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HEREDITARY AND SEMIHEREDITARY REPRESENTATIONS OF QUIVERS

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Abstract. In this paper, we investigate hereditary and semihereditary representations of quivers over an arbitrary ring. As consequences hereditary and semihereditary category of representations of quivers over an arbitrary ring are characterized.

1. Introduction

In this paper, we are interested in hereditary and semihereditary representations as well as hereditary and semihereditary category of representations of an acyclic quiver over a ring. Recall that the hereditary and semihereditary modules are introduced as a generalization of the classical hereditary and semihereditary rings by Shrikhande in [14] and then independently by Hill in [10]. There are modules whose all submodules (resp., finitely generated submodules) are projective. They have been widely studied by many authors (sometimes without mention the name as in [2], [11], [17]).

In a category with enough projectives, one can define hereditary (resp. semihereditary) objects as the ones whose all subobjects (resp., all finitely generated subobjects) are projective (see [3]). Thus, a category with enough projectives is hereditary (resp. semihereditary) if every projective object is hereditary (resp. semihereditary). Our aim in this paper is to investigate these notions in the category of representations of an acyclic quiver over a ring. In Section 3, we study hereditary representations as well as hereditary category of representations of quivers over arbitrary rings. Our first main result, Theorem 3.2, characterizes hereditary representations of left rooted quivers (see also Proposition 3.1). In general the equivalence of Theorem 3.2 fails if we drop the assumption that Q is a left rooted quiver (see Example 3.6). The second part of Section 3 is devoted to the study of hereditary category of representations. It is a well-known fact that, if R is a field and Q is acyclic, connected and has finite number of vertices and arrows, then the category (Q,R-Mod) is hereditary (see for instance

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[15, Theorem 1.7, page 248]). In this paper, as a consequence of Theorem 3.2, we show that the category (Q,R-Mod) is hereditary if and only if R is semisimple whenever Q is non-discrete and left rooted. As consequences, we find that, for a non-discrete finite acyclic quiver Q and a ring R, the path algebra RQ is hereditary if and only if R is semisimple. Then, in particular, for an integer $n \geq 2$, the triangular matrix algebra $T_n(R)$ is hereditary if and only if R is semisimple (see Corollaries 3.8 and 3.9).

Section 4 studies semihereditary representations and semihereditary category of representations. We start with Proposition 4.1 which gives a general characterization of a semihereditary representation that holds true even in any Grothendieck category having a family of finitely generated generators (see also Proposition 4.2). Then, for an acyclic quiver Q, we give necessary conditions of the fact that X is a semihereditary representation (see Proposition 4.3). In Theorem 4.4, we prove that the converse holds whenever the quiver is acyclic and has finite number of vertices and arrows. The last part of the paper studies semihereditary category of representations. In Proposition 4.5, we characterize when the category (Q,R-Mod) is semihereditary in terms of flatness and coherence for any quiver Q. Then, for a non-discrete acyclic quiver Q, we show that R is von Neumann regular if the category (Q,R-Mod) is semihereditary (see Proposition 4.6). For finite quivers, we get an equivalence (see Proposition 4.7). That is, for a non-discrete acyclic quiver Q which has a finite number of vertices and arrows and a ring R, the path algebra RQ is semihereditary if and only if R is von Neumann regular. In particular, for an integer $n \geq 2$, the triangular matrix algebra $T_n(R)$ is semihereditary if and only if R is von Neumann regular (see Corollary 4.8).

In the following section we give some preliminaries on the category of representations of a quiver over a ring.

2. Preliminaries

All rings considered in this paper will be associative with identity. Throughout this paper, R denotes a ring and all modules are left R-modules. The category of left R-modules is denoted by R-Mod.

