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ESTIMATES FOR EIGENVALUES OF NEUMANN AND NAVIER PROBLEM

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Abstract. In this paper, we firstly prove some general inequalities for the Neumann eigenvalues for domains contained in a Euclidean n-space $\mathbb{R}^n$. Using the general inequalities, we can derive some new Neumann eigenvalues estimates which include an upper bound for the $(k+1)^{th}$ eigenvalue and a new estimate for the gap of the consecutive eigenvalues. Moreover, we give sharp lower bound for the first eigenvalue of two kinds of eigenvalue problems of the biharmonic operator with Navier boundary condition on compact Riemannian manifolds with boundary and positive Ricci curvature.

1. Introduction

Let Ω be a bounded domain with boundary $\partial\Omega$ in an n-dimensional Euclidean space $\mathbb{R}^n$, and let Δ be the Laplace operator on $\mathbb{R}^n$. We can consider the following Dirichlet problem

$$\begin{cases} \Delta u = -\lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

It is well known that this eigenvalue problem has a discrete spectrum,

$$0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots ,$$

where each eigenvalue is repeated with its multiplicity. When Ω is a bounded domain in a 2-dimensional Euclidean space $\mathbb{R}^2$, Payne-Pólya-Weinberger [7] in 1956 gave an upper bound of the gap of consecutive eigenvalues of the problem

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(1.1) that

$$\lambda_{k+1} - \lambda_k \le \frac{4}{kn} \sum_{i=1}^{k} \lambda_i. \tag{1.2}$$

In 1969, Thompson [9] extended (1.2) to n-dimensional case and obtained

$$\lambda_{k+1} - \lambda_k \le \frac{4}{kn} \sum_{i=1}^{k} \lambda_i. \tag{1.3}$$

In 1980, Hile-Protter [4] strengthened (1.3), and proved

$$\frac{kn}{4} \le \sum_{i=1}^{k} \frac{\lambda_i}{\lambda_{k+1} - \lambda_i}. \tag{1.4}$$

In 1991, Yang [11] gave the following much stronger inequality

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \le \frac{4}{n} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)\lambda_i. \tag{1.5}$$

From (1.5), we can get a weaker but explicit form

$$\lambda_{k+1} \le \left(1 + \frac{4}{n}\right) \frac{1}{k} \left(\sum_{i=1}^{k} \lambda_i\right). \tag{1.6}$$

These inequalities (1.2)-(1.6) are called universal inequalities because they do not involve domain dependence.

Let Ω be a bounded domain with boundary $\partial\Omega$ in an n-dimensional Euclidean space $\mathbb{R}^n$, we consider the following Neumann problem

$$\begin{cases} \Delta u = -\mu u, & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

where ν is the outward unit normal vector field of the boundary $\partial\Omega$. The Neumann problem (1.7) also has a discrete spectrum:

$$0 = \mu_1 < \mu_2 \le \cdots \le \mu_k \le \cdots,$$

where each eigenvalue is repeated with its multiplicity.

For the Neumann eigenvalues, some mathematicians want to derive similar results with (1.2). In [3], Harrell-Michel gave a estimate for the gap of the consecutive eigenvalues of (1.7)

$$\mu_{n+1} - \mu_n \le (n+1) \left(\frac{A}{n} \sum_{l=1}^{n} \mu_l + B\right), \tag{1.8}$$

where A and B depend only on the inradius r and Ω.

Inspired by the work of Harrell-Michel [3] and Chung-Grigor'ya-Yau [2], Levitin-Parnovski [6] gave a new bound for the gap of the consecutive eigenvalues of (1.7) as following

$$\mu_{k+1}-\mu_k \le \frac{C_3|\Omega|}{\sum_{p=1}^{q} r_p^{-2+n}}\left(r_q^{-2}\sum_{i=1}^{k}\mu_i+\sum_{p=1}^{q} r_p^{-4}\right), \tag{1.9}$$

where $|\Omega|$ is the volume of Ω, C_3 is a constant depending only on n, and $\{r_j\}_{j=1}^q$ are radii of q balls $B_p = B(x_p, r_p)$ inside Ω such that $r_1 \ge r_2 \ge \cdots \ge r_q$ and these balls do not intersect each other. Assuming that all the radii $\{r_j\}_{j=1}^q$ are the same, they got

$$\mu_{k+1}-\mu_k \le C_4|\Omega| r_q^{-n}\left(\frac{1}{q}\sum_{i=1}^{k}\mu_i + r_q^{-2}\right), \tag{1.10}$$

where C_4 is also a constant depending only on n.

