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A WEIERSTRASS SEMIGROUP AT A GENERALIZED FLEX ON A PLANE CURVE

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Abstract. We consider a Weierstrass semigroup at a generalized flex on a smooth plane curve. We find the candidates of a Weierstrass semigroup at a 2-flex of higher multiplicity on a smooth plane curve of degree $d \geq 5$, and give some examples to show the existence of them.

1. Introduction and Preliminaries

Let C be a smooth complex projective plane curve of degree $d \geq 4$. Let P be a point on C. We divide the lines on the plane into three types according to the intersection multiplicity at P:

(1) $I(C \cap \ell_0, P) = 0$;
(2) $I(C \cap \ell_1, P) = 1$;
(3) $I(C \cap \ell_2, P) \geq 2$;

where $I(C \cap \ell, P)$ means the intersection multiplicity of C and ℓ at P. We call ℓ_2 the tangent line to C at P and denote it by $T_P C$. If $I(C \cap T_P C, P) > 2$, then we call P the inflection point or a flex on C. One can generalize the notion of this concept by replacing the lines by curves of some given degree m. At each point P, for each natural number $m < d$, there exists a curve F_m of degree m which have the highest order of contact with C. We call such a curve F_m as an osculating curve of degree m at P to C. Note that an osculating curve F_m need not be irreducible. We are interested in the case that F_m is irreducible. The point P on C is called an *m-flex* if F_m is irreducible and $I(C \cap F_m, P) > \frac{m(m+3)}{2}$ where F_m is an osculating curve of

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degree m. Note that the number $\frac{m(m+3)}{2}$ is the dimension of the system of curves of degree m. Obviously, a 1-flex means a flex in our notation.

In this paper, we consider a Weierstrass semigroup at 2-flexes, which is also known as sextactic points.

The following are well known;

Lemma 1.1. *On a smooth plane curve of degree $d \geq 4$, the canonical series is cut out by the system of all curves of degree $d-3$.*

Lemma 1.2 ([5], Bertini's theorem)**.** *The generic element of a linear system is smooth away from the base locus of the system.*

Lemma 1.3 ([2], Bezout's theorem)**.** *Let C_m and C_n be plane curves of degree m and n, respectively. If they have no common component, then we have*

$$\sum_{P \in C_m \cap C_n} I(C_m \cap C_n) = mn.$$

Lemma 1.4 ([4], Namba's lemma)**.** *Let C_1, C_2 and C be three plane curves, and let P be a smooth point on C. If $I(C \cap C_1, P) \geq m$ and $I(C \cap C_2, P) \geq m$, then $I(C_1 \cap C_2, P) \geq m$.*

Corollary 1.5. *Let C be a plane curve and P a smooth point on C. Let C_1 and C_2 be plane curves defined by the polynomial h_1 and h_2, respectively. If $\min\{I(C \cap C_1, P), I(C \cap C_2, P)\} > (\deg h_1)(\deg h_2)$ and h_2 is irreducible, then h_1 is a multiple of h_2.*

Proof. By Namba's lemma, $I(C_1 \cap C_2, P) \geq \min\{I(C \cap C_1, P), I(C \cap C_2, P)\} > (\deg h_1)(\deg h_2)$. By Bezout's theorem, C_1 and C_2 have a common component. Since C_2 is irreducible, C_2 is the common component of them. □

For a point P on a smooth curve C of genus g, P is a Weierstrass point if the gap sequence $G_P = \{n \in \mathbb{N}_0 \mid$ there exists a canonical divisor K with $I(C \cap K, P) = n-1\}$ is different from $\{1, 2 \longrightarrow g\}$([1]). We call the sequence $\{I(C \cap K, P) \mid K$ is a canonical divisor of $C\}$ as an order sequence of canonical divisors at P. Thus P is a Weierstrass point if the order sequence of canonical divisors at P is not $\{0, 1 \longrightarrow g-1\}$. Recall that there are only finite number of Weierstrass points on C, which means that the order sequence of canonical divisors at a point is exactly $\{0, 1 \longrightarrow g-1\}$ except for a finite number of points, i.e., Weierstrass points. For a smooth plane curve C of degree d, by Lemma 1.1, the order sequence of canonical divisors at P is the set $\{I(C \cap f_{d-3}, P) \mid f_{d-3}$ is a polynomial of degree $d-3\}$.