We assume the reader has some familiarity with the theory of quiver representations by modules. Throughout this paper Q denotes a quiver and the category of representations of a Q by R-modules is denoted by $(Q, R\text{-Mod})$. Here, we use the notations of the book [8] (see also [15] for more details on the classical representation theory). Namely, see [8] for the definition of the useful functors $f_v^*, {g^*}_v$ and ${t^*}_v$ (resp., $f_v', {g'}_v$ and ${t'}_v$). They are used to prove among other things that $(Q, R\text{-Mod})$ has enough projectives (see also [7, Proposition 5.1.2]). For the reader convenience we recall the characterization of projective (resp. flat) representation given for left rooted quivers. Recall first that a quiver Q is said to be left rooted [8, Definition 3.5], if there exists an ordinal number λ such that the set of vertices $V = \cup_{\alpha<\lambda} V_\alpha$, where the subsets V_α are defined by transfinite induction as follows: $V_0 = \{v \in V$: there is no arrow of Q with $t(a) = v\}$. For a successor ordinal α, $V_\alpha = \{v \in V^{\alpha-1}$: there is no arrow a of $Q^{\alpha-1}$ with $t(a) = v\}$, where $Q^{\alpha-1} = (V^{\alpha-1}, E^{\alpha-1})$ is the subquiver of Q with $V^{\alpha-1} = V \setminus V_{\alpha-1}$ and $E^{\alpha-1} = E \setminus \{a \in E : i(a) \in V_{\alpha-1}\}$. For a limit ordinal ω, $V_\omega = \varinjlim V_\alpha = \cup_{\alpha<\omega} V_\alpha$. From [8, Proposition 3.6], a quiver Q is left rooted if and only if there exists no path of the form $\cdots \to \bullet \to \bullet$ in Q.

Theorem 2.1 ([5], Theorem 3.1)**.** *If a representation P of Q is projective, then the following assertions hold*:

(1) *$P(v)$ is a projective R-module for any vertex v of Q.*
(2) *For every vertex v, the morphism $\oplus_{t(a)=v} P(s(a)) \longrightarrow P(v)$, where the morphism $P(s(a)) \longrightarrow P(v)$ is $P(a)$, is a split monomorphism.*

The converse holds if Q is a left rooted quiver.

For flat representations we have the following characterization.

Theorem 2.2 ([8], Theorem 3.7)**.** *If a representation F of Q is flat, then the following assertions hold*:

(1) *$F(v)$ is a flat R-module for any vertex v of Q.*
(2) *For every vertex v the morphism $\oplus_{t(a)=v} F(s(a)) \longrightarrow F(v)$, where the morphism $F(s(a)) \longrightarrow F(v)$ is $F(a)$, is a pure monomorphism.*

The converse holds if Q is a left rooted quiver.

3. Hereditary representations

In this section we investigate hereditary representations and hereditary category of representations. We start with the following result which establishes a necessary conditions so that a representation be hereditary.

Proposition 3.1. *Let Q be an acyclic quiver and X be a hereditary representation of Q. Then the following statements hold*:

(1) *$X(v)$ is a hereditary R-module for any vertex $v \in V$.*
(2) *$X(v)$ is a semisimple R-module for any vertex v which is not a sink.*
(3) *For every vertex v, the morphism $\oplus_{t(a)=v} X(s(a)) \longrightarrow X(v)$, where the morphism $X(s(a)) \longrightarrow X(v)$ is $X(a)$, is a split monomorphism.*

Proof. (1) Let v be a vertex and H be an R-submodule of $X(v)$ and consider the canonical injection $h : H \longrightarrow X(v)$. Let us define a representation $\overline{H}$ as follows:

$$\begin{cases} \overline{H}(v) = H, & \\ \overline{H}(u) = X(u) & \text{if there exists a path } p \text{ from } v \text{ to } u, \\ 0 & \text{otherwise.} \end{cases}$$

And

$$\begin{cases} \overline{H}(a) = X(a)h & \text{if } s(a) = v, \\ \overline{H}(a) = X(a) & \text{if there exists a path } p \text{ from } v \text{ to } s(a), \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $\overline{H}$ is a subrepresentation of X. By hypothesis, $\overline{H}$ is a projective representation. Then, H is a projective R-module.

(2) Suppose that v is not a sink. Then, there exists an arrow $a : v \to w$ and so the induced morphism $\oplus_{t(\alpha)=w}\overline{H}(u) \longrightarrow \overline{H}(w)$ exists. Then, by Theorem 2.1, it is a split monomorphism. Then, for any arrow α such that $t(\alpha) = w$, $\overline{H(\alpha)}$ is also a split monomorphism. Then, $\overline{H(a)} = X(a)h$ is a split monomorphism, and so is h. This shows that $X(v)$ is a semisimple R-module.

(3) holds by Theorem 2.1 since X is projective. □

The previous proposition shows that the conditions (1), (2) and (3) are necessary for a representation to be hereditary for any acyclic quiver. Now, we are going to show that such conditions are sufficient for any left rooted quivers. Afterwards, we will give an example of a non-left rooted quiver with which the conditions fails to be sufficient.

