarrow \mathcal{H}$ is of non trivial closed range, the remaining assertions follow. □

Remark 5.2. Let $\{f_k\}_{k=1}^{\infty}$ be a frame in $\mathcal{K}$ with bounds A and B and $U : \mathcal{K} \longrightarrow \mathcal{H}$ be a bounded linear surjective operator. Then $\{Uf_k\}_{k=1}^{\infty}$ is a frame in $\mathcal{H}$ with bounds $A \parallel U^{\dagger} \parallel^{-2}$ and $B \parallel U \parallel^2$.

Let $\mathcal{H}$ be a finite dimensional Hilbert space and let $\{x_1,\ x_2,\ \ldots,\ x_n\}$ be an orthonormal basis for $\mathcal{H}$. The map U defined by $U(ej) = x_j$ for $j = 1,\ 2, \ldots,\ n$, where $\{e_j : j = 1,\ 2, \ldots,\ n\}$ is the standard basis of $\mathbb{C}^N$ which defines a unitary map from $\mathbb{C}^N$ to $\mathcal{H}$.

Now, for a Gabor frame $\{M_l T_k g : (k,l) \in \Lambda\}$, $\Lambda \subseteq \mathbb{Z}_N \times \mathbb{Z}_N$ in $l^2(\mathbb{Z}_N)$, by Remark 5.2, $\{UM_l T_k g : (k,l) \in \Lambda\}$, $\Lambda \subseteq \mathbb{Z}_N \times \mathbb{Z}_N$ is a frame in $\mathcal{H}$. Further $UM_l U^{-1}$ and $UT_k U^{-1}$ are bounded linear operators on $\mathcal{H}$ and
$U(M_l T_k g) = (UM_l U^{-1})(UT_k U^{-1})U(g)$ for each $(k,l) \in \Lambda$.

Denote $M_l^U = UM_l U^{-1}$ and $T_k^U = UT_k U^{-1}$. Then $M_l^U M_m^U = M_{l+m}^U$ and $T_k^U T_{k'}^U = T_{k+k'}^U$ for each $\{l,\ m,\ k,\ k'\} \subseteq \mathbb{Z}_N$. Thus $\{UM_l T_k g : (k,l) \in \Lambda\} = \{M_l^U T_k^U U(g) : (k,l) \in \Lambda\}$, $\Lambda \subseteq \mathbb{Z}_N \times \mathbb{Z}_N$ is a frame in $\mathcal{H}$.

Definition. Let $\mathcal{H}$ be a Hilbert space of dimension N. For $l, k \in \mathbb{Z}_N$ and a unitary linear map $U : l^2(\mathbb{Z}_N) \mapsto \mathcal{H}$, let $M_l^U = UM_l U^{-1}$ and $T_k^U = UT_k U^{-1}$ where T_k and M_l are respectively, the translation and modulation operators on $l^2(\mathbb{Z}_N)$. The unitary operators M_l^U and T_k^U are called *generalised modulation* and *generalised translation* respectively on $\mathcal{H}$ associated to the unitary operator U. A frame of the form $\{M_l^U T_k^U h : (k,l) \in \Lambda\}$ where $\Lambda \subseteq \mathbb{Z}_N \times \mathbb{Z}_N$ and $h \in \mathcal{H}$ is called a *Gabor type frame* in $\mathcal{H}$. If $\Lambda = \Lambda_1 \times \Lambda_2$ with Λ_1 and Λ_2 are subgroups of $\mathbb{Z}_N$, then the *Gabor type frame* $\{M_l^U T_k^U h : (k,l) \in \Lambda\}$ is called a *regular Gabor type frame.*

Here $\{M_l^U T_k^U : (k,l) \in \Lambda \subseteq \mathbb{Z}_N \times \mathbb{Z}_N\}$ is a subfamily of the group of unitary operators on $\mathcal{H}$.

Obviously, every Gabor frame in $l^2(\mathbb{Z}_N)$ is also a Gabor type frame in $l^2(\mathbb{Z}_N)$. The following simple observation is interesting in this context.

Proposition 5.3. *The Fourier transform $\mathcal{F}$ on $l^2(\mathbb{Z}_N)$ maps Gabor frames in $l^2(\mathbb{Z}_N)$ into Gabor type frames in $l^2(\mathbb{Z}_N)$ (which are not Gabor frames).*

Proof. The Fourier transform $\mathcal{F}$ on $l^2(\mathbb{Z}_N)$ is a unitary map that converts translations into modulations and modulations into translations on $l^2(\mathbb{Z}_N)$. Let $\mathcal{G}(g,\Lambda) = \{M_l T_k g : (k,l) \in \Lambda = \Lambda_1 \times \Lambda_2\}$ be a Gabor frame in $l^2(\mathbb{Z}_N)$. The commutator relations $\mathcal{F}T_k = M_{-k}\mathcal{F}$ and $\mathcal{F}M_l = T_l\mathcal{F}$ yield that the image $\mathcal{F}(\mathcal{G}(g,\Lambda))$ of this Gabor frame under $\mathcal{F}$ takes the form $\{T_l M_{-k}\mathcal{F}(g) : (k,l) \in \Lambda_1 \times \Lambda_2\}$ which is not of the Gabor frame structure that we follow.