In this paper, we will give some new estimates for the eigenvalues of the Neumann problem (1.7).

Theorem 1.1. *Let Ω be a bounded domain in an n-dimensional Euclidean space $\mathbb{R}^n$ with piecewise smooth boundary $\partial\Omega$, and let μ_i be the i^{th} eigenvalue of the Neumann problem* (1.7). *Then we have*

$$\begin{aligned}\left(\sum_{i=1}^{k} f(\mu_i)\right)^2 \le\; & \frac{|\Omega|^2}{\xi^2\left(\sum_{p=1}^{q} r_p^{-2+n}\right)^2}\left(\sum_{i=1}^{k} g(\mu_i) r_q^{-2}\right)\\ & \times\left(\sum_{i=1}^{k}\frac{f^2(\mu_i)}{g(\mu_i)(\mu_{k+1}-\mu_i)}\left(8r_q^{-2}\mu_i+\sum_{p=1}^{q}2\xi^2 r_p^{-4}\right)\right),\end{aligned} \tag{1.11}$$

where (f, g) is a family of couples of functions which is defined in Definition 1, and ξ is a constant. Especially, if $f \equiv g$, we have

$$\sum_{i=1}^{k} f(\mu_i) \le \frac{2|\Omega|}{\xi\sum_{p=1}^{q} r_p^{-2+n}}\sum_{i=1}^{k}\frac{f(\mu_i)}{(\mu_{k+1}-\mu_i)}\left(4r_q^{-2}\mu_i+\sum_{p=1}^{q}\xi^2 r_p^{-4}\right). \tag{1.12}$$

By choosing different $(f(\mu_i), g(\mu_i))$, we can get different eigenvalue inequalities. For example, if $f(\mu_i) = (\mu_{k+1}-\mu_i)^2$ in (1.12), we can get:

Corollary 1.2. *Under the same assumption in Theorem 1.1, we have*

$$\sum_{i=1}^{k}(\mu_{k+1}-\mu_i)^2 \le \frac{2|\Omega|}{\xi\sum_{p=1}^{q} r_p^{-2+n}}\sum_{i=1}^{k}(\mu_{k+1}-\mu_i)\left(4r_q^{-2}\mu_i+\sum_{p=1}^{q}\xi^2 r_p^{-4}\right). \tag{1.13}$$

Remark 1.3. i) Let $A = \frac{4|\Omega|}{\xi r_q^2 \sum_{p=1}^q r_p^{-2+n}}$, $B = \frac{\xi^2 r_q^2 \sum_{p=1}^q r_p^{-4}}{4}$, and let $\eta_i = \mu_i + B$, we infer from (1.13) that

$$\sum_{i=1}^{k} (\eta_{k+1} - \eta_i)^2 \leq 2A \sum_{i=1}^{k} (\eta_{k+1} - \eta_i)\, \eta_i, \tag{1.14}$$

solving quadratic inequality of η_{k+1}, we have

$$\eta_{k+1} \leq \frac{A+1}{k} \sum_{i=1}^{k} \eta_i - \sqrt{\left(\frac{A+1}{k} \sum_{i=1}^{k} \eta_i\right)^2 - \frac{1+2A}{k} \sum_{i=1}^{k} \eta_i^2}. \tag{1.15}$$

ii) Taking $f(\mu_i) = g(\mu_i) = (\mu_{k+1} - \mu_i)$, we have

$$\sum_{i=1}^{k} (\mu_{k+1} - \mu_i) \leq \frac{2|\Omega|}{\xi \sum_{p=1}^q r_p^{-2+n}} \sum_{i=1}^{k} \left(4 r_q^{-2} \mu_i + \sum_{p=1}^{q} \xi^2 r_p^{-4}\right). \tag{1.16}$$

Since $\sum_{i=1}^k (\mu_{k+1} - \mu_k) \leq \sum_{i=1}^k (\mu_{k+1} - \mu_i)$, then we get

$$\mu_{k+1} - \mu_k \leq \frac{2|\Omega|}{k\xi \sum_{p=1}^q r_p^{-2+n}} \sum_{i=1}^{k} \left(4 r_q^{-2} \mu_i + \sum_{p=1}^{q} \xi^2 r_p^{-4}\right). \tag{1.17}$$

iii) The inequalities (1.13)-(1.17) are not the universal inequalities because they are not domain independent.