Theorem 3.2. *Let Q be a left rooted quiver. Then, a representation X is hereditary if and only if the following statements hold*:

(1) *$X(v)$ is a hereditary R-module for any vertex $v \in V$.*
(2) *$X(v)$ is a semisimple R-module for any vertex v does not being a sink.*
(3) *For every vertex v the morphism $\oplus_{t(a)=v}X(s(a)) \longrightarrow X(v)$, where the morphism $X(s(a)) \longrightarrow X(v)$ is $X(a)$, is a split monomorphism.*

Proof. (⇒) is Proposition 3.1.
(⇐) We suppose that X satisfies (1), (2) and (3). Let T be a subrepresentation of X and consider the canonical injection $\phi : T \longrightarrow X$. Let v be a vertex of Q. Then $T(v)$ is an R-submodule of $X(v)$, so $T(v)$ is a projective R-module (since X satisfies (1)). Now, for any arrow $a : u \to v$ we have the following diagram:

$$\begin{array}{ccc} T(u) & \xrightarrow{\phi_u} & X(u) \\ \downarrow{\scriptstyle T(a)} & & \downarrow{\scriptstyle X(a)} \\ T(v) & \xrightarrow[\phi_v]{} & X(v) \end{array}$$

Thus, we get the induced diagram:

$$\begin{array}{ccc} \oplus_{t(a)=v}T(u) & \xrightarrow{f} & \oplus_{t(a)=v}X(u) \\ \downarrow{\scriptstyle \psi_{T(v)}} & & \downarrow{\scriptstyle \psi_{X(v)}} \\ T(v) & \xrightarrow[\phi_v]{} & X(v) \end{array}$$

where $f := \oplus_{t(a)=v}\phi_{s(a)}$ and $\psi_{X(v)}$ and $\psi_{T(v)}$ are the induced morphisms by $X(a) : X(s(a)) \longrightarrow X(v)$ and $T(a) : T(s(a)) \longrightarrow T(v)$. Thus, $\psi_{X(v)}f = \phi_v\psi_{T(v)}$. From (2), $\phi_{s(a)}$ is a split monomorphism for any arrow $a : u \to v$. Thus, f is also a split monomorphism. From (3), $\psi_{X(v)}$ is also a split monomorphism. Then, $\psi_{T(v)}$ is a split monomorphism, and so T is projective. Therefore, X is hereditary. □

Corollary 3.3. *For the quiver $Q \equiv \bullet \to \bullet$, a representation $X \equiv X(a) : X(v) \to X(w)$ is hereditary if and only if $X(v)$ and $X(w)$ are hereditary R-modules, $X(w)$ is a semisimple R-module and $X(a)$ is a split monomorphism.*

Corollary 3.4. *Let Q be an acyclic quiver, v a vertex of Q and M an R-module.*

(1) *If v is not a sink vertex, then $f'_v(M)$ is a hereditary representation if and only if M is a semisimple R-module.*
(2) *If v is a sink vertex, then $f'_v(M)$ is a hereditary representation if and only if M is a hereditary R-module.*

Now, we are going to give an example to show that the characterization of hereditary representations given in Theorem 3.2 is in general not true if Q is not a left rooted quiver. For that we recall the characterization of projective representations of the quiver $\mathbb{A}_\infty \equiv \cdots \bullet \to \bullet \to \bullet$.

Proposition 3.5 ([5], Theorem 4.1)**.** *A representation*

$$P \equiv \cdots P_{n+1} \xrightarrow{f_n} P_n \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} P_1 \xrightarrow{f_0} P_0$$

of $\mathbb{A}_\infty$ is projective if and only if the following conditions hold:

(1) *For every $n \in \mathbb{N}$, P_n is a projective R-module.*
(2) *For every $n \in \mathbb{N}$, the morphism $f_n : P_{n+1} \longrightarrow P_n$ is a split monomorphism.*
(3) *For every $n \in \mathbb{N}$, there exists a retraction α_n of f_n such that, if $x \in P_n$, then there exists a positive integer $k \geq 1$ such that $\alpha_{n+k} \cdots \alpha_{n+1}(x) = 0$.*

Example 3.6. Let R be a semisimple ring. We consider the representation

$$X_R \equiv \ldots R \xrightarrow{id} R \xrightarrow{id} R$$

of $\mathbb{A}_\infty$ which obviously satisfies the conditions of Theorem 3.2, however X_R is not projective due to Proposition 3.5. Therefore, X_R is not hereditary.

The category (Q,R-Mod) is said to be hereditary if every projective representation of Q is hereditary.