2. A 2-flex which is a Weierstrass Point

Let C be a smooth plane curve of degree $d \geq 4$. For each natural number $1 \leq k \leq d-1$, the number $i_k := i_k(P)$ means the number $I(C \cap F_k, P)$ where F_k is the osculating curve of degree k at P to C.

Lemma 2.1. *Let C be a smooth plane curve of degree $d \geq 4$ and P a 2-flex on C. Then $i_1 = 2$ and $i_2 > 5$.*

Proof. Since P is a 2-flex, the osculating conic F_2 at P to C is irreducible and $i_2 = I(C \cap F_2, P) > 5$. Let ℓ_2 be the tangent line to C at P. Then $I(C \cap \ell_2, P) \geq 2$. If $I(C \cap \ell_2, P) \geq 3$, then $I(F_2 \cap \ell_2, P) \geq 3 > (\deg F_2)(\deg \ell_2)$, F_2 contains ℓ_2 as a component, which is a contradiction since F_2 is an irreducible conic. Thus $i_1 = I(C \cap \ell_2, P) = 2$. □

We are interested in a 2-flex point P which is a Weierstrass point.

If $d = 4$, then the genus of C is 3 and the lines cut out the canonical series whose order sequence at P is $\{0, 1, 2\}$. Thus P is not a Weierstrass point. Thus we consider only $d \geq 5$.

Remark 2.2. If $d \geq 5$ and $m = 2$ then F_2 is unique. If F_2 and G_2 are two different osculating conics to C at P then $\min\{I(C \cap F_2, P), I(C \cap G_2, P)\} > 5$ so $I(F_2 \cap G_2, P) > 5$ by Nambs's lemma. But $I(F_2 \cap G_2, P) \leq 4$ by Bertini's theorem, which is a contradiction.

Theorem 2.3. *Let C be a smooth plane curve of degree $d \geq 5$ and P a 2-flex on C. If $i_2(P) = I(C \cap F_2, P) \geq 2\lfloor \frac{d}{2} \rfloor + 2$ for an irreducible conic F_2, then P is a Weierstrass point of C.*

Proof. It suffices to show that there exists a polynomial f_{d-3} such that $I(C \cap f_{d-3}, P) \geq g = \frac{(d-1)(d-2)}{2}$.

When d is odd, we let $d = 2k+1$, $k \geq 2$. Since $d - 3 = 2(k-1)$, the degree of F_2^{k-1} is a polynomial of degree $d-3$. We have

$$I(C \cap F_2^{k-1}, P) \geq (k-1)(2\lfloor \frac{d}{2} \rfloor + 2) = \frac{d-3}{2}(d+1) \geq g.$$

When d is even, we let $d = 2k$, $k \geq 3$. Since $d-3 = 2(k-2)+1$, the degree of $F_2^{k-2}\ell_2$ is a polynomial of degree $d-3$, where ℓ_2 is the tangent line to C at P. We have

$$I(C \cap F_2^{k-2}\ell_2, P) \geq (k-2)(2\lfloor \frac{d}{2} \rfloor + 2) + 2 = (\frac{d}{2} - 2)(d+2) + 2 \geq g,$$

since $d \geq 6$ for even d.

Thus P is a Weierstrass point in both cases. □

In next theorem, we give the order sequence at P when $i_2(P)$ is a high value, i.e., $i_2(P) \geq 2(d-3)+1$.

Theorem 2.4. *Let C be a smooth plane curve of degree $d \geq 5$ and P a 2-flex point on C. If $i_2(P) \geq 2(d-3)+1$, then the order sequences at P is*

$$\bigcup_{0 \leq \alpha \leq \frac{d-3}{2}} \{\alpha i_2 \longrightarrow \alpha i_2 + 2(d - 2\alpha - 3)\}.$$

Moreover, such a curve C and a point $P \in C$ exist, indeed the following curve and the point P satisfy the conditions.

$C_d :\ \lambda_1(y-x^2) + \lambda_2(y-x^2)(x^{d-2} + y^{d-2}) + \lambda_3 y^d + \lambda_4 x^{i_2 - 2\lfloor \frac{i_2}{2} \rfloor} y^{\lfloor \frac{i_2}{2} \rfloor}$ *and* $P = (0,0)$.