However, again from commutator relations $\mathcal{F}T_k = M_{-k}\mathcal{F}$ and $\mathcal{F}M_l = T_l\mathcal{F}$, we observe that $M_{-k} = \mathcal{F}T_k\mathcal{F}^{-1}$ and $T_l = \mathcal{F}M_l\mathcal{F}^{-1}$ respectively. Thus T_l and M_{-k} appearing in the image frame are respectively the generalised modulations and generalised translations on $l^2(\mathbb{Z}_N)$. In particular, $\mathcal{F}(\mathcal{G}(g,\Lambda))$ is a Gabor type frame in $l^2(\mathbb{Z}_N)$. Thus images of Gabor frames in $l^2(\mathbb{Z}_N)$ under the Fourier transform $\mathcal{F}$ are Gabor type frames in $l^2(\mathbb{Z}_N)$. □

Remark 5.4. More generally, every unitary linear map from $l^2(\mathbb{Z}_N)$ onto a Hilbert space $\mathcal{H}$ sends Gabor frames in $l^2(\mathbb{Z}_N)$ into a Gabor type frames in $\mathcal{H}$.

The following proposition gives an interesting property of the family of unitary operators on a finite dimensional Hilbert space.

Proposition 5.5. *The family of all Gabor type frames in a given finite dimensional Hilbert space $\mathcal{H}$ is invariant under every unitary linear map on $\mathcal{H}$.*

Proof. Let $\mathcal{H}$ be an N-dimensional Hilbert space and $\{M_l^U T_k^U h : (k,l) \in \Lambda \subseteq \mathbb{Z}_N \times \mathbb{Z}_N\}$, where $h \in \mathcal{H}$, be a Gabor type frame in $\mathcal{H}$. Then there is a $g \in l^2(\mathbb{Z}_N)$ such that $Ug = h$. Now consider a unitary map V on $\mathcal{H}$. Then VU is a unitary map from $l^2(\mathbb{Z}_N)$ onto $\mathcal{H}$. Also

$$\begin{aligned}\{VM_l^U T_k^U h : (k,l) \in \Lambda\} &= \{V(UM_lU^{-1})V^{-1}V(UT_kU^{-1})V^{-1}VUg : (k,l) \in \Lambda\} \\ &= \{VUM_l(VU)^{-1}VUT_k(VU)^{-1}VUg : (k,l) \in \Lambda\} \\ &= \{M_l^{VU} T_k^{VU} VUg : (k,l) \in \Lambda\}\end{aligned}$$

Clearly $\{M_l^{VU}T_k^{VU}VUg : (k,l) \in \Lambda\}$ is a Gabor type frame in $\mathcal{H}$ with generating function VUg and generating set Λ. This completes the proof. $\square$

Theorem 5.6. *A linear operator on a finite dimensional Hilbert space $\mathcal{H}$ of dimension N is a frame operator of a regular Gabor type frame in $\mathcal{H}$ if and only if it is positive, invertible and commutes with unitary operators of the form M_l^U and T_k^U on $\mathcal{H}$ for some unitary operator U from $l^2(\mathbb{Z}_N)$ onto $\mathcal{H}$ for all $(k,l) \in \Lambda_1 \times \Lambda_2$ where Λ_1 and Λ_2 are subgroups of $\mathbb{Z}_N$ with $|\Lambda_1 \times \Lambda_2| \geq N$.*

Proof. Let $\{M_l^U T_k^U h : (k,l) \in \Lambda_1 \times \Lambda_2\}$, where $h \in \mathcal{H}$, be a regular Gabor type frame in $\mathcal{H}$ and S be a the frame operator of the regular Gabor frame $\{M_l T_k g : (k,l) \in \Lambda_1 \times \Lambda_2\}$ in $l^2(\mathbb{Z}_N)$, with $Ug = h$. Then S is a positive and invertible linear operator on $l^2(\mathbb{Z}_N)$. Moreover S commutes with the translations $T_k, k \in \Lambda_1$ and modulations $M_l, l \in \Lambda_2$. Now $S' = USU^{-1}$ is the frame operator of the regular Gabor type frame $\{M_l^U T_k^U h : (k,l) \in \Lambda_1 \times \Lambda_2\}$, $\Lambda_1 \times \Lambda_2 \subseteq \mathbb{Z}_N \times \mathbb{Z}_N$. Obviously, S' is positive and invertible. Also, for any $l \in \Lambda_2$

$$\begin{aligned} S'M_l^U &= (USU^{-1})(UM_lU^{-1}) \\ &= USM_lU^{-1} \\ &= UM_lSU^{-1} \\ &= (UM_lU^{-1})(USU^{-1}) \\ &= M_l^U S' \end{aligned}$$

Similarly we have $S'T_k^U = T_k^U S'$, for any $k \in \lambda_1$. Thus S' commutes with all the operators M_l^U, $l \in \lambda_2$ and T_k^U, $k \in \lambda_1$.