Proposition 3.7. *Let Q be a non-discrete left rooted quiver. Then, the category $(Q, R\text{-Mod})$ is hereditary if and only if the ring R is semisimple.*

Proof. ($\Longrightarrow$) Let v be a vertex of Q. We have $f'_v(R)$ is hereditary, since such a representation is projective. As Q is not discrete, we can suppose v to be not a sink. Therefore, $f'_v(R)(v) = R$ is semisimple.

($\Longleftarrow$) Let X be a projective representation. We prove that X is hereditary. We prove that X satisfies the three conditions of Theorem 3.2. Indeed, for any vertex v of the quiver Q, $X(v)$ is a semisimple R-module. Then, $X(v)$ is a hereditary R-module, since $X(v)$ is projective. This shows that the conditions (1) and (2) are satisfied. The condition (3) is also satisfied since X is projective. Therefore, X is hereditary. $\square$

It is a well-known fact that the path algebra kQ, for a field k, is hereditary, when Q is a finite acyclic quiver [15, Theorem 1.7, page 248]. Now using Proposition 3.7 this result is generalized as follows.

Corollary 3.8. *Let Q be a non-discrete finite acyclic quiver. The path algebra RQ is hereditary if and only if R is semisimple.*

Proof. It suffices to recall that, under the assumption, the category RQ-Mod is equivalent to (Q,R-Mod) and then apply Proposition 3.7. $\square$

We point out that we can also get the necessary condition of Corollary 3.8 using [1, Corollary 3.2].

It is well-known that, for a positive integer n, the triangular matrix algebra over R

$$T_n(R) := \begin{pmatrix} R & R & \dots & R \\ \vdots & \ddots & & \vdots \\ 0 & \dots & R & R \\ 0 & \dots & 0 & R \end{pmatrix}$$

is isomorphic to the path algebra RA_n, where A_n is the line quiver

$$A_n: \quad 1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n-1 \longrightarrow n.$$

Thus, as a direct consequence of Corollary 3.8, we get again a characterization of when the triangular matrix algebra $T_n(R)$ over a ring R is hereditary. Notice that the inverse implication of the following result is a classical fact (see [4, Proposition 13]).

Corollary 3.9. *For an integer $n \geq 2$, the triangular matrix algebra $T_n(R)$ is hereditary if and only if R is semisimple.*

4. Semihereditary representations

This section is devoted to the study of semihereditary representations and semihereditary category of representations.

Let us start with a general characterization of semihereditary representations. Notice that this characterization holds for any semihereditary object in a Grothendieck category having a family of finitely generated generators.

Proposition 4.1. *Let Q be a quiver. A representation X is semihereditary if and only if the following three conditions are satisfied*:

(1) *X is a flat representation.*
(2) *Any subrepresentation T of X is flat.*
(3) *Any finitely generated subrepresentation T of X is finitely presented.*

Proof. For the "IF" part. We suppose that X is semihereditary. We know that (Q,R-Mod) is a Grothendieck category and has a family of finitely generated generators (see for instance [6, page 3]). Then, X is the sum of a family of its finitely generated subrepresentations ([12, Lemma 2, page 205]). By hypothesis, they are projectives, so the representation X is flat and hence the first condition holds. The second condition holds too, since, for any subrepresentation T of X, any finitely generated subrepresentation of T is also a finitely generated subrepresentation of X. Consequently, T is flat since it is the sum of a family of projective representations. To prove the third condition, consider T to be a finitely generated representation of Q. Then, there exists an epimorphism ϕ of representations of the form

$$\phi : \bigoplus_{i \in I} f'_{v_i}(R^{n_i}) \longrightarrow T\,,$$

where I is a finite set and, for any $i \in I$, v_i is a vertex of Q and n_i is an integer. We suppose that T is a subrepresentation of X. Then, by hypothesis, ϕ is a split epimorphism, then $\bigoplus_{i \in I} f'_{v_i}(R^{n_i}) = K \bigoplus T$, where K is the kernel of ϕ. Therefore, K is finitely generated, as desired.

For the "ONLY IF" part, consider T to be any finitely generated subrepresentation of X. Then, using the conditions (2) and (3), we deduce that T is a finitely presented flat representation. Therefore, by [16, Theorem 3 and Lemma 7(ii)], T is projective. □

For any left rooted quiver, we get the following characterization of representations that satisfy the conditions (1) and (2) of Proposition 4.1. We need to recall the following notion.