Proof. Note that the canonical series is cut out by the curves of degree $d-3$. First, we obtain the orders at P using polynomials of the form $F_2^{\alpha}\ell_0^{\beta_0}\ell_1^{\beta_1}\ell_2^{\beta_2}$ with $2\alpha + \beta_0 + \beta_1 + \beta_2 = d-3$. Here ℓ_2 is the tangent line at P to C, ℓ_0 is any line not passing through P, and ℓ_1 is any line, distinct from ℓ_2, passing through P. We have $I(F_2^{\alpha}\ell_0^{\beta_0}\ell_1^{\beta_1}\ell_2^{\beta_2} \cap C, P) = \alpha i_2 + \beta_1 + 2\beta_2$. For fixed α, $0 \leq \alpha \leq \frac{d-3}{2}$, we obtain

$$\{I(F_2^{\alpha}\ell_0^{\beta_0}\ell_1^{\beta_1}\ell_2^{\beta_2} \cap C, P) \mid \beta_0 + \beta_1 + \beta_2 = d-3-2\alpha\} = \{\alpha i_2 \longrightarrow \alpha i_2 + 2(d-3-2\alpha)\}.$$

Since $i_2(P) \geq 2(d-3)+1$, we can check that

$$\bigcup_{0 \leq \alpha \leq \frac{d-3}{2}} \{\alpha i_2 \longrightarrow \alpha i_2 + 2(d-3-2\alpha)\}$$

is the disjoint union. Hence the cardinality of it is

$$\sum_{0 \leq \alpha \leq \frac{d-3}{2}} (2(d-3-2\alpha)+1) = \frac{(d-1)(d-2)}{2} = g$$

where g is the genus of C. Thus it is exactly the order sequence of the point P.

Since the order sequence is completely determined by the values $i_1 = 2$ and $i_2 \geq 2(d-3)+1$, it suffices to find a smooth curve of degree d admitting such values. In fact, by the Bertini's theorem, C_d is smooth for general nonzero λ_i's. If we let

$F_2 = y - x^2$, then F_2 is the osculating conic and $I(F_2 \cap C_d, P) = i_2$. Thus C_d is a desired curve. □

3. At a 2-flex of Order of Contact $2(d-3)$

In the Theorem 2.4, if $i_2(P) \leq 2(d-3)$, then we check that $\bigcup_{0 \leq \alpha \leq \frac{d-3}{2}} \{\alpha i_2 \longrightarrow \alpha i_2 + 2(d - 2\alpha - 3)\}$ is not a disjoint union, by counting the elements of sets. So the number of orders at P appeared in the union is less than the genus g of C. Then we must find more orders at P not appeared in the union.

In this section, we consider the case $i_2(P) = 2(d-3)$. In this case, the cardinality of the union is exactly $g-1$ because the last element $2(d-3)$ of the first set and the first element i_2 in the second set are coincide. Thus we must find one more order at P.

For $d = 5$, $2(d-3) = 4$ can not be i_2 since $i_2 > 5$.

For $d = 6$, we have $i_2 = 2(d-3) = 6$. From [3], we obtain the order sequences $\{0 \longrightarrow 8\} \cup \{r\}$ with $r \in \{10 \longrightarrow 16\} \cup \{18\}$. Here $r = i_3$. Moreover, they proved that i_3 can not be 17.

So we deal with the cases for $d = 7, 8$ and 9 in this section.

3.1. On a curve of degree 7 Let $d = 7$ and $i_2 = 2(d-3) = 8$. In this case the orders determined by lines and the power of osculating conic are

$$\{0 \longrightarrow 8\} \cup \{8 \longrightarrow 12\} \cup \{16\} = \{0 \longrightarrow 12\} \cup \{16\}$$

and its cardinality is $g - 1 = 14$. Hence we need to find one more order.

Lemma 3.1. *We have $i_1 = 2$, $i_3 = 10$, and $i_4 \geq 16$.*

Proof. By Lemma 2.1, $i_1 = 2$.

Let ℓ_2 be the tangent line to C at P. Since $I(C \cap F_2\ell_2, P) = 10$, we have $i_3 \geq 10$. Suppose that $i_3 > 10$ and let f_3 be a cubic such that $I(C \cap f_3, P) = i_3 > 10$. By Corollary 1.5, $f_3 = F_2\ell$ for some line ℓ. Then $I(C \cap f_3, P) \leq 10$ which is a contradiction. Thus $i_3 = 10$.

Since $I(C \cap F_2^2, P) = 16$, we have $i_4 \geq 16$. □

Remark 3.2. Let f_4 be a quartic such that $i_4 = I(C \cap f_4, P)$. By Namba's lemma, we have $I(f_4 \cap F_2, P) = 8 = (\deg f_4)(\deg F_2)$. Here we can not apply Corollary 1.5, i.e., we can not say that F_2 is a component of f_4.

