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ADMISSIBLE BALANCED PAIRS OVER FORMAL TRIANGULAR MATRIX RINGS

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Abstract. Suppose that $T = \left(\begin{smallmatrix} A & 0 \\ U & B \end{smallmatrix}\right)$ is a formal triangular matrix ring, where A and B are rings and U is a (B, A)-bimodule. Let $\mathfrak{C}_1$ and $\mathfrak{C}_2$ be two classes of left A-modules, $\mathfrak{D}_1$ and $\mathfrak{D}_2$ be two classes of left B-modules. We prove that $(\mathfrak{C}_1, \mathfrak{C}_2)$ and $(\mathfrak{D}_1, \mathfrak{D}_2)$ are admissible balanced pairs if and only if $(\mathbf{p}(\mathfrak{C}_1, \mathfrak{D}_1), \mathbf{h}(\mathfrak{C}_2, \mathfrak{D}_2))$ is an admissible balanced pair in T-Mod. Furthermore, we describe when $(\mathfrak{P}_{\mathfrak{D}_1}^{\mathfrak{C}_1}, \mathfrak{I}_{\mathfrak{D}_2}^{\mathfrak{C}_2})$ is an admissible balanced pair in T-Mod. As a consequence, we characterize when T is a left virtually Gorenstein ring.

1. Introduction

Let A and B be rings and U be a (B, A)-bimodule. The ring $T = \left(\begin{smallmatrix} A & 0 \\ U & B \end{smallmatrix}\right)$ is known as a *formal triangular matrix ring* or *generalized triangular matrix ring* with usual matrix addition and multiplication. Formal triangular matrix rings play an important role in ring theory and the representation theory of algebras. This kind of rings are often used to construct examples and counterexamples. As a consequence of the classical results by Green [11], the module category over the formal triangular matrix ring T can be, via some functors, reconstructed from the categories of modules over A and B. Using these functors, one can describe classes of modules over the formal triangular matrix ring T from the corresponding classes of modules over A and B. So the properties of formal triangular matrix rings and modules over them make the theory of rings and modules more abundant and concrete, and have deserved more and more interest (see [1,8], [11–16]).

On the other hand, the concepts of preenvelopes and precovers (approximations) of modules were introduced independently in the early eighties of the 20th century by Enochs [5] and Auslander-Smalø [2]. Let $\mathfrak{C}$ be a class of left R-modules and M be a left R-module. Recall that a homomorphism $\phi : M \to C$

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with $C \in \mathfrak{C}$ is a $\mathfrak{C}$-*preenvelope* of M [5] if for any homomorphism $f\colon M \to C'$ with $C' \in \mathfrak{C}$, there is a homomorphism $g : C \to C'$ such that $g\phi = f$. Dually we have the definition of a $\mathfrak{C}$-precover. The class $\mathfrak{C}$ is called a (resp. *epic*) *precovering class* of the category R-Mod of left R-modules if every left R-module has a (resp. epic) $\mathfrak{C}$-precover. The class $\mathfrak{C}$ is called a (resp. *monic*) *preenveloping class* of the category R-Mod if every left R-module has a (resp. monic) $\mathfrak{C}$-preenvelope. Using precovering classes and preenveloping classes, Enochs and Jenda [6] introduced the notion of a balanced functor, which plays an important role in relative homological algebra. In [4], Chen called a pair $(\mathfrak{F}, \mathfrak{L})$ of classes of left R-modules a *balanced pair* if the functor $\mathrm{Hom}_R(-,-)$ is right balanced on R-Mod $\times$ R-Mod by $\mathfrak{F} \times \mathfrak{L}$ in the sense of [6]. We use ${}_R\mathcal{P}$ and ${}_R\mathcal{I}$ to denote the classes of projective left R-mdules and injective left R-mdules, respectively. It is well known that $({}_R\mathcal{P}, {}_R\mathcal{I})$ is a balanced pair, which is called the *classical balanced pair*. In general, the concept of a balanced pair inherits many similar properties from the classical one (see [4]) and so has gained attention in recent years in the context of relative homological algebra.

In this paper, we will investigate how to construct balanced pairs over a formal triangular matrix ring $T = \left(\begin{smallmatrix} A & 0 \\ U & B \end{smallmatrix}\right)$. Let $\mathfrak{C}_1$ and $\mathfrak{C}_2$ be two classes of left A-modules, $\mathfrak{D}_1$ and $\mathfrak{D}_2$ be two classes of left B-modules. We prove that $(\mathfrak{C}_1, \mathfrak{C}_2)$ and $(\mathfrak{D}_1, \mathfrak{D}_2)$ are admissible balanced pairs if and only if $(\mathbf{p}(\mathfrak{C}_1, \mathfrak{D}_1), \mathbf{h}(\mathfrak{C}_2, \mathfrak{D}_2))$ is an admissible balanced pair in T-Mod (see Theorem 2.2). Moreover, we describe when $(\mathfrak{P}^{\mathfrak{C}_1}_{\mathfrak{D}_1}, \mathfrak{I}^{\mathfrak{C}_2}_{\mathfrak{D}_2})$ is an admissible balanced pair in T-Mod (see Theorem 2.5). As an application, we characterize when T is a left virtually Gorenstein ring (see Theorem 2.8).

Throughout this paper, all rings are nonzero associative rings with identity and all modules are unitary. For a ring R, we write R-Mod (resp. Mod-R) for the category of left (resp. right) R-modules. ${}_RM$ (resp. M_R) denotes a left (resp. right) R-module. All classes of modules are assumed to be closed under isomorphisms and contain 0.

Next let us recall some basic facts about formal triangular matrix rings. $T = \left(\begin{smallmatrix} A & 0 \\ U & B \end{smallmatrix}\right)$ always means a formal triangular matrix ring, where A and B are rings and U is a (B, A)-bimodule. By [11, Theorem 1.5], the category T-Mod of left T-modules is equivalent to the category Ω whose objects are triples $M = \left(\begin{smallmatrix} M_1 \\ M_2 \end{smallmatrix}\right)_{\varphi^M}$, where $M_1 \in A$-Mod, $M_2 \in B$-Mod and $\varphi^M : U \otimes_A M_1 \to M_2$ is a B-morphism, and whose morphisms from $\left(\begin{smallmatrix} M_1 \\ M_2 \end{smallmatrix}\right)_{\varphi^M}$ to $\left(\begin{smallmatrix} N_1 \\ N_2 \end{smallmatrix}\right)_{\varphi^N}$ are pairs $\left(\begin{smallmatrix} f_1 \\ f_2 \end{smallmatrix}\right)$ such that $f_1 \in \mathrm{Hom}_A(M_1, N_1)$, $f_2 \in \mathrm{Hom}_B(M_2, N_2)$ satisfying that the following diagram is commutative:

$$\begin{array}{ccc} U \otimes_A M_1 & \xrightarrow{1 \otimes f_1} & U \otimes_A N_1 \\ {\scriptstyle \varphi^M}\downarrow & & \downarrow{\scriptstyle \varphi^N} \\ M_2 & \xrightarrow{f_2} & N_2 \end{array}$$

Given a triple $M = \binom{M_1}{M_2}_{\varphi^M}$ in Ω, we will denote by $\widetilde{\varphi^M}$ the A-morphism from M_1 to $\mathrm{Hom}_B(U, M_2)$ given by $\widetilde{\varphi^M}(x)(u) = \varphi^M(u \otimes x)$ for each $u \in U$ and $x \in M_1$. In the rest of the paper we will identify T-Mod with this category Ω and, whenever there is no possible confusion, we will omit the morphism φ^M.

Note that a sequence $0 \to \binom{M_1'}{M_2'}_{\varphi^{M'}} \to \binom{M_1}{M_2}_{\varphi^M} \to \binom{M_1''}{M_2''}_{\varphi^{M''}} \to 0$ of left T-modules is exact if and only if the two sequences $0 \to M_1' \to M_1 \to M_1'' \to 0$ and $0 \to M_2' \to M_2 \to M_2'' \to 0$ are exact.