Definition ([9])**.** We say that an R-module M is Fieldhouse-regular if all its submodules are pure.

Proposition 4.2. *Let Q be a left rooted quiver and X be a representation. Then, X satisfies* (1) *and* (2) *of Proposition 4.1 if and only if the following three conditions are satisfied*:

(1) *For any vertex $v \in Q$, $X(v)$ is a flat R-module and any submodule of it is also flat.*
(2) *For any vertex $v \in Q$ does not being a sink, $X(v)$ is a Fieldhouse-regular R-module.*
(3) *The induced homomorphism $\psi_{X(v)} : \bigoplus_{t(a)=v} X(s(a)) \longrightarrow X(v)$ is a pure monomorphism.*

Proof. ($\Rightarrow$) (1) Let H be an R-submodule of $X(v)$, and consider the canonical injection $h : H \longrightarrow X(v)$. We consider the representation $\overline{H}$ defined as follows:

$$\begin{cases} \overline{H}(v) = H, & \\ \overline{H}(u) = X(u) & \text{if there exists a path } p \text{ from } v \text{ to } u, \\ 0 & \text{otherwise.} \end{cases}$$

And

$$\begin{cases} \overline{H}(a) = X(a)h & \text{if } s(a) = v, \\ \overline{H}(a) = X(a) & \text{if there exists a path } p \text{ from } v \text{ to } s(a), \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $\overline{H}$ is a subrepresentation of X, so by hypothesis $\overline{H}$ is a flat representation, then H is a flat R-module (by Theorem 2.2). Therefore, (1) holds since, by Theorem 2.2, $X(v)$ is also flat.

(2) Suppose that v is not a sink. Then, there exists an arrow of the form $a : v \to w$. Then, the induced morphism $\oplus_{t(\alpha)=w}\overline{H}(u) \longrightarrow \overline{H}(w)$ is a pure monomorphism. So, for any arrow α such that $t(\alpha) = w$, $\overline{H}(\alpha)$ is also a pure monomorphism, then $\overline{H}(a) = X(a)h$ is a pure monomorphism. Therefore, h is also a pure monomorphism. This proves the second condition.

(3) holds since X is flat.

($\Leftarrow$) Let T be a subrepresentation of X and consider the canonical injection $\phi : T \longrightarrow X$. For a vertex v of Q, $T(v)$ is an R-submodule of $X(v)$, so $T(v)$ is a flat R-module. Now, for any arrow $a : u \to v$, we have the following diagram:

$$\begin{array}{ccc} T(u) & \xrightarrow{\phi_u} & X(u) \\ \downarrow{\scriptstyle T(a)} & & \downarrow{\scriptstyle X(a)} \\ T(v) & \xrightarrow[\phi_v]{} & X(v) \end{array}$$

Thus, we get the induced diagram:

$$\begin{array}{ccc} \oplus_{t(a)=v}T(u) & \xrightarrow{f} & \oplus_{t(a)=v}X(u) \\ \downarrow{\scriptstyle \psi_{T(v)}} & & \downarrow{\scriptstyle \psi_{X(v)}} \\ T(v) & \xrightarrow[\phi_v]{} & X(v) \end{array}$$

where $f := \oplus_{t(a)=v}\phi_{s(a)}$ and $\psi_{X(v)}$ and $\psi_{T(v)}$ are the induced morphisms by $X(a) : X(s(a)) \longrightarrow X(v)$ and $T(a) : T(s(a)) \longrightarrow T(v)$. Thus, we have $\psi_{X(v)}f = \phi_v\psi_{T(v)}$. Using (2), $\phi_{s(a)}$ is a pure monomorphism. Therefore, f also is a pure monomorphism. And using (3), $\psi_{X(v)}$ is a pure monomorphism. Then, $\psi_{T(v)}$ is a pure monomorphism. Therefore, T is flat and consequently X satisfies (1) and (2) of Proposition 4.1. □

Now we give necessary conditions of a representation to be semihereditary for any acyclic quiver. We will show that such conditions are sufficient for an important class of quivers.

Proposition 4.3. *Let Q be an acyclic quiver and X be a semihereditary representation. Then, the three following conditions are satisfied*:

(1) *For any vertex $v \in Q$, $X(v)$ is a semihereditary R-module.*

(2) *For any vertex $v \in Q$ does not being a sink, $X(v)$ is a Fieldhouse-regular R-module.*
(3) *The induced homomorphism $\psi_{X(v)} : \bigoplus_{t(a)=v} X(s(a)) \longrightarrow X(v)$ is a pure monomorphism.*

Proof. From Propositions 4.1 and 4.2, it is clear that (2) and (3) are satisfied.