Lemma 3.3. *The order sequence at* P *is* $\{0 \longrightarrow 12\} \cup \{16\} \cup \{r\}$ *for some* $r \in \{13, 14, 15\} \cup \{17 \longrightarrow 28\}$. *Moreover, such* r *is attained by an irreducible polynomial of degree 4.*

Proof. Since the degree of canonical divisor is $2g - 2 = 28$, any order of canonical divisor at P is in the set $\{0 \longrightarrow 28\}$. Thus one more order r is an element in the set $\{13, 14, 15\} \cup \{17 \longrightarrow 28\}$. Let the divisor rP be cut out by a curve f. If $\deg f \leq 3$, then $I(f \cap C, P) = r \geq 10$. Namba's lemma implies that $I(f \cap F_2, P) \geq 8$, which is bigger than $(\deg f)(\deg F_2) \leq 6$. By Bezout's theorem, and that f is a multiple of F_2, say $f = F_2 h$ with $\deg h \leq 1$. Then $I(h \cap C, P) = r - 8 \geq 5$, which is a contradiction. Thus the degree of f is 4 since any canonical divisor is cut out by a curve of degree 4. We can also prove that f is irreducible. Indeed, if f is reducible, then f is factored into polynomials of lower degree than 4. If f is factored into four lines, then $0 \leq I(C \cap f, P) \leq 8$. If f is factored into one line and a cubic, then $I(C \cap f, P) \leq 12$. If f is factored into two lines and a conic, then $I(C \cap f, P) \leq 12$. If f is factored into two conics, then $I(C \cap f, P) = 16$ for the case $f = F_2^2$, and $I(C \cap f, P) \leq 12$ for the case $f = F_2 f_2$, where f_2 is a conic different from F_2. Note that $I(C \cap f_2, P) \leq 4$ because $I(F_2 \cap f_2, P) \leq 4 = (\deg F_2)(\deg f_2)$ by Bezout's theorem. □

Lemma 3.4. *There is no smooth plane curve of degree 7 with a point at which the order sequence is* $\{0 \longrightarrow 12\} \cup \{16\} \cup \{r\}$ *with* $27 \leq r \leq 28$.

Proof. Suppose that such a curve exists. Then the Weierstrass gap sequence $G(P) = \{1 \longrightarrow 13\} \cup \{17\} \cup \{r+1\}$ and $H(P) = \{0\} \cup \{14, 15, 16\} \cup \{18 \longrightarrow\} - \{r+1\}$. Since 14 and 15 are elements in $H(P)$ and $H(P)$ is a semigroup, 28 and 29 are elements of $H(P)$. Thus $r + 1 \neq 28, 29$, which contradicts the assumption. □

Lemma 3.5. *There exists a smooth plane curve of degree 7 with a point at which the order sequence is* $\{0 \longrightarrow 12\} \cup \{16\} \cup \{r\}$ *with* $24 \leq r \leq 26$.

Proof. Let $P = (0, 0)$ and

$$\begin{aligned} f_2 &= y - x^2 \\ f_4 &= \lambda_1 f_2 + \lambda_2 x^2 f_2 + \lambda_3 y^4 \\ f_7 &= \mu_1 f_4 + \mu_2 f_4(x^3 + y^3) + \mu_3 (f_2)^3 x^{r-24-2\lfloor \frac{r-24}{2} \rfloor} y^{\lfloor \frac{r-24}{2} \rfloor}, \end{aligned}$$

and let C_7 be the curve with the equation f_7.

Then, for general nonzero λ_i's and μ_i's we can check the following:

(1) C_7 is a smooth curve of degree 7, since the base locus of the system is only one point $\{(0,0,1)\}$ at which a generic member is smooth. By Bertini's theorem, C_7 is smooth.
(2) $I(f_2 \cap C_7, P) = 8$, since $I(f_2 \cap C_7, P) = I(f_2 \cap f_4, P) = I(f_2 \cap y^4, P)$.
(3) $I(T_P C_7 \cap C_7, P) = 2$, since $I(T_P C \cap C_7, P) = I(y \cap C_7, P)$.
(4) $I(f_4 \cap C_7, P) = r$, since

$$\begin{aligned} I(f_4 \cap C_7, P) &= I(f_4 \cap (f_2)^3 x^{r-24-2\lfloor\frac{r-24}{2}\rfloor} y^{\lfloor\frac{r-24}{2}\rfloor}) \\ = 3I(f_4 \cap f_2, P) &+ (r-24-2\lfloor\frac{r-24}{2}\rfloor) I(f_4 \cap x, P) + (\lfloor\frac{r-24}{2}\rfloor) I(f_4 \cap y, P) \\ = 12 I(y \cap f_2, P) &+ (r-24-2\lfloor\frac{r-24}{2}\rfloor) + 2(\lfloor\frac{r-24}{2}\rfloor) \end{aligned}$$