Recall that the *product category* A-Mod $\times B$-Mod is defined as follows: An object of A-Mod $\times$ B-Mod is a pair (M, N) with $M \in A$-Mod and $N \in B$-Mod, a morphism from (M, N) to (M', N') is a pair (f, g) with $f \in \mathrm{Hom}_A(M, M')$ and $g \in \mathrm{Hom}_B(N, N')$. There are some functors between the category T-Mod and the product category A-Mod $\times$ B-Mod as follows:

(1) $\mathbf{p}$: A-Mod $\times$ B-Mod $\to$ T-Mod is defined as follows: For each object (M_1, M_2) of A-Mod $\times$ B-Mod, let $\mathbf{p}(M_1, M_2) = \binom{M_1}{(U \otimes_A M_1) \oplus M_2}$ with the obvious map and for any morphism (f_1, f_2) in A-Mod $\times$ B-Mod, let $\mathbf{p}(f_1, f_2) = \binom{f_1}{(U \otimes_A f_1) \oplus f_2}$.

(2) $\mathbf{h}$: A-Mod $\times$ B-Mod $\to$ T-Mod is defined as follows: For each object (M_1, M_2) of A-Mod $\times$ B-Mod, let $\mathbf{h}(M_1, M_2) = \binom{M_1 \oplus \mathrm{Hom}_B(U, M_2)}{M_2}$ with the obvious map and for any morphism (f_1, f_2) in A-Mod $\times$ B-Mod, let $h(f_1, f_2) = \binom{f_1 \oplus \mathrm{Hom}_B(U, f_2)}{f_2}$.

(3) $\mathbf{q}$: T-Mod $\to$ A-Mod $\times$ B-Mod is defined, for each left T-module $\binom{M_1}{M_2}_{\varphi^M}$ as $\mathbf{q}\binom{M_1}{M_2}_{\varphi^M} = (M_1, M_2)$, and for each morphism $\binom{f_1}{f_2}$ in T-Mod as $\mathbf{q}\binom{f_1}{f_2} = (f_1, f_2)$.

It is easy to see that $\mathbf{p}$ is a left adjoint of $\mathbf{q}$ and $\mathbf{h}$ is a right adjoint of $\mathbf{q}$.

2. Admissible balanced pairs over formal triangular matrix rings

Let $\mathfrak{F}$ and $\mathfrak{L}$ be two classes of left R-modules. Following [6], a complex $\cdots \to A_1 \to A_0 \to A^0 \to A^1 \to \cdots$ of left R-modules is called $\mathrm{Hom}_R(\mathfrak{F}, -)$-*exact* if the induced sequence $\cdots \to \mathrm{Hom}_R(C, A_1) \to \mathrm{Hom}_R(C, A_0) \to \mathrm{Hom}_R(C, A^0) \to \mathrm{Hom}_R(C, A^1) \to \cdots$ is exact for any $C \in \mathfrak{F}$, and it is called $\mathrm{Hom}_R(-, \mathfrak{L})$-*exact* if the induced sequence $\cdots \to \mathrm{Hom}_R(A^1, D) \to \mathrm{Hom}_R(A^0, D) \to \mathrm{Hom}_R(A_0, D) \to \mathrm{Hom}_R(A_1, D) \to \cdots$ is exact for any $D \in \mathfrak{L}$. A complex $\cdots \to A_1 \to A_0 \to M \to 0$ of left R-modules with each $A_i \in \mathfrak{F}$ is called a *left* $\mathfrak{F}$-*resolution* of M if it is $\mathrm{Hom}_R(\mathfrak{F}, -)$-exact. Dually, a complex $0 \to M \to A^0 \to A^1 \to \cdots$ of left R-modules with each $A^i \in \mathfrak{L}$ is called a *right* $\mathfrak{L}$-*resolution* of M if it is $\mathrm{Hom}_R(-, \mathfrak{L})$-exact. Obviously, $\mathfrak{F}$ is a precovering class if and only if each left R-module has a left $\mathfrak{F}$-resolution, $\mathfrak{L}$ is a preenveloping class if and only if each left R-module has a right $\mathfrak{L}$-resolution.

A pair $(\mathfrak{F}, \mathfrak{L})$ of classes of left R-modules is called a *balanced pair* [4] if the following conditions are satisfied: (1) $\mathfrak{F}$ is a precovering class and $\mathfrak{L}$ is a preenveloping class; (2) for each left R-module M, there is a left $\mathfrak{F}$-resolution which is $\mathrm{Hom}_R(-, \mathfrak{L})$-exact; (3) for each left R-module M, there is a right $\mathfrak{L}$-resolution which is $\mathrm{Hom}_R(\mathfrak{F}, -)$-exact.

A balanced pair $(\mathfrak{F}, \mathfrak{L})$ is called *admissible* [4] provided that $\mathfrak{F}$ is an epic precovering class and $\mathfrak{L}$ is a monic preenveloping class.

Lemma 2.1. *Let $\mathfrak{F}$ be an epic precovering class and $\mathfrak{L}$ be a monic preenveloping class in R-Mod. Then the following conditions are equivalent*:

(1) *$(\mathfrak{F}, \mathfrak{L})$ is an admissible balanced pair.*
(2) *An exact sequence in R-Mod is $\mathrm{Hom}_R(\mathfrak{F}, -)$-exact if and only if it is $\mathrm{Hom}_R(-, \mathfrak{L})$-exact.*
(3) *For each left R-module M, there are two exact sequences $0 \to K \to F \to M \to 0$ and $0 \to M \to L \to G \to 0$ with $F \in \mathfrak{F}$ and $L \in \mathfrak{L}$, which are both $\mathrm{Hom}_R(\mathfrak{F}, -)$-exact and $\mathrm{Hom}_R(-, \mathfrak{L})$-exact.*

Proof. It is easy by [4, Proposition 2.2] or [7, Lemma 3.1]. □

Let $\mathfrak{C}$ be a class of left A-modules and $\mathfrak{D}$ be a class of left B-modules. We write $\mathbf{p}(\mathfrak{C}, \mathfrak{D}) = \{\mathbf{p}(M_1, M_2) : M_1 \in \mathfrak{C}, M_2 \in \mathfrak{D}\}$ and $\mathbf{h}(\mathfrak{C}, \mathfrak{D}) = \{\mathbf{h}(M_1, M_2) : M_1 \in \mathfrak{C}, M_2 \in \mathfrak{D}\}$.

Let $\mathfrak{C}_1$ and $\mathfrak{C}_2$ be two classes of left A-modules, $\mathfrak{D}_1$ and $\mathfrak{D}_2$ be two classes of left B-modules. We first characterize when $(\mathbf{p}(\mathfrak{C}_1, \mathfrak{D}_1), \mathbf{h}(\mathfrak{C}_2, \mathfrak{D}_2))$ is an admissible balanced pair in T-Mod.

Theorem 2.2. *Let $\mathfrak{C}_1$ and $\mathfrak{C}_2$ be two classes of left A-modules, $\mathfrak{D}_1$ and $\mathfrak{D}_2$ be two classes of left B-modules. The following conditions are equivalent*:

(1) *$(\mathfrak{C}_1, \mathfrak{C}_2)$ and $(\mathfrak{D}_1, \mathfrak{D}_2)$ are admissible balanced pairs.*
(2) *$(\boldsymbol{p}(\mathfrak{C}_1, \mathfrak{D}_1), \boldsymbol{h}(\mathfrak{C}_2, \mathfrak{D}_2))$ is an admissible balanced pair.*

Proof. (1) ⇒ (2) By Lemma 2.1, for any left T-module $M = \binom{M_1}{M_2}_{\varphi^M}$, there is an exact sequence $0 \to K_1 \xrightarrow{\lambda_1} F_1 \xrightarrow{f_1} M_1 \to 0$ with $F_1 \in \mathfrak{C}_1$ which is $\mathrm{Hom}_A(\mathfrak{C}_1, -)$-exact and $\mathrm{Hom}_A(-, \mathfrak{C}_2)$-exact. Also there is an exact sequence $0 \to N \xrightarrow{\iota} F_2 \xrightarrow{\gamma} M_2 \to 0$ with $F_2 \in \mathfrak{D}_1$ which is $\mathrm{Hom}_B(\mathfrak{D}_1, -)$-exact and $\mathrm{Hom}_B(-, \mathfrak{D}_2)$-exact.