Conversely suppose that S' be a positive and invertible linear operator on $\mathcal{H}$, such that it commutes with all the operators M_l^U, $l \in \lambda_2$ and T_k^U, $k \in \lambda_1$ for some unitary operator U from $l^2(\mathbb{Z}_N)$ onto $\mathcal{H}$ and for some subgroups Λ_1 and Λ_2 of $\mathbb{Z}_N$ with $|\Lambda_1 \times \Lambda_2| \geq N$. Then $S = U^{-1}S'U$ is a positive and invertible linear operator on $l^2(\mathbb{Z}_N)$ and hence by Proposition 3.4 it is the frame operator of a regular Gabor frame say $\{M_l T_k g : (k,l) \in \Lambda_1 \times \Lambda_2\}$, $g \in l^2(\mathbb{Z}_N)$. Now $S' = USU^{-1}$ and hence it is the frame operator of the regular Gabor type frame $\{M_l^U T_k^U h : (k,l) \in \Lambda_1 \times \Lambda_2\}$ in $\mathcal{H}$. $\square$

Analogous to our discussions on irregular Gabor frames in $\mathbb{C}^N$, irregular Gabor type frames can be considered in $\mathcal{H}$ and also, regular Gabor type frames can be constructed in $\mathcal{H}$ from irregular ones.

References

1. Z. Amiri, M. A. Dehghan, E. Rahimib & L. Soltania: Bessel subfusion sequences and subfusion frames. Iran. J. Math. Sci. Inform. **8** (2013), 31-38.
2. P.G. Cazassa & G. Kutyniok: Frames of subspaces, wavelets, frames and operator theory. American Mathematical Society, Contemporary Mathematics Publishers **345** (2004), 87-113.
3. P.G. Cazassa & G. Kutyniok: Finite Frames Theory and Applications. Applied and Numerical Harmonic Analysis Series, Birkhäuser, Boston, 2013.
4. O. Christensen: An Introduction to Frames and Riesz Bases. Second Edition, Birkhäuser, Boston, 2016.
5. O. Christensen and Y.C. Eldar: Oblique dual frames and shift-invariant spaces. Appl. Comput. Harmon. Anal. **17** (2004), 48-68.
6. I. Daubechies, A. Grossmann & Y. Meyer: Painless nonorthogonal expansions. J. Math. Phys. **27** (1986), 1271-1283.
7. R.J. Duffin & A.C. Schaeffler: A class of non-harmonic Fourier series. Trans. Amer. Math. Soc. **72** (1952), 341-366.
8. T.C. Easwaran Nambudiri & K. Parthasarathy: Generalised Weyl-Heisenberg frame operators. Bull. Sci. Math. **136** (2012), 44-53.
9. T.C. Easwaran Nambudiri & K. Parthasarathy: Characterization of Weyl-Heisenberg frame operators. Bull. Sci. Math. **137** (2013), 322-324.
10. D. Gabor: Theory of communication. Journal of Institution of Electrical Engineers **93** (1946), 429-457.
11. K. Gröchenig: Foundations of Time Frequency Analysis. Birkhäuser, Boston, 2001.
12. M. Janssen: Gabor representation of generalized functions. J. Math. Anal. Appl. **83** (1981), 377-394.
13. J. Laurence: Linear independence of Gabor systems in finite dimensional vector spaces. J. Fourier Anal. Appl. **11** (2005), 715-726.
14. S. Li & H. Ogawa: Pseudoframes for subspaces with applications. J. Fourier Anal. Appl. **10** (2004), 409-431.
15. R.D. Malikiosis: A note on Gabor frames in finite dimensions. Appl. Comp. Harmonic Anal. **38** (2015), 318-330.
16. G.E. Pfander: Gabor frames in finite dimensions. Birkhäuser, Boston, 2010.

[a]Department of Mathematics, Government College Kottayam, Nattakom, Kerala, India 686013,
Email address: `madhavangck@gmail.com`

[b]Department of Mathematics, Government Brennen College Thalassery, Kannur, Kerala, India 670106
Email address: `easwarantc@gmail.com`

[c]Research scholar, St. Thomas College Palai, Arunapuram, Kottayam, Kerala, India 686574
Email address: jineeshthomas@gmail.com