Thus the C_7 is a desired curve. □

Lemma 3.6. *There exists a smooth plane curve of degree 7 with a point at which the order sequence is* $\{0 \longrightarrow 12\} \cup \{16\} \cup \{r\}$ *with* $17 \leq r \leq 22$.

Proof. Let $P = (0,0)$ and

$$\begin{aligned} f_2 &= y - x^2 \\ f_4 &= \lambda_1 f_2 + \lambda_2 x^2 f_2 + \lambda_3 y^4 \\ f_7 &= \mu_1 f_4 + \mu_2 f_4 (x^3 + y^3) + \mu_3 (f_2)^2 x^{r-16-2\lfloor\frac{r-16}{2}\rfloor} y^{\lfloor\frac{r-16}{2}\rfloor}, \end{aligned}$$

and let C_7 be the curve with the equation f_7.

Then, for general nonzero λ_i's and μ_i's we can check the following:
(1) C_7 is a smooth curve of degree 7.
(2) $I(f_2 \cap C_7, P) = 8$.
(3) $I(T_P C_7 \cap C_7, P) = 2$.
(4) $I(f_4 \cap C_7, P) = r$.

Thus the C_7 is a desired curve. □

Lemma 3.7. *There exists a smooth plane curve of degree 7 with a point at which the order sequence is* $\{0 \longrightarrow 12\} \cup \{16\} \cup \{r\}$ *with* $13 \leq r \leq 15$.

Proof. Let $P = (0,0)$ and

$$\begin{aligned} f_2 &= y - x^2 \\ f_4 &= \lambda_1 f_2 + \lambda_2 x^2 f_2 + \lambda_3 y^4 \\ f_7 &= \mu_1 f_4 + \mu_2 f_4 (x^3 + y^3) + \mu_3 (f_2) x^{r-8-2\lfloor\frac{r-8}{2}\rfloor} y^{\lfloor\frac{r-8}{2}\rfloor}, \end{aligned}$$

and let C_7 be the curve with the equation f_7.

Then, for general nonzero λ_i's and μ_i's we can check the following:

(1) C_7 is a smooth curve of degree 7.
(2) $I(f_2 \cap C_7, P) = 8$.
(3) $I(T_P C_7 \cap C_7, P) = 2$.
(4) $I(f_4 \cap C_7, P) = r$.

Thus the C_7 is a desired curve. □

Theorem 3.8. *Let P be a 2-flex of order of contact 8 on a smooth plane curve of degree 7. Then the order sequence at P is one of $\{0 \longrightarrow 12\} \cup \{16\} \cup \{r\}$ for $r \in \{13 \longrightarrow 15\} \cup \{17 \longrightarrow 26\}$.*

Also there exists a smooth plane curve of degree 7 with a 2-flex point P at which the order sequence is $\{0 \longrightarrow 12\} \cup \{16\} \cup \{r\}$ for $r \in \{13 \longrightarrow 15\} \cup \{17 \longrightarrow 22\} \cup \{24 \longrightarrow 26\}$.

Remark 3.9. In the set $\{13, 14, 15\} \cup \{17 \longrightarrow 28\}$ of all candidates for r, we proved or disproved the existence of a smooth curve of degree 7 corresponding to each integer except for the number 23.

3.2. On a curve of degree 8 In this case the orders determined by lines and the power of osculating conic are

$$\{0 \longrightarrow 10\} \cup \{10 \longrightarrow 16\} \cup \{20 \longrightarrow 22\} = \{0 \longrightarrow 16\} \cup \{20 \longrightarrow 22\}$$

and its cardinality is $g - 1 = 20$. Hence we need to find one more order.

Using Bezout's theorem and Namba's Lemma, we have $i_1 = 2$, $i_2 = 10$, $i_3 = 12$, $i_4 = 20$, and $i_5 \geq 22$.