Define $f_2 : (U \otimes_A F_1) \oplus F_2 \to M_2$ by $f_2(u \otimes x_1, x_2) = \varphi^M(u \otimes f_1(x_1)) + \gamma(x_2)$ for $u \in U, x_1 \in F_1, x_2 \in F_2$. Then f_2 is clearly an epimorphism. So we get an epimorphism $\binom{f_1}{f_2} : \mathbf{p}(F_1, F_2) \to M$. Let $K = \binom{K_1}{K_2}_{\varphi^K} = \ker \binom{f_1}{f_2}$. Then we get the exact sequence

$$0 \to K \xrightarrow{\binom{\lambda_1}{\lambda_2}} \mathbf{p}(F_1, F_2) \xrightarrow{\binom{f_1}{f_2}} M \to 0$$

with $\mathbf{p}(F_1, F_2) \in \mathbf{p}(\mathfrak{C}_1, \mathfrak{D}_1)$. Let $\mathbf{p}(C_1, D_1) \in \mathbf{p}(\mathfrak{C}_1, \mathfrak{D}_1)$. Then we get the epimorphism $(f_1)_* : \mathrm{Hom}_A(C_1, F_1) \to \mathrm{Hom}_A(C_1, M_1)$. Let $i : F_2 \to (U \otimes_A F_1) \oplus F_2$

be the injection. Then $\gamma = f_2 i$. Since $\gamma_* : \mathrm{Hom}_B(D_1, F_2) \to \mathrm{Hom}_B(D_1, M_2)$ is an epimorphism, we obtain that $(f_2)_* : \mathrm{Hom}_B(D_1, (U \otimes_A F_1) \oplus F_2) \to \mathrm{Hom}_B(D_1, M_2)$ is an epimorphism. From the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{Hom}_T(\mathbf{p}(C_1, D_1), \mathbf{p}(F_1, F_2)) & \longrightarrow & \mathrm{Hom}_T(\mathbf{p}(C_1, D_1), M) \\
\downarrow \cong & & \downarrow \cong \\
\mathrm{Hom}_A(C_1, F_1) \oplus \mathrm{Hom}_B(D_1, (U \otimes_A F_1) \oplus F_2) & \xrightarrow{(f_1)_* \oplus (f_2)_*} & \mathrm{Hom}_A(C_1, M_1) \oplus \mathrm{Hom}_B(D_1, M_2),
\end{array}$$

we infer that $0 \to K \to \mathbf{p}(F_1, F_2) \to M \to 0$ is $\mathrm{Hom}_T(\mathbf{p}(\mathfrak{C}_1, \mathfrak{D}_1), -)$-exact.

There is the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N & \xrightarrow{\iota} & F_2 & \xrightarrow{\gamma} & M_2 & \longrightarrow & 0 \\
 & & \vdots & & \downarrow i & & \downarrow & & \\
0 & \longrightarrow & K_2 & \xrightarrow{\lambda_2} & (U \otimes_A F_1) \oplus F_2 & \xrightarrow{f_2} & M_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U \otimes_A F_1 & \longrightarrow & U \otimes_A F_1 & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Let $\mathbf{h}(C_2, D_2) \in \mathbf{h}(\mathfrak{C}_2, \mathfrak{D}_2)$. Applying $\mathrm{Hom}_B(-, D_2)$ to the above commutative diagram, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}_B(U \otimes_A F_1, D_2) & \longrightarrow & \mathrm{Hom}_B((U \otimes_A F_1) \oplus F_2, D_2) & \longrightarrow & \mathrm{Hom}_B(F_2, D_2) & \longrightarrow & 0 \\
 & & \| & & \downarrow \lambda_2^* & & \downarrow \iota^* & & \\
0 & \longrightarrow & \mathrm{Hom}_B(U \otimes_A F_1, D_2) & \longrightarrow & \mathrm{Hom}_B(K_2, D_2) & \longrightarrow & \mathrm{Hom}_B(N, D_2). & &
\end{array}$$

Notice that ι^* is an epimorphism, so λ_2^* is an epimorphism by the Snake lemma. Also $\lambda_1^* : \mathrm{Hom}_A(F_1, C_2) \to \mathrm{Hom}_A(K_1, C_2)$ is an epimorphism. Thus the commutative diagram:

$$\begin{array}{ccc}
\mathrm{Hom}_T(\mathbf{p}(F_1, F_2), \mathbf{h}(C_2, D_2)) & \longrightarrow & \mathrm{Hom}_T(K, \mathbf{h}(C_2, D_2)) \\
\downarrow \cong & & \downarrow \cong \\
\mathrm{Hom}_A(F_1, C_2) \oplus \mathrm{Hom}_B((U \otimes_A F_1) \oplus F_2, D_2) & \xrightarrow{\lambda_1^* \oplus \lambda_2^*} & \mathrm{Hom}_A(K_1, C_2) \oplus \mathrm{Hom}_B(K_2, D_2)
\end{array}$$

implies that $0 \to K \to \mathbf{p}(F_1, F_2) \to M \to 0$ is $\mathrm{Hom}_T(-, \mathbf{h}(\mathfrak{C}_2, \mathfrak{D}_2))$-exact.

On the other hand, there are two exact sequences $0 \to M_1 \overset{\theta}{\to} X \overset{\rho}{\to} Q \to 0$ with $X \in \mathfrak{C}_2$ which is $\mathrm{Hom}_A(\mathfrak{C}_1, -)$-exact and $\mathrm{Hom}_A(-, \mathfrak{C}_2)$-exact, and $0 \to M_2 \overset{g_2}{\to} Y \overset{\pi_2}{\to} L_2 \to 0$ with $Y \in \mathfrak{D}_2$ which is $\mathrm{Hom}_B(\mathfrak{D}_1, -)$-exact and $\mathrm{Hom}_B(-,\mathfrak{D}_2)$-exact. Define $g_1 : M_1 \to X \oplus \mathrm{Hom}_B(U,Y)$ by $g_1(x) = (\theta(x), \mathrm{Hom}_B(U,g_2)\widetilde{\varphi^M}(x))$ for $x \in M_1$. Since g_1 is clearly a monomorphism, we get a monomorphism $\binom{g_1}{g_2} : M \to \mathbf{h}(X,Y)$. Let $L = \binom{L_1}{L_2}_{\varphi^L} = \mathrm{coker}\,\binom{g_1}{g_2}$. Then we get the exact sequence

$$0 \to M \xrightarrow{\binom{g_1}{g_2}} \mathbf{h}(X,Y) \xrightarrow{\binom{\pi_1}{\pi_2}} L \to 0$$

with $\mathbf{h}(X,Y) \in \mathbf{h}(\mathfrak{C}_2,\mathfrak{D}_2)$. Let $j : X \oplus \mathrm{Hom}_B(U,Y) \to X$ be the projection. Then $\theta = jg_1$. Let $\mathbf{h}(C_2,D_2) \in \mathbf{h}(\mathfrak{C}_2,\mathfrak{D}_2)$. Since $\theta^* : \mathrm{Hom}_A(X,C_2) \to \mathrm{Hom}_A(M_1,C_2)$ is an epimorphism, we obtain the epimorphism $g_1^* : \mathrm{Hom}_A(X \oplus \mathrm{Hom}_B(U,Y),C_2) \to \mathrm{Hom}_A(M_1,C_2)$. Also $g_2^* : \mathrm{Hom}_B(Y,D_2) \to \mathrm{Hom}_B(M_2,D_2)$ is an epimorphism. Thus from the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Hom}_T(\mathbf{h}(X,Y),\mathbf{h}(C_2,D_2)) & \longrightarrow & \mathrm{Hom}_T(M,\mathbf{h}(C_2,D_2)) \\ \downarrow{\cong} & & \downarrow{\cong} \\ \mathrm{Hom}_A(X\oplus \mathrm{Hom}_B(U,Y),C_2)\oplus\mathrm{Hom}_B(Y,D_2) & \xrightarrow{g_1^*\oplus g_2^*} & \mathrm{Hom}_A(M_1,C_2)\oplus\mathrm{Hom}_B(M_2,D_2), \end{array}$$

we infer that $0 \to M \to \mathbf{h}(X,Y) \to L \to 0$ is $\mathrm{Hom}_T(-,\mathbf{h}(\mathfrak{C}_2,\mathfrak{D}_2))$-exact.