We prove the assertion (1). Let $v \in Q$ be a vertex and T_v be a finitely generated R-submodule of $X(v)$. We associate a representation $\overline{T_v}$ to T_v as follows: For any vertex w, $\overline{T_v}(w) = \sum_{p \in Y_w^v} X(p)(T_v)$ and for any arrow $a : w_1 \to w_2$, the morphism $\overline{T_v}(a) : \overline{T_v}(w_1) \longrightarrow \overline{T_v}(w_2)$ is defined by the formula

$$\overline{T_v}(a)(t_p) = X(a)X(p)(t'_v)$$

for any $t_p = X(p)(t'_v) \in X(p)(T_v)$, where $p \in Y_w^v$. Let us prove that $\overline{T_v}$ is a finitely generated subrepresentation of X. First, we prove that $\overline{T_v}$ is a subrepresentation of X. Indeed, let w be a vertex of Q. It is clear that $\overline{T_v}(w) = \sum_{p \in Y_w^v} X(p)(T_v)$ is a submodule of $X(w)$, since $X(p)(T_v)$ is a submodule of $X(w)$ for any $p \in Y_w^v$. Now, we show that for any arrow $a : w_1 \to w_2$ of Q, the following diagram

$$\begin{array}{ccc} \overline{T_v}(w_1) & \xrightarrow{\iota_{w_1}} & X(w_1) \\ \downarrow{\scriptstyle \overline{T_v}(a)} & & \downarrow{\scriptstyle X(a)} \\ \overline{T_v}(w_2) & \xrightarrow[\iota_{w_2}]{} & X(w_2) \end{array}$$

is commutative, where ι_{w_1} and ι_{w_2} are the canonical injections. Let $t'_v \in T_v$ and $t_p = X(p)(t'_v) \in X(p)(T_v)$, where $p \in Y_w^v$. Then,

$$\begin{aligned} X(a)\iota_{w_1}(t_v) &= X(a)\iota_{w_1}X(p)(t'_v) \\ &= X(a)X(p)(t'_v) \\ &= X(a)X(p)(t'_v) \\ &= \iota_{w_1}X(ap)(t'_v) \\ &= \iota_{w_1}\overline{T_v}(w_2)(t_v). \end{aligned}$$

Then, $X(a)\iota_{w_1} = \iota_{w_1}\overline{T_v}(w_2)$. Thus, the diagram above is commutative. Consequently, the representation $\overline{T_v}$ is a subrepresentation of X. Now, we show that $\overline{T_v}$ is finitely generated. We have already that T_v is finitely generated, then there exists an epimorphism $\phi_v : R^n \longrightarrow T_v$ such that n is a positive integer. For any vertex w of Q, we set $\phi_w := \psi_w \bigoplus_{p \in Y_w^v} \phi_v$, where $\psi_w : \bigoplus_{p \in Y_w^v} T_v \longrightarrow \overline{T_v}(w)$ is the homomorphism induced by $\overline{T_v}(p) : T_v \longrightarrow \overline{T_v}(w)$. To show that ϕ_w is epic it suffices to remark that ϕ_w is the composition of two epic homomorphisms. Now we show that $\phi := \{\phi_u\}_{u \in Q} : f'_v(R^n) \longrightarrow \overline{T_v}$ is a homomorphism of representations. That is, for any arrow $a : w_1 \to w_2$, the following diagram

$$\begin{array}{ccc} \bigoplus_{p\in Y^v_{w_1}} R^n & \xrightarrow{\phi_{w_1}} & \overline{T_v}(w_1) \\ {\scriptstyle h(a)}\downarrow & & \downarrow{\scriptstyle \overline{T_v}(a)} \\ \bigoplus_{p\in Y^v_{w_2}} R^n & \xrightarrow[\phi_{w_2}]{} & \overline{T_v}(w_2) \end{array}$$

is commutative. We note by $h(a)$ the corresponding homomorphism of $f'_v(R^n)$ at the arrow $a : w_1 \longrightarrow w_2$. Indeed, for any $t \in R^n$, we denote by $t_p = (0, \dots, t, 0, \dots) \in p \in Y^v_w R^n$ with t is in the p-th position. Then,