Since the degree of canonical divisor is $2g - 2 = 40$, any order of canonical divisor at P is in the set $\{0 \longrightarrow 40\}$. Thus one more order r is an element in the set $\{17, 18, 19\} \cup \{23 \longrightarrow 40\}$. Let the divisor rP be cut out by a curve f. If $\deg f \leq 4$, then $I(f \cap C, P) = r \geq 17$. Namba's lemma implies that $I(f \cap F_2, P) \geq 10$, which is bigger than $(\deg f)(\deg F_2) \leq 8$. By Bezout's theorem, and that f is a multiple of F_2, say $f = F_2 h$ with $\deg h \leq 2$. Then $I(h \cap C, P) = r - 10 \geq 7$ and hence $h = F_2$ and $r = 20$ by Bezout's theorem, which is a contradiction to the choice of r. Thus the degree of f is 5 since any canonical divisor is cut out by a curve of degree 5. We can also prove that f is irreducible in a similar way.

Lemma 3.10. *There exists a smooth plane curve of degree 8 with a point at which the order sequence is* $\{0 \longrightarrow 16\} \cup \{20 \longrightarrow 22\} \cup \{40\}$, *i.e.*, $r = 40$.

Proof. Let $P = (0, 0)$ and

$$\begin{aligned} f_2 &= y - x^2 \\ f_5 &= \lambda_1(y - x^2) + \lambda_2 x^3(y - x^2) + \lambda_3 y^5 \\ f_8 &= \mu_1 f_5 + \mu_2 f_5(x^3 + y^3) + \mu_3(f_2)^4, \end{aligned}$$

and let C_8 be the curve defined by the equation f_8.

Then, for general nonzero λ_i's and μ_i's we can check the following:

(1) C_8 is a smooth curve of degree 8.
(2) $I(f_2 \cap C_8, P) = 10$.
(3) $I(T_P C_8 \cap C_8, P) = 2$.
(4) $I(f_5 \cap C_8, P) = 40$.

Thus the C_8 is a desired curve. □

Lemma 3.11. *There is no smooth plane curve of degree 8 with a point at which the order sequence is* $\{0 \longrightarrow 16\} \cup \{20 \longrightarrow 22\} \cup \{r\}$ *for* $35 \leq r \leq 39$.

Proof. Suppose that such a curve exists. Then the Weierstrass gap sequence $G(P) = \{1 \longrightarrow 17\} \cup \{21 \longrightarrow 23\} \cup \{r+1\}$ and $H(P) = \{0\} \cup \{18, 19, 20\} \cup \{24 \longrightarrow\} - \{r+1\}$. Since 18, 19 and 20 are elements in $H(P)$ and $H(P)$ is a semigroup, every integers from 36 to 40 are elements of $H(P)$. However, since the number $r + 1$ belongs to this set, it is a contradiction. □

Lemma 3.12. *There exists a smooth plane curve of degree 8 with a point at which the order sequence is* $\{0 \longrightarrow 16\} \cup \{20 \longrightarrow 22\} \cup \{r\}$ *for* $30 \leq r \leq 34$.

Proof. Let $P = (0, 0)$ and

$$\begin{aligned} f_2 &= y - x^2 \\ f_5 &= \lambda_1(y - x^2) + \lambda_2 x^3(y - x^2) + \lambda_3 y^5 \\ f_8 &= \mu_1 f_5 + \mu_2 f_5(x^3 + y^3) + \mu_3(f_2)^3 x^{r-30-2\lfloor \frac{r-30}{2} \rfloor} y^{\lfloor \frac{r-30}{2} \rfloor}, \end{aligned}$$

and let C_8 be the curve with the equation f_8.

Then, for general nonzero λ_i's and μ_i's we can check the following:

(1) C_8 is a smooth curve of degree 8.
(2) $I(f_2 \cap C_8, P) = 10$.

(3) $I(T_PC_8 \cap C_8, P) = 2$.
(4) $I(f_5 \cap C_8, P) = r$.

Thus the C_8 is a desired curve. □

Lemma 3.13. *There exists a smooth plane curve of degree 8 with a point at which the order sequence is* $\{0 \longrightarrow 16\} \cup \{20 \longrightarrow 22\} \cup \{r\}$ *for* $23 \leq r \leq 28$.

Proof. Let $P = (0,0)$ and

$$\begin{aligned} f_2 &= y - x^2 \\ f_5 &= \lambda_1(y-x^2) + \lambda_2 x^3(y-x^2) + \lambda_3 y^5 \\ f_8 &= \mu_1 f_5 + \mu_2 f_5(x^3+y^3) + \mu_3 (f_2)^2 x^{r-20-2\lfloor\frac{r-20}{2}\rfloor} y^{\lfloor\frac{r-20}{2}\rfloor}, \end{aligned}$$

and let C_8 be the curve with the equation f_8.