Let $\mathbf{p}(C_1,D_1) \in \mathbf{p}(\mathfrak{C}_1,\mathfrak{D}_1)$. Applying $\mathrm{Hom}_A(C_1,-)$ to the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & 0 & \longrightarrow & \mathrm{Hom}_B(U,Y) & \longrightarrow & \mathrm{Hom}_B(U,Y) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M_1 & \xrightarrow{g_1} & X\oplus\mathrm{Hom}_B(U,Y) & \xrightarrow{\pi_1} & L_1 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{j} & & \vdots & & \\ 0 & \longrightarrow & M_1 & \xrightarrow{\theta} & X & \xrightarrow{\rho} & Q & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}_A(C_1,\mathrm{Hom}_B(U,Y)) & \longrightarrow & \mathrm{Hom}_A(C_1,X\oplus\mathrm{Hom}_B(U,Y)) & \longrightarrow & \mathrm{Hom}_A(C_1,X) & \longrightarrow & 0 \\ & & \| & & \downarrow{(\pi_1)_*} & & \downarrow{\rho_*} & & \\ 0 & \longrightarrow & \mathrm{Hom}_A(C_1,\mathrm{Hom}_B(U,Y)) & \longrightarrow & \mathrm{Hom}_A(C_1,L_1) & \longrightarrow & \mathrm{Hom}_A(C_1,Q). & & \end{array}$$

Notice that ρ_* is an epimorphism, so $(\pi_1)_*$ is an epimorphism by the Snake lemma. Also $(\pi_2)_* : \mathrm{Hom}_B(D_1,Y) \to \mathrm{Hom}_B(D_1,L_2)$ is an epimorphism. So the commutative diagram:

$$\begin{array}{ccc}
\mathrm{Hom}_T(\mathbf{p}(C_1,D_1),\mathbf{h}(X,Y)) & \longrightarrow & \mathrm{Hom}_T(\mathbf{p}(C_1,D_1),L) \\
\downarrow \cong & & \downarrow \cong \\
\mathrm{Hom}_A(C_1,X\oplus \mathrm{Hom}_B(U,Y))\oplus \mathrm{Hom}_B(D_1,Y) & \xrightarrow{(\pi_1)_*\oplus(\pi_2)_*} & \mathrm{Hom}_A(C_1,L_1)\oplus \mathrm{Hom}_B(D_1,L_2)
\end{array}$$

implies that $0\to M\to \mathbf{h}(X,Y)\to L\to 0$ is $\mathrm{Hom}_T(\mathbf{p}(\mathfrak{C}_1,\mathfrak{D}_1),-)$-exact.

Thus $(\mathbf{p}(\mathfrak{C}_1,\mathfrak{D}_1),\mathbf{h}(\mathfrak{C}_2,\mathfrak{D}_2))$ is an admissible balanced pair by Lemma 2.1.

(2) $\Rightarrow$ (1) We first prove that $(\mathfrak{C}_1,\mathfrak{C}_2)$ is an admissible balanced pair. For any left A-module M_1, by Lemma 2.1, there is an exact sequence $0\to J\to \mathbf{p}(X_1,X_2)\to \mathbf{h}(M_1,0)\to 0$ with $\mathbf{p}(X_1,X_2)\in \mathbf{p}(\mathfrak{C}_1,\mathfrak{D}_1)$ and $J=\left(\begin{smallmatrix}J_1\\J_2\end{smallmatrix}\right)_{\varphi^J}$, which is $\mathrm{Hom}_T(\mathbf{p}(\mathfrak{C}_1,\mathfrak{D}_1),-)$-exact and $\mathrm{Hom}_T(-,\mathbf{h}(\mathfrak{C}_2,\mathfrak{D}_2))$-exact. Let $C_1\in\mathfrak{C}_1$ and $C_2\in\mathfrak{C}_2$. Then we get two commutative diagrams with exact rows:

$$\begin{array}{ccccc}
\mathrm{Hom}_T(\mathbf{p}(C_1,0),\mathbf{p}(X_1,X_2)) & \longrightarrow & \mathrm{Hom}_T(\mathbf{p}(C_1,0),\mathbf{h}(M_1,0)) & \longrightarrow & 0 \\
\downarrow \cong & & \downarrow \cong & & \\
\mathrm{Hom}_A(C_1,X_1) & \longrightarrow & \mathrm{Hom}_A(C_1,M_1) & \longrightarrow & 0
\end{array}$$

and

$$\begin{array}{ccccc}
\mathrm{Hom}_T(\mathbf{p}(X_1,X_2),\mathbf{h}(C_2,0)) & \longrightarrow & \mathrm{Hom}_T(J,\mathbf{h}(C_2,0)) & \longrightarrow & 0 \\
\downarrow \cong & & \downarrow \cong & & \\
\mathrm{Hom}_A(X_1,C_2) & \longrightarrow & \mathrm{Hom}_A(J_1,C_2) & \longrightarrow & 0.
\end{array}$$

So $0\to J_1\to X_1\to M_1\to 0$ is $\mathrm{Hom}_A(\mathfrak{C}_1,-)$-exact and $\mathrm{Hom}_B(-,\mathfrak{C}_2)$-exact.

On the other hand, by Lemma 2.1, there is an exact sequence

$$0\to \mathbf{h}(M_1,0)\to \mathbf{h}(Y_1,Y_2)\overset{\left(\begin{smallmatrix}\phi\\ \theta\end{smallmatrix}\right)}{\to} E\to 0$$

with $\mathbf{h}(Y_1,Y_2)\in \mathbf{h}(\mathfrak{C}_2,\mathfrak{D}_2)$ and $E=\left(\begin{smallmatrix}E_1\\E_2\end{smallmatrix}\right)_{\varphi^E}$, which is $\mathrm{Hom}_T(\mathbf{p}(\mathfrak{C}_1,\mathfrak{D}_1),-)$-exact and $\mathrm{Hom}_T(-,\mathbf{h}(\mathfrak{C}_2,\mathfrak{D}_2))$-exact. It is obvious that $\theta: Y_2\to E_2$ is an isomorphism. For $C_1\in\mathfrak{C}_1$ and $C_2\in\mathfrak{C}_2$, we get the following commutative diagrams with exact rows:

$$\begin{array}{ccccc}
\mathrm{Hom}_T(\mathbf{p}(C_1,0),\mathbf{h}(Y_1,Y_2)) & \longrightarrow & \mathrm{Hom}_T(\mathbf{p}(C_1,0),E) & \longrightarrow & 0 \\
\downarrow \cong & & \downarrow \cong & & \\
\mathrm{Hom}_A(C_1,Y_1\oplus \mathrm{Hom}_B(U,Y_2)) & \longrightarrow & \mathrm{Hom}_A(C_1,E_1) & \longrightarrow & 0
\end{array}$$

and

$$\begin{array}{ccccc}
\mathrm{Hom}_T(\mathbf{h}(Y_1,Y_2),\mathbf{h}(C_2,0)) & \longrightarrow & \mathrm{Hom}_T(\mathbf{h}(M_1,0),\mathbf{h}(C_2,0)) & \longrightarrow & 0 \\
\downarrow \cong & & \downarrow \cong & & \\
\mathrm{Hom}_A(Y_1\oplus \mathrm{Hom}_B(U,Y_2),C_2) & \longrightarrow & \mathrm{Hom}_A(M_1,C_2) & \longrightarrow & 0.
\end{array}$$