$$\begin{aligned} \phi_{w_2} h(a)(t_p) &= \psi_{w_2} \bigoplus_{p\in Y^v_{w_2}} \phi_v h(a)(t_p) \\ &= \psi_{w_2} \bigoplus_{p\in Y^v_{w_2}} \phi_v(t_{ap}) \\ &= \psi_{w_2}(0, \dots, \phi_v(t), 0, \dots) \\ &= X(ap)(\phi_v)(t) \\ &= X(a)X(p)(\phi_v)(t) \\ &= \overline{T_v}(a)\psi_{w_1}(((\phi_v)(t))_p) \\ &= \overline{T_v}(a)\psi_{w_1}(\bigoplus_{p\in Y^v_{w_1}} \phi_v)(t_p) \\ &= \overline{T_v}(a)\phi_{w_1}(t_p). \end{aligned}$$

Then, $\phi_{w_2} h(a) = \overline{T_v}(a)\phi_{w_1}$. Thus, the diagram above is commutative and so $\phi : f'_v(R^n) \longrightarrow \overline{T_v}$ is a homomorphism of representations. In addition, ϕ is epic. Hence, $\overline{T_v}$ is a finitely generated representation. Also, $\overline{T_v}$ is a subrepresentation of X, so $\overline{T_v}$ is projective. Thus, T_v is a projective R-module. Therefore, $X(v)$ is a semihereditary R-module. This completes the proof. □

When Q is an acyclic quiver which has a finite number of vertices and arrows, a representation T of Q is finitely generated (resp., finitely presented) if and only if $T(v)$ is a finitely generated (resp., a finitely presented) R-module for any vertex $v \in Q$. Using this fact, we have a situation where the converse implication of Proposition 4.3 holds as shown by the following result.

Theorem 4.4. *Let Q be an acyclic quiver which has a finite number of vertices and arrows and X be a representation of Q. Then, X is semihereditary if and only if the following three conditions are satisfied*:

(1) *For any vertex $v \in Q$, $X(v)$ is a semihereditary R-module.*
(2) *For any vertex $v \in Q$ does not being a sink, $X(v)$ is a Fieldhouse-regular R-module.*

(3) *The induced homomorphism* $\psi_{X(v)} : \bigoplus_{t(a)=v} X(s(a)) \longrightarrow X(v)$ *is a pure monomorphism.*

Proof. From Proposition 4.3, it remains to prove the converse implication. To this end, we prove that X satisfies the three conditions of Proposition 4.1. For any vertex $v \in Q$, $X(v)$ is a semihereditary R-module, then $X(v)$ is a flat R-module and all its R-submodules are also flat. In addition, Q has a finite number of vertices and arrows, then Q is left rooted, so from Proposition 4.2, X is flat and all its subrepresentations are also flat. It remains to show that any finitely generated subrepresentation of X is finitely presented as well. Let T be a finitely generated subrepresentation of X. Then, for any vertex v of Q, $T(v)$ is a finitely generated R-submodule of $X(v)$. Thus, $T(v)$ is a finitely presented R-module for any vertex v of Q, since $X(v)$ is a semihereditary R-module. Then, the representation T is finitely presented. This proves that X is semihereditary, as desired. □

The category (Q,R-Mod) is said to be semihereditary if every projective representation of Q is semihereditary. As in ring theory context, semihereditary categories is related with coherence property.

Definition ([13], Definition 1, page 203)**.** An object C in a Grothendieck category $\mathfrak{C}$ is said to be coherent, if C is finitely generated and the kernel of any morphism $f : C' \longrightarrow C$ is also finitely generated.

The category $\mathfrak{C}$ is said to be locally coherent, if it has a family of coherent generators.

Proposition 4.5. *Let Q be a quiver. Then, the category $(Q, R\text{-}Mod)$ is semihereditary if and only if $(Q, R\text{-}Mod)$ is locally coherent and any subrepresentation of flat representation is also flat.*

Proof. ⇒) We suppose that (Q,R-Mod) is semihereditary. Let X be a flat representation of Q. Let T be a subrepresentation of X and consider the canonical injection $\phi : T \longrightarrow X$. We prove that T is flat. To this end, we prove that any homomorphism $f : S \longrightarrow T$, with S is a finitely presented representation, factorizes through a projective representation. Since X is flat, there exists a projective representation P such that ϕf is factorized through P; that is, there exist two homomorphisms $g : S \longrightarrow P$ and $\varphi : P \longrightarrow X$ such that $\phi f = \varphi g$. Consider the pullback of ϕ and φ:

$$\begin{array}{ccc} G & \xrightarrow{\varphi'} & P \\ {\scriptstyle \phi'}\downarrow & & \downarrow{\scriptstyle \varphi} \\ T & \xrightarrow[\phi]{} & X \end{array}$$