Then, for general nonzero λ_i's and μ_i's we can check the following:

(1) C_8 is a smooth curve of degree 8.
(2) $I(f_2 \cap C_8, P) = 10$.
(3) $I(T_PC_8 \cap C_8, P) = 2$.
(4) $I(f_5 \cap C_8, P) = r$.

Thus the C_8 is a desired curve. □

Lemma 3.14. *There exists a smooth plane curve of degree 8 with a point at which the order sequence is* $\{0 \longrightarrow 16\} \cup \{20 \longrightarrow 22\} \cup \{r\}$ *for* $17 \leq r \leq 19$.

Proof. Let $P = (0,0)$ and

$$\begin{aligned} f_2 &= y - x^2 \\ f_5 &= \lambda_1(y-x^2) + \lambda_2 x^3(y-x^2) + \lambda_3 y^5 \\ f_8 &= \mu_1 f_5 + \mu_2 f_5(x^3+y^3) + \mu_3 f_2 (x^{r-10-2\lfloor\frac{r-10}{2}\rfloor} y^{\lfloor\frac{r-10}{2}\rfloor}), \end{aligned}$$

and let C_8 be the curve with the equation f_8.

Then, for general nonzero λ_i's and μ_i's we can check the following:

(1) C_8 is a smooth curve of degree 8.
(2) $I(f_2 \cap C_8, P) = 10$.
(3) $I(T_PC_8 \cap C_8, P) = 2$.
(4) $I(f_5 \cap C_8, P) = r$.

Thus the C_8 is a desired curve. □

Theorem 3.15. *Let P be a 2-flex of order of contact 10 on a smooth plane curve of degree 8. Then the order sequence at P is one of* $\{0 \longrightarrow 16\} \cup \{20 \longrightarrow 22\} \cup \{r\}$ *for* $r \in \{17 \longrightarrow 19\} \cup \{23 \longrightarrow 34\} \cup \{40\}$.

Also there exists a smooth plane curve of degree 8 with a 2-flex point P at which the order sequence is $\{0 \longrightarrow 16\} \cup \{20 \longrightarrow 22\} \cup \{r\}$ *for* $r \in \{17 \longrightarrow 19\} \cup \{23 \longrightarrow 28\} \cup \{30 \longrightarrow 34\} \cup \{40\}$.

Remark 3.16. In the set $\{17, 18, 19\} \cup \{23 \longrightarrow 40\}$ of all candidates for r, we proved or disproved the existence of a smooth curve of degree 8 corresponding to each integer except for the number 29.

3.3. On a curve of degree 9 In this case the orders determined by lines and the power of osculating conic are

$$\begin{aligned}&\{0 \longrightarrow 12\} \cup \{12 \longrightarrow 20\} \cup \{24 \longrightarrow 28\} \cup \{36\}\\ &= \{0 \longrightarrow 20\} \cup \{24 \longrightarrow 28\} \cup \{36\}\end{aligned}$$

and its cardinality is $g - 1 = 27$. Hence we need to find one more order.

Using Bezout's theorem and Namba's Lemma, we have $i_1 = 2$, $i_2 = 12$, $i_3 = 14$, $i_4 = 24$, $i_5 = 26$, and $i_6 \geq 36$.

One more order r is an element in the set

$$\{21, 22, 23\} \cup \{29 \longrightarrow 35\} \cup \{37 \longrightarrow 54\}.$$

Let the divisor rP be cut out by a curve f. If $\deg f \leq 5$, then $I(f \cap C, P) = r \geq 21$. Namba's lemma implies that $I(f \cap F_2, P) \geq 12$, and that f is a multiple of F_2 by Bezout's theorem. Let $f = hF_2$, $\deg h \leq 3$. Then we have $I(h \cap C, P) = r - 12 \geq 9$. By Namba's theorem again, we conclude that h is multiple of F_2. Thus $f = F_2^2$ or $f = F_2^2\ell$ where ℓ is a line. Hence $I(f \cap C, P) = 24$, 25 or 26. Then this is a contradiction. Thus the degree of f is 6. We can also prove that f is irreducible in a similar way.

Lemma 3.17. *There is no smooth plane curve of degree 9 with a point at which the order sequence is* $\{0 \longrightarrow 20\} \cup \{24 \longrightarrow 28\} \cup \{36\} \cup \{r\}$ *for* $r \in \{43 \longrightarrow 47\} \cup \{51 \longrightarrow 54\}$.