Also, there is the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_1 & \dashrightarrow & Y_1 & \overset{\rho}{\dashrightarrow} & Q & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_1 & \longrightarrow & Y_1 \oplus \mathrm{Hom}_B(U, Y_2) & \xrightarrow{\phi} & E_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & \mathrm{Hom}_B(U, Y_2) & \xrightarrow{\theta_*} & \mathrm{Hom}_B(U, E_2) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Applying $\mathrm{Hom}_A(-, C_2)$ and $\mathrm{Hom}_A(C_1, -)$ to the above diagram, we obtain that $0 \to M_1 \to Y_1 \to Q \to 0$ is $\mathrm{Hom}_A(-, \mathcal{C}_2)$-exact and get the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Hom}_A(C_1, Y_1) & \longrightarrow & \mathrm{Hom}_A(C_1, Y_1 \oplus \mathrm{Hom}_B(U, Y_2)) & \longrightarrow & \mathrm{Hom}_A(C_1, \mathrm{Hom}_B(U, Y_2)) \\
 & & \downarrow{\scriptstyle \rho_*} & & \downarrow{\scriptstyle \phi_*} & & \downarrow{\scriptstyle \cong} \\
0 & \longrightarrow & \mathrm{Hom}_A(C_1, Q) & \longrightarrow & \mathrm{Hom}_A(C_1, E_1) & \longrightarrow & \mathrm{Hom}_A(C_1, \mathrm{Hom}_B(U, E_2)).
\end{array}$$

Notice that ϕ_* is an epimorphism, so ρ_* is an epimorphism. Thus $0 \to M_1 \to Y_1 \to Q \to 0$ is $\mathrm{Hom}_A(\mathcal{C}_1, -)$-exact.

It follows that $(\mathfrak{C}_1, \mathfrak{C}_2)$ is an admissible balanced pair by Lemma 2.1. By a similar proof, $(\mathfrak{D}_1, \mathfrak{D}_2)$ is an admissible balanced pair. □

Remark 2.3. In the proof of (2) ⇒ (1) in Theorem 2.2, we may also obtain that $(\mathfrak{D}_1, \mathfrak{D}_2)$ is an admissible balanced pair by applying [9, Corollary 2.4]. In fact, let $f = (0, B) \otimes_T -$, $i = \mathbf{p}(0, -)$, $g = \mathrm{Hom}_T(\left(\begin{smallmatrix} 0 \\ B \end{smallmatrix}\right), -)$, $r = \mathbf{h}(-, 0)$, $e = \mathrm{Hom}_T(\left(\begin{smallmatrix} A \\ U \end{smallmatrix}\right), -)$, $l = \mathbf{p}(-, 0)$. Then by [9, Lemma 3.2], we get the recollement

$$B\text{–Mod} \;\begin{smallmatrix} \xleftarrow{f} \\ \xrightarrow{i} \\ \xleftarrow[g]{} \end{smallmatrix}\; T\text{–Mod} \;\begin{smallmatrix} \xleftarrow{l} \\ \xrightarrow{e} \\ \xleftarrow{r} \end{smallmatrix}\; A\text{–Mod}.$$

If $(\mathbf{p}(\mathfrak{C}_1, \mathfrak{D}_1), \mathbf{h}(\mathfrak{C}_2, \mathfrak{D}_2))$ is an admissible balanced pair in T-Mod, then by [9, Corollary 2.4], $(\mathfrak{D}_1, \mathfrak{D}_2) = (f(\mathbf{p}(\mathfrak{C}_1, \mathfrak{D}_1)), g(\mathbf{h}(\mathfrak{C}_2, \mathfrak{D}_2)))$ is an admissible balanced pair in B-Mod.

Let $\mathfrak{C}$ be a class of left A-modules and $\mathfrak{D}$ be a class of left B-modules. We will denote by $\mathfrak{P}^{\mathfrak{C}}_{\mathfrak{D}}$ the class of left T-modules $\{\left(\begin{smallmatrix} M_1 \\ M_2 \end{smallmatrix}\right)_{\varphi^M} : M_1 \in \mathfrak{C}$ and

$M_2/\mathrm{im}(\varphi^M) \in \mathfrak{D}, \varphi^M$ is a monomorphism}, denote by $\mathfrak{I}^{\mathfrak{C}}_{\mathfrak{D}}$ the class of left T-modules $\{\binom{M_1}{M_2}_{\varphi^M} : \ker(\widetilde{\varphi^M}) \in \mathfrak{C}$ and $M_2 \in \mathfrak{D}$, $\widetilde{\varphi^M}$ is an epimorphism} and denote by $\mathfrak{A}^{\mathfrak{C}}_{\mathfrak{D}}$ the class of left T-modules $\{\binom{M_1}{M_2}_{\varphi^M} : M_1 \in \mathfrak{C}$ and $M_2 \in \mathfrak{D}\}$.

Let $\mathfrak{C}_1$ and $\mathfrak{C}_2$ be two classes of left A-modules, $\mathfrak{D}_1$ and $\mathfrak{D}_2$ two classes of left B-modules. Next we study when $(\mathfrak{P}^{\mathfrak{C}_1}_{\mathfrak{D}_1}, \mathfrak{I}^{\mathfrak{C}_2}_{\mathfrak{D}_2})$ is an admissible balanced pair.

Proposition 2.4. *Let $\mathfrak{C}_1$ and $\mathfrak{C}_2$ be two classes of left A-modules, $\mathfrak{D}_1$ and $\mathfrak{D}_2$ be two classes of left B-modules. If $(\mathfrak{P}^{\mathfrak{C}_1}_{\mathfrak{D}_1}, \mathfrak{I}^{\mathfrak{C}_2}_{\mathfrak{D}_2})$ is an admissible balanced pair, then $(\mathfrak{C}_1, \mathfrak{C}_2)$ and $(\mathfrak{D}_1, \mathfrak{D}_2)$ are admissible balanced pairs.*

Proof. We first prove that $(\mathfrak{C}_1, \mathfrak{C}_2)$ is an admissible balanced pair. For any left A-module M_1, by Lemma 2.1, there is an exact sequence $0 \to J \to X \to \mathbf{h}(M_1, 0) \to 0$ with $X = \binom{X_1}{X_2}_{\varphi^X} \in \mathfrak{P}^{\mathfrak{C}_1}_{\mathfrak{D}_1}$ and $J = \binom{J_1}{J_2}_{\varphi^J}$, which is $\mathrm{Hom}_T(\mathfrak{P}^{\mathfrak{C}_1}_{\mathfrak{D}_1}, -)$-exact and $\mathrm{Hom}_T(-, \mathfrak{I}^{\mathfrak{C}_2}_{\mathfrak{D}_2})$-exact. Let $C_1 \in \mathfrak{C}_1$ and $C_2 \in \mathfrak{C}_2$. Note that $\mathbf{p}(\mathfrak{C}_1, \mathfrak{D}_1) \subseteq \mathfrak{P}^{\mathfrak{C}_1}_{\mathfrak{D}_1}$ and $\mathbf{h}(\mathfrak{C}_2, \mathfrak{D}_2) \subseteq \mathfrak{I}^{\mathfrak{C}_2}_{\mathfrak{D}_2}$, hence we get the following commutative diagrams with exact rows:

$$\begin{array}{ccccc}
\mathrm{Hom}_T(\mathbf{p}(C_1,0), X) & \longrightarrow & \mathrm{Hom}_T(\mathbf{p}(C_1,0), \mathbf{h}(M_1,0)) & \longrightarrow & 0 \\
\downarrow \cong & & \downarrow \cong & & \\
\mathrm{Hom}_A(C_1, X_1) & \longrightarrow & \mathrm{Hom}_A(C_1, M_1) & \longrightarrow & 0
\end{array}$$

and

$$\begin{array}{ccccc}
\mathrm{Hom}_T(X, \mathbf{h}(C_2,0)) & \longrightarrow & \mathrm{Hom}_T(J, \mathbf{h}(C_2,0)) & \longrightarrow & 0 \\
\downarrow \cong & & \downarrow \cong & & \\
\mathrm{Hom}_A(X_1, C_2) & \longrightarrow & \mathrm{Hom}_A(J_1, C_2) & \longrightarrow & 0.
\end{array}$$