Then, there exists a homomorphism $\pi : S \longrightarrow G$ such that $\varphi'\pi = g$ and $\phi'\pi = f$. Thus, $\varphi' : G \longrightarrow P$ is a monomorphism, since $\phi : T \longrightarrow X$ is a monomorphism. Then, G is a flat representation, since G is a subrepresentation of a projective representation. Then, $\pi : S \longrightarrow G$ is factorized through a projective representation P'. Therefore, f is factorized through P' since $\phi'\pi = f$. Now, we show that (Q,R-Mod) is locally coherent; that is, the projective representation $f'_v(R)$ is coherent for any vertex v of Q. Indeed, let T be a finitely generated subrepresentation of $f'_v(R)$. By hypothesis, $f'_v(R)$ is semihereditary, then T is projective, and so T is finitely presented. Therefore, $f'_v(R)$ is coherent.

$\Leftarrow$) We suppose that (Q,R-Mod) is locally coherent and any subrepresentation of a flat representation is also flat. Let X be a projective representation and T be a finitely generated subrepresentation of X. We prove that T is also projective. We know that the family $\{f'_v(R)\}$ generates the category (Q,R-Mod), so there exists an epimorphism $\psi : \bigoplus_{i\in I} f'_{v_i}(R) \longrightarrow X$. Then, ψ is a split epimorphism, since X is projective. So, X is isomorphic to a subrepresentation of $\bigoplus_{i\in I} f'_{v_i}(R)$. Also, T is isomorphic to a subrepresentation of $\bigoplus_{i\in I} f'_{v_i}(R)$. So, T is flat since the representation $\bigoplus_{i\in I} f'_{v_i}(R)$ is projective. In addition, since T is finitely generated, there exists a finite subset J of I such that T is isomorphic to a subrepresentation of $\bigoplus_{i\in J} f'_{v_i}(R)$. Then, T is finitely presented. Therefore, T is projective. □

The following result provides a sufficient condition so that (Q,R-Mod) be semihereditary.

Proposition 4.6. *Let Q be a non-discrete acyclic quiver. If the category (Q, R-Mod) is semihereditary, then R is a von Neumann regular ring.*

Proof. Let v be a vertex of Q. We have $f'_v(R)$ is semihereditary since such a representation is projective. Since Q is non-discrete we can suppose v to not be a sink. Then, $f'_v(R)(v) = R$ is a Fieldhouse-regular R-module. Consequently, R is von Neumann regular. □

When Q is finite, we get an equivalence.

Proposition 4.7. *Let Q be a non-discrete acyclic quiver which has a finite number of vertices and arrows. Then, the path algebra RQ is semihereditary if and only if R is von Neumann regular.*

Proof. Showing that RQ is semihereditary is equivalent to show that the category (Q,R-Mod) is semihereditary. We only need to prove the converse implication. So, suppose that R is a von Neumann regular ring. We prove that (Q,R-Mod) is locally coherent and any subrepresentation of a flat representation is flat too. We first prove that (Q,R-Mod) is locally coherent; equivalently, $f'_v(R)$ is coherent for any vertex v of Q. Let v be a vertex of Q and X be a finitely generated subrepresentation of $f'_v(R)$. Then, for any vertex w of Q, $X(w)$ is a finitely generated submodule of a finitely generated free R-module. Since R is von Neumann regular, it is coherent, and so $X(w)$ is a finitely presented R-module. Then, X is a finitely presented representation. This, implies that $f'_v(R)$ is a coherent representation, and therefore the category (Q,R-Mod) is locally coherent. Now, consider a flat representation X. By hypothesis, $X(v)$ is a Fieldhouse-regular R-module for any vertex v of Q. So, it is clear that X satisfies the conditions of Proposition 4.2. Then, any subrepresentation of X is flat. Therefore, from Proposition 4.5, (Q,R-Mod) is semihereditary. □

Thus, as a direct consequence of Corollary 3.8, we get a characterization of when the triangular matrix algebra $T_n(R)$ over R is semihereditary. Unlike the hereditary case, [4, Proposition 13], the authors were unable to find a reference which investigates this question even though it seems to be natural.

Corollary 4.8. *For an integer $n \geq 2$, the triangular matrix algebra $T_n(R)$ is semihereditary if and only if R is von Neumann regular.*

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