Proof. Suppose that such a curve exists. Then the Weierstrass gap sequence $G(P) = \{1 \longrightarrow 21\} \cup \{25 \longrightarrow 29\} \cup \{37\} \cup \{r + 1\}$ and $H(P) = \{0\} \cup \{22, 23, 24\} \cup \{30 \longrightarrow 36\} \cup \{38 \longrightarrow\} - \{r + 1\}$. Since $\{22, 23, 24, 30 \longrightarrow 36\}$ is a subset of $H(P)$ and $H(P)$

is a semigroup, $\{44 \longrightarrow 48, 52 \longrightarrow 55\}$ is a subset of $H(P)$. However, since the gap $r+1$ belongs to this set, it is a contradiction. $\square$

Lemma 3.18. *There exists a smooth plane curve of degree 9 with a point at which the order sequence is* $\{0 \longrightarrow 20\} \cup \{24 \longrightarrow 28\} \cup \{36\} \cup \{r\}$ *for* $r \in \{21 \longrightarrow 23\} \cup \{29 \longrightarrow 34\} \cup \{37 \longrightarrow 42\} \cup \{48 \longrightarrow 50\}$.

Proof. Let $P = (0,0)$ and

$$\begin{aligned}
f_2 &= y - x^2 \\
f_6 &= \lambda_1(y-x^2) + \lambda_2 x^4(y-x^2) + \lambda_3 y^6 \\
f_9 &= \mu_1 f_6 + \mu_2 f_6(x^3+y^3) + \mu_3(f_2)^4 \\
f_{9,r} &= \begin{cases} \nu_1 f_9 + \nu_2(f_2)^4 x^{r-48-2\lfloor\frac{r-48}{2}\rfloor} y^{\lfloor\frac{r-48}{2}\rfloor}, & \text{if } 48 \le r \le 50 \\ \nu_1 f_9 + \nu_2(f_2)^3 x^{r-36-2\lfloor\frac{r-36}{2}\rfloor} y^{\lfloor\frac{r-36}{2}\rfloor}, & \text{if } 37 \le r \le 42 \\ \nu_1 f_9 + \nu_2(f_2)^2 x^{r-24-2\lfloor\frac{r-24}{2}\rfloor} y^{\lfloor\frac{r-24}{2}\rfloor}, & \text{if } 29 \le r \le 34 \\ \nu_1 f_9 + \nu_2(f_2) x^{r-12-2\lfloor\frac{r-12}{2}\rfloor} y^{\lfloor\frac{r-12}{2}\rfloor}, & \text{if } 21 \le r \le 23 \end{cases}
\end{aligned}$$

and let $C_{9,r}$ be the curve with the equation $f_{9,r}$.

Then, for general nonzero λ_i's, μ_i's and ν_i's we can check the following:

(1) $C_{9,r}$ is a smooth curve of degree 9.
(2) $I(f_2 \cap C_{9,r}, P) = 12$.
(3) $I(T_P C_{9,r} \cap C_{9,r}, P) = 2$.
(4) $I(f_5 \cap C_{9,r}, P) = r$.

Thus the $C_{9,r}$ is a desired curve. $\square$

Theorem 3.19. *Let P be a 2-flex of order of contact 12 on a smooth plane curve of degree 10. Then the order sequence at P is one of* $\{0 \longrightarrow 20\} \cup \{24 \longrightarrow 28\} \cup \{36\} \cup \{r\}$ *for* $r \in \{21 \longrightarrow 23\} \cup \{29 \longrightarrow 35\} \cup \{37 \longrightarrow 42\} \cup \{48 \longrightarrow 50\}$.

Also there exists a smooth plane curve of degree 10 with a 2-flex point P at which the order sequence is $\{0 \longrightarrow 20\} \cup \{24 \longrightarrow 28\} \cup \{36\} \cup \{r\}$ *for* $r \in \{21 \longrightarrow 23\} \cup \{29 \longrightarrow 34\} \cup \{37 \longrightarrow 42\} \cup \{48 \longrightarrow 50\}$.

Remark 3.20. In the set $\{21, 22, 23\} \cup \{29 \longrightarrow 35\} \cup \{37 \longrightarrow 54\}$ of all candidates for r, we proved or disproved the existence of a smooth curve of degree 9 corresponding to each integer except for the number 35.

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