So $0 \to J_1 \to X_1 \to M_1 \to 0$ is $\mathrm{Hom}_A(\mathfrak{C}_1, -)$-exact and $\mathrm{Hom}_A(-, \mathfrak{C}_2)$-exact. On the other hand, by Lemma 2.1, there is an exact sequence $0 \to \mathbf{h}(M_1, 0) \to Y \xrightarrow{\binom{\phi}{\theta}} E \to 0$ with $Y = \binom{Y_1}{Y_2}_{\varphi^Y} \in \mathfrak{I}^{\mathfrak{C}_2}_{\mathfrak{D}_2}$ and $E = \binom{E_1}{E_2}_{\varphi^E}$, which is $\mathrm{Hom}_T(\mathfrak{P}^{\mathfrak{C}_1}_{\mathfrak{D}_1}, -)$-exact and $\mathrm{Hom}_T(-, \mathfrak{I}^{\mathfrak{C}_2}_{\mathfrak{D}_2})$-exact. Note that $\theta : Y_2 \to E_2$ is an isomorphism. For $C_1 \in \mathfrak{C}_1$ and $C_2 \in \mathfrak{C}_2$, we have the following commutative diagrams with exact rows:

$$\begin{array}{ccccc}
\mathrm{Hom}_T(\mathbf{p}(C_1,0), Y) & \longrightarrow & \mathrm{Hom}_T(\mathbf{p}(C_1,0), E) & \longrightarrow & 0 \\
\downarrow \cong & & \downarrow \cong & & \\
\mathrm{Hom}_A(C_1, Y_1) & \xrightarrow{\phi_*} & \mathrm{Hom}_A(C_1, E_1) & \longrightarrow & 0
\end{array}$$

and

$$\begin{array}{ccccc}
\mathrm{Hom}_T(Y,\mathbf{h}(C_2,0)) & \longrightarrow & \mathrm{Hom}_T(\mathbf{h}(M_1,0),\mathbf{h}(C_2,0)) & \longrightarrow & 0 \\
\downarrow{\cong} & & \downarrow{\cong} & & \\
\mathrm{Hom}_A(Y_1,C_2) & \longrightarrow & \mathrm{Hom}_A(M_1,C_2) & \longrightarrow & 0.
\end{array}$$

Applying $\mathrm{Hom}_A(-,C_2)$ and $\mathrm{Hom}_A(C_1,-)$ to the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_1 & \dashrightarrow & \ker(\widetilde{\varphi^Y}) & \overset{\rho}{\dashrightarrow} & Q & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_1 & \longrightarrow & Y_1 & \overset{\phi}{\longrightarrow} & E_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & \mathrm{Hom}_B(U,Y_2) & \longrightarrow & \mathrm{Hom}_B(U,E_2) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

we obtain that $0\to M_1\to \ker(\widetilde{\varphi^Y})\to Q\to 0$ is $\mathrm{Hom}_A(-,\mathcal{C}_2)$-exact and get the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Hom}_A(C_1,\ker(\widetilde{\varphi^Y})) & \longrightarrow & \mathrm{Hom}_A(C_1,Y_1) & \longrightarrow & \mathrm{Hom}_A(C_1,\mathrm{Hom}_B(U,Y_2)) \\
 & & {\rho_*}\downarrow & & {\phi_*}\downarrow & & {\cong}\downarrow \\
0 & \longrightarrow & \mathrm{Hom}_A(C_1,Q) & \longrightarrow & \mathrm{Hom}_A(C_1,E_1) & \longrightarrow & \mathrm{Hom}_A(C_1,\mathrm{Hom}_B(U,E_2)).
\end{array}$$

Since ϕ_* is an epimorphism, ρ_* is an epimorphism. Thus $0\to M_1\to \ker(\widetilde{\varphi^Y})\to Q\to 0$ is $\mathrm{Hom}_A(\mathcal{C}_1,-)$-exact. So $(\mathfrak{C}_1,\mathfrak{C}_2)$ is an admissible balanced pair by Lemma 2.1. By a similar proof, $(\mathfrak{D}_1,\mathfrak{D}_2)$ is an admissible balanced pair. □

Given a class $\mathfrak{G}$ of left R-modules, we recall that a monomorphism $\alpha: M\to N$ with $N\in\mathfrak{G}$ is a *special $\mathfrak{G}$-preenvelope* of M [6] if $\mathrm{Ext}^1_R(\mathrm{coker}(\alpha),C)=0$ for all $C\in\mathfrak{G}$. Dually we have the definition of a special $\mathfrak{G}$-precover. Write $\mathfrak{G}^\perp=\{X:\mathrm{Ext}^1_R(C,X)=0 \text{ for all } C\in\mathfrak{G}\}$ and ${}^\perp\mathfrak{G}=\{L:\mathrm{Ext}^1_R(L,C)=0 \text{ for all } C\in\mathfrak{G}\}$. A pair $(\mathfrak{F},\mathfrak{G})$ of classes of left R-modules is called a *cotorsion pair* [10] if $\mathfrak{F}^\perp=\mathfrak{G}$ and $\mathfrak{F}={}^\perp\mathfrak{G}$. A cotorsion pair $(\mathfrak{F},\mathfrak{G})$ is called *complete* [10] if every left R-module has a special $\mathfrak{G}$-preenvelope, equivalently, every left R-module has a special $\mathfrak{F}$-precover by [10, Lemma 2.2.6]. A cotorsion pair $(\mathfrak{F},\mathfrak{G})$ is called *hereditary* [10] if whenever $0\to L^{'}\to L\to L^{''}\to 0$ is exact with

$L, L^{''} \in \mathfrak{F}$, then $L^{'}$ is also in $\mathfrak{F}$, or equivalently, if $0 \to C^{'} \to C \to C^{''} \to 0$ is exact with $C^{'}, C \in \mathfrak{G}$, then $C^{''}$ is also in $\mathfrak{G}$.

Theorem 2.5. *Suppose that* $(\mathfrak{C}_1, \mathfrak{C}_1^{\perp})$ *and* $({}^{\perp}\mathfrak{C}_2, \mathfrak{C}_2)$ *are complete hereditary cotorsion pairs in A-Mod with* $\mathrm{Tor}_i^A(U, \mathfrak{C}_1) = 0$ *for any* $i \geq 1$, $(\mathfrak{D}_1, \mathfrak{D}_1^{\perp})$ *and* $({}^{\perp}\mathfrak{D}_2, \mathfrak{D}_2)$ *are complete hereditary cotorsion pairs in B-Mod with* $\mathrm{Ext}_B^i(U, \mathfrak{D}_2) = 0$ *for any* $i \geq 1$, $\mathfrak{C}_1 \bigcap \mathfrak{C}_1^{\perp} \subseteq {}^{\perp}\mathfrak{C}_2$, $\mathfrak{C}_2 \bigcap {}^{\perp}\mathfrak{C}_2 \subseteq \mathfrak{C}_1^{\perp}$, $\mathfrak{D}_1 \bigcap \mathfrak{D}_1^{\perp} \subseteq {}^{\perp}\mathfrak{D}_2$ *and* $\mathfrak{D}_2 \bigcap {}^{\perp}\mathfrak{D}_2 \subseteq \mathfrak{D}_1^{\perp}$. *The following conditions are equivalent*:

(1) $(\mathfrak{C}_1, \mathfrak{C}_2)$ *and* $(\mathfrak{D}_1, \mathfrak{D}_2)$ *are admissible balanced pairs.*
(2) $(\mathfrak{P}_{\mathfrak{D}_1}^{\mathfrak{C}_1}, \mathfrak{I}_{\mathfrak{D}_2}^{\mathfrak{C}_2})$ *is an admissible balanced pair.*
(3) $\mathfrak{C}_1^{\perp} = {}^{\perp}\mathfrak{C}_2$ *and* $\mathfrak{D}_1^{\perp} = {}^{\perp}\mathfrak{D}_2$.
(4) $(\mathfrak{P}_{\mathfrak{D}_1}^{\mathfrak{C}_1})^{\perp} = {}^{\perp}\mathfrak{I}_{\mathfrak{D}_2}^{\mathfrak{C}_2}$.

Proof. (1) $\Rightarrow$ (3) Since $(\mathfrak{C}_1, \mathfrak{C}_2)$ is an admissible balanced pair, $(\mathfrak{C}_1, \mathfrak{C}_1^{\perp})$ and $({}^{\perp}\mathfrak{C}_2, \mathfrak{C}_2)$ are complete hereditary cotorsion pairs, $\mathfrak{C}_1 \bigcap \mathfrak{C}_1^{\perp} \subseteq {}^{\perp}\mathfrak{C}_2$, $\mathfrak{C}_2 \bigcap {}^{\perp}\mathfrak{C}_2 \subseteq \mathfrak{C}_1^{\perp}$, we have $\mathfrak{C}_1^{\perp} = {}^{\perp}\mathfrak{C}_2$ by [7, Corollary 4.8]. Similarly, since $(\mathfrak{D}_1, \mathfrak{D}_2)$ is an admissible balanced pair, $(\mathfrak{D}_1, \mathfrak{D}_1^{\perp})$ and $({}^{\perp}\mathfrak{D}_2, \mathfrak{D}_2)$ are complete hereditary cotorsion pairs, $\mathfrak{D}_1 \bigcap \mathfrak{D}_1^{\perp} \subseteq {}^{\perp}\mathfrak{D}_2$, $\mathfrak{D}_2 \bigcap {}^{\perp}\mathfrak{D}_2 \subseteq \mathfrak{D}_1^{\perp}$, we have $\mathfrak{D}_1^{\perp} = {}^{\perp}\mathfrak{D}_2$.

(3) $\Rightarrow$ (2) Since $(\mathfrak{C}_1, \mathfrak{C}_1^{\perp})$ and $(\mathfrak{D}_1, \mathfrak{D}_1^{\perp})$ are complete hereditary cotorsion pairs with $\mathrm{Tor}_i^A(U, \mathfrak{C}_1) = 0$ for any $i \geq 1$, $(\mathfrak{P}_{\mathfrak{D}_1}^{\mathfrak{C}_1}, \mathfrak{A}_{\mathfrak{D}_1^{\perp}}^{\mathfrak{C}_1^{\perp}})$ is a complete hereditary cotorsion pair by [15, Theorem 5.6(1)]. Since $({}^{\perp}\mathfrak{C}_2, \mathfrak{C}_2)$ and $({}^{\perp}\mathfrak{D}_2, \mathfrak{D}_2)$ are complete hereditary cotorsion pairs with $\mathrm{Ext}_B^i(U, \mathfrak{D}_2) = 0$ for any $i \geq 1$, $(\mathfrak{A}_{{}^{\perp}\mathfrak{D}_2}^{{}^{\perp}\mathfrak{C}_2}, \mathfrak{I}_{\mathfrak{D}_2}^{\mathfrak{C}_2})$ is a complete hereditary cotorsion pair by [15, Theorem 5.6(2)]. Since $\mathfrak{A}_{\mathfrak{D}_1^{\perp}}^{\mathfrak{C}_1^{\perp}} = \mathfrak{A}_{{}^{\perp}\mathfrak{D}_2}^{{}^{\perp}\mathfrak{C}_2}$ by (3), $(\mathfrak{P}_{\mathfrak{D}_1}^{\mathfrak{C}_1}, \mathfrak{I}_{\mathfrak{D}_2}^{\mathfrak{C}_2})$ is an admissible balanced pair by [4, Proposition 2.6] or [7, Proposition 4.2].

(2) $\Rightarrow$ (1) follows from Proposition 2.4.

(3) $\Leftrightarrow$ (4) By [15, Theorem 4.2], $(\mathfrak{P}_{\mathfrak{D}_1}^{\mathfrak{C}_1})^{\perp} = \mathfrak{A}_{\mathfrak{D}_1^{\perp}}^{\mathfrak{C}_1^{\perp}}$ and ${}^{\perp}\mathfrak{I}_{\mathfrak{D}_2}^{\mathfrak{C}_2} = \mathfrak{A}_{{}^{\perp}\mathfrak{D}_2}^{{}^{\perp}\mathfrak{C}_2}$. So $(\mathfrak{P}_{\mathfrak{D}_1}^{\mathfrak{C}_1})^{\perp} = {}^{\perp}\mathfrak{I}_{\mathfrak{D}_2}^{\mathfrak{C}_2}$ if and only if $\mathfrak{C}_1^{\perp} = {}^{\perp}\mathfrak{C}_2$ and $\mathfrak{D}_1^{\perp} = {}^{\perp}\mathfrak{D}_2$. □

Corollary 2.6. *Let R be a ring,* $T(R) = \left(\begin{smallmatrix} R & 0 \\ R & R \end{smallmatrix}\right)$, $(\mathfrak{C}_1, \mathfrak{C}_1^{\perp})$ *and* $({}^{\perp}\mathfrak{C}_2, \mathfrak{C}_2)$ *be complete hereditary cotorsion pairs in R-Mod. The following conditions are equivalent*:

(1) $(\mathfrak{C}_1, \mathfrak{C}_2)$ *is an admissible balanced pair,* $\mathfrak{C}_1 \bigcap \mathfrak{C}_1^{\perp} \subseteq {}^{\perp}\mathfrak{C}_2$, $\mathfrak{C}_2 \bigcap {}^{\perp}\mathfrak{C}_2 \subseteq \mathfrak{C}_1^{\perp}$.
(2) $(\mathfrak{P}_{\mathfrak{C}_1}^{\mathfrak{C}_1}, \mathfrak{I}_{\mathfrak{C}_2}^{\mathfrak{C}_2})$ *is an admissible balanced pair in* $T(R)$*-Mod.*
(3) $\mathfrak{C}_1^{\perp} = {}^{\perp}\mathfrak{C}_2$.
(4) $(\mathfrak{P}_{\mathfrak{C}_1}^{\mathfrak{C}_1})^{\perp} = {}^{\perp}\mathfrak{I}_{\mathfrak{C}_2}^{\mathfrak{C}_2}$.

Proof. (4) $\Leftrightarrow$ (3) $\Rightarrow$ (1) $\Rightarrow$ (2) follow from Theorem 2.5 by letting $\mathfrak{D}_1 = \mathfrak{C}_1$ and $\mathfrak{D}_2 = \mathfrak{C}_2$.

(2) $\Rightarrow$ (3) Let $M \in {}^{\perp}\mathfrak{C}_2$. For $X = \binom{X_1}{X_2}_{\varphi^X} \in \mathfrak{I}^{\mathfrak{C}_2}_{\mathfrak{C}_2}$, we get the following commutative diagram with the second row exact:

$$\begin{array}{ccccc}
\mathrm{Hom}_T(\mathbf{p}(M,0),X) & \longrightarrow & \mathrm{Hom}_T(\mathbf{p}(0,M),X) & & \\
\downarrow \cong & & \downarrow \cong & & \\
\mathrm{Hom}_R(M,X_1) & \longrightarrow & \mathrm{Hom}_R(M,X_2) & \longrightarrow & \mathrm{Ext}^1_R(M,\ker(\widetilde{\varphi^X}))=0.
\end{array}$$

So the exact sequence $0 \to \mathbf{p}(0,M) \to \mathbf{p}(M,0) \to \mathbf{h}(M,0) \to 0$ is $\mathrm{Hom}_T(-,\mathfrak{I}^{\mathfrak{C}_2}_{\mathfrak{C}_2})$-exact. By Lemma 2.1, it is also $\mathrm{Hom}_T(\mathfrak{P}^{\mathfrak{C}_1}_{\mathfrak{C}_1},-)$-exact. For $N \in \mathfrak{C}_1$, there is an exact sequence $0 \to K_1 \xrightarrow{\lambda} K_2 \to N \to 0$ with K_2 projective. Let $K = \binom{K_1}{K_2}_{\lambda}$. Then $K \in \mathfrak{P}^{\mathfrak{C}_1}_{\mathfrak{C}_1}$. Note that $\mathbf{p}(M,0) \cong \mathbf{h}(0,M)$. Hence $\mathrm{Hom}_T(K,\mathbf{h}(0,M)) \to \mathrm{Hom}_T(K,\mathbf{h}(M,0)) \to 0$ is exact. So $\mathrm{Hom}_R(K_2,M) \to \mathrm{Hom}_R(K_1,M) \to 0$ is exact. Hence $M \in N^{\perp}$. Thus ${}^{\perp}\mathfrak{C}_2 \subseteq \mathfrak{C}_1^{\perp}$.

Let $F \in \mathfrak{C}_1^{\perp}$. For $Y = \binom{Y_1}{Y_2}_{\varphi^Y} \in \mathfrak{P}^{\mathfrak{C}_1}_{\mathfrak{C}_1}$, we get the following commutative diagram with the second row exact:

$$\begin{array}{ccccc}
\mathrm{Hom}_T(Y,\mathbf{h}(0,F)) & \longrightarrow & \mathrm{Hom}_T(Y,\mathbf{h}(F,0)) & & \\
\downarrow \cong & & \downarrow \cong & & \\
\mathrm{Hom}_R(Y_2,F) & \longrightarrow & \mathrm{Hom}_R(Y_1,F) & \longrightarrow & \mathrm{Ext}^1_R(Y_2/\mathrm{im}(\varphi^Y),F)=0.
\end{array}$$

So the exact sequence $0 \to \mathbf{p}(0,F) \to \mathbf{h}(0,F) \to \mathbf{h}(F,0) \to 0$ is $\mathrm{Hom}_T(\mathfrak{P}^{\mathfrak{C}_1}_{\mathfrak{C}_1},-)$-exact. By Lemma 2.1, it is also $\mathrm{Hom}_T(-,\mathfrak{I}^{\mathfrak{C}_2}_{\mathfrak{C}_2})$-exact. For $G \in \mathfrak{C}_2$, there is an exact sequence $0 \to G \to L_1 \xrightarrow{\rho} L_2 \to 0$ with L_1 injective. Let $L = \binom{L_1}{L_2}_{\rho}$. Then $L \in \mathfrak{I}^{\mathfrak{C}_2}_{\mathfrak{C}_2}$. Note that $\mathbf{p}(F,0) \cong \mathbf{h}(0,F)$. Thus $\mathrm{Hom}_T(\mathbf{p}(F,0),L) \to \mathrm{Hom}_T(\mathbf{p}(0,F),L) \to 0$ is exact. So $\mathrm{Hom}_R(F,L_1) \to \mathrm{Hom}_R(F,L_2) \to 0$ is exact. Hence $F \in {}^{\perp}G$ and so $\mathfrak{C}_1^{\perp} \subseteq {}^{\perp}\mathfrak{C}_2$.

It follows that $\mathfrak{C}_1^{\perp} = {}^{\perp}\mathfrak{C}_2$. $\square$

Finally, as an application, we consider the balanced pairs of Gorenstein T-modules.

Recall that a left R-module M is *Gorenstein projective* [6] if there is an exact sequence $\cdots \to P^{-2} \to P^{-1} \to P^0 \to P^1 \to \cdots$ of projective left R-modules with $M = \ker(P^0 \to P^1)$, which remains exact after applying $\mathrm{Hom}_R(-,P)$ for any projective left R-module P.

Dually, a left R-module N is called *Gorenstein injective* [6] if there is an exact sequence $\cdots \to E^{-2} \to E^{-1} \to E^0 \to E^1 \to \cdots$ of injective left R-modules with $N = \ker(E^0 \to E^1)$, which remains exact after applying $\mathrm{Hom}_R(E,-)$ for any injective left R-module E.

We denote by ${}_R\mathcal{GP}$ (resp. ${}_R\mathcal{GI}$) the class of Gorenstein projective (resp. Gorenstein injective) left R-modules.

Suppose that U_A has finite flat dimension and ${}_BU$ has finite projective dimension. By [16, Remarks 3.11 and 4.11], a left T-module $M = \binom{M_1}{M_2}_{\varphi^M}$ is Gorenstein projective if and only if M_1 is a Gorenstein projective left A-module, $M_2/\mathrm{im}(\varphi^M)$ is a Gorenstein projective left B-module and φ^M is a monomorphism; $M = \binom{M_1}{M_2}_{\varphi^M}$ is Gorenstein injective if and only if M_2 is a Gorenstein injective left B-module, $\ker(\widetilde{\varphi^M})$ is a Gorenstein injective left A-module and $\widetilde{\varphi^M}$ is an epimorphism.

Corollary 2.7. *Suppose that U_A has finite flat dimension and ${}_BU$ has finite projective dimension. If $({}_T\mathcal{GP}, {}_T\mathcal{GI})$ is an admissible balanced pair, then $({}_A\mathcal{GP}, {}_A\mathcal{GI})$ and $({}_B\mathcal{GP}, {}_B\mathcal{GI})$ are admissible balanced pairs.*

Proof. It is an immediate consequence of Proposition 2.4. □

Recall that a ring R is *left virtually Gorenstein* [3] if $({}_R\mathcal{GP})^\perp = {}^\perp({}_R\mathcal{GI})$. Such rings were first introduced and studied in the context of representation theory of artin algebras by Beligiannis and Reiten. Examples of virtually Gorenstein rings include Iwanaga-Gorenstein rings and artin algebras of finite representation type.

Theorem 2.8. *Suppose that U_A is flat and ${}_BU$ is projective, A and B are left noetherian rings with finite left self-injective dimensions. The following conditions are equivalent*:

(1) *$({}_A\mathcal{GP}, {}_A\mathcal{GI})$ and $({}_B\mathcal{GP}, {}_B\mathcal{GI})$ are admissible balanced pairs.*
(2) *$({}_T\mathcal{GP}, {}_T\mathcal{GI})$ is an admissible balanced pair.*
(3) *$(\boldsymbol{p}({}_A\mathcal{GP}, {}_B\mathcal{GP}), \boldsymbol{h}({}_A\mathcal{GI}, {}_B\mathcal{GI}))$ is an admissible balanced pair.*
(4) *A and B are left virtually Gorenstein rings.*
(5) *T is a left virtually Gorenstein ring.*

Proof. Because A and B are left noetherian rings with finite left self-injective dimensions, all projective left A-modules and projective left B-modules have finite injective dimensions. By [18, Theorem 4.2], $({}_A\mathcal{GP}, ({}_A\mathcal{GP})^\perp)$ and $({}_B\mathcal{GP}, ({}_B\mathcal{GP})^\perp)$ are complete hereditary cotorsion pairs. Also by [17, Theorem 5.6], $({}^\perp({}_A\mathcal{GI}), {}_A\mathcal{GI})$ and $({}^\perp({}_B\mathcal{GI}), {}_B\mathcal{GI})$ are complete hereditary cotorsion pairs. By [7, p. 78], we have ${}_A\mathcal{GP}\bigcap({}_A\mathcal{GP})^\perp = {}_A\mathcal{P} \subseteq {}^\perp({}_A\mathcal{GI})$, ${}_A\mathcal{GI}\bigcap{}^\perp({}_A\mathcal{GI}) = {}_A\mathcal{I} \subseteq ({}_A\mathcal{GP})^\perp$, ${}_B\mathcal{GP}\bigcap({}_B\mathcal{GP})^\perp = {}_B\mathcal{P} \subseteq {}^\perp({}_B\mathcal{GI})$ and ${}_B\mathcal{GI}\bigcap{}^\perp({}_B\mathcal{GI}) = {}_B\mathcal{I} \subseteq ({}_B\mathcal{GI})^\perp$. Thus the result follows immediately from Theorems 2.2 and 2.5. □

Corollary 2.9. *Let R be a left noetherian ring with finite left self-injective dimension and $T(R) = \left(\begin{smallmatrix} R & 0 \\ R & R \end{smallmatrix}\right)$. The following conditions are equivalent*:

(1) *$({}_R\mathcal{GP}, {}_R\mathcal{GI})$ is an admissible balanced pair.*
(2) *$({}_{T(R)}\mathcal{GP}, {}_{T(R)}\mathcal{GI})$ is an admissible balanced pair.*
(3) *$(\boldsymbol{p}({}_R\mathcal{GP}, {}_R\mathcal{GP}), \boldsymbol{h}({}_R\mathcal{GI}, {}_R\mathcal{GI}))$ is an admissible balanced pair.*
(4) *R is a left virtually Gorenstein ring.*
(5) *$T(R)$ is a left virtually Gorenstein ring.*

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