In the second part of this section, we will study the eigenvalue problem of biharmonic operator with Navier boundary condition. Let M be an n-dimensional compact connected Riemannian manifold with smooth boundary ∂M, we consider the following equation

$$\begin{cases} \Delta^2 u - \tau \Delta u = \Lambda u, & \text{in } M, \\ u = (1-\sigma)\frac{\partial^2 u}{\partial^2 \nu} + \sigma \Delta u = 0, & \text{on } \partial M, \end{cases} \tag{1.18}$$

where τ is a non-negative constant related to the lateral tension of the plate, and σ is also a constant and which is between $-\frac{1}{n-1}$ and 1. Problem (1.18) is called the Navier problem of biharmonic operator and has a discrete spectrum:

$$0 = \Lambda_1 < \Lambda_2 \leq \cdots \leq \Lambda_k \leq \cdots,$$

where each eigenvalue is repeated with its multiplicity. For the first nonzero eigenvalue of Problem (1.18), we can get:

Theorem 1.4. *Let M be an $n(\geq 2)$-dimensional compact connected Riemannian manifold M^n with smooth boundary ∂M and let $\tau \geq 0$, $-\frac{1}{n-1} < \sigma < 1$. Assume that the Ricci curvature of M is bounded below by $(n-1)$ and that the mean curvature of ∂M is non-negative. Let Λ_1 be the first nonzero eigenvalue of the Navier problem* (1.18)*. Then we have*

$$\Lambda_1 \geq (n+\tau)\lambda_1, \tag{1.19}$$

where λ_1 is the first Dirichlet eigenvalue. The equality holds if and only if M is isometric to an n-dimensional Euclidean unit semi-sphere.

At the end of this section, we will give estimates for the first nonzero eigenvalue of buckling problem with Navier boundary condition.

Theorem 1.5. *Let M be an $n(\geq 2)$-dimensional compact connected Riemannian manifold M^n with smooth boundary ∂M. Assume that the Ricci curvature of M is bounded below by $(n-1)$ and that the mean curvature of ∂M is nonnegative. Let Γ_1 be the first nonzero eigenvalue of the problem*

$$\begin{cases} \Delta^2 u = \Gamma \Delta u, & \text{in } M, \\ u = (1-\sigma)\frac{\partial^2 u}{\partial^2 \nu} + \sigma \Delta u = 0, & \text{on } \partial M, \end{cases} \tag{1.20}$$

where $-\frac{1}{n-1} < \sigma < 1$. Then we have

$$\Gamma_1 \geq n, \tag{1.21}$$

the equality holds if and only if M is isometric to an n-dimensional Euclidean unit semi-sphere.

Remark 1.6. When $\tau = 0$, $\sigma = 0$, Theorem 1.4 can cover the result of [1, Theorem 1.7], when $\sigma = 0$, Theorem 1.5 can cover the result of [1, Theorem 1.8].

2. Preliminaries

In this section, we will give some results which play a key role in the proof of the main results which are listed in Section 1. Firstly, we shall introduce a family of couples of functions, more details can be seen in [5].

Definition 1. Let $\lambda > 0$, a couple (f, g) of functions defined on $]0, \lambda[$ belongs to $\Im_\lambda$ as that

(i) f and g are positive,

(ii) f and g satisfy the following condition, for any $x, y \in]0, \lambda[$, such that $x \neq y$,

$$\left(\frac{f(x)-f(y)}{x-y}\right)^2 + \left(\frac{(f(x))^2}{g(x)(\lambda - x)} + \frac{(f(y))^2}{g(y)(\lambda - y)}\right)\left(\frac{g(x)-g(y)}{x-y}\right) \leq 0.$$

Remark 2.1. We know from the definition that $g(x)$ is a nonincreasing function, and there are many couples $(f, g) \in \Im_{\lambda_{k+1}}$, e.g.

(i) $\{1, (\lambda_{k+1} - x)^\alpha \,|\, \alpha \geq 0\} \in \Im_{\lambda_{k+1}}$;

(ii) $\{(\lambda_{k+1} - x), (\lambda_{k+1} - x)^\beta \,|\, \beta \geq \frac{1}{2}\} \in \Im_{\lambda_{k+1}}$;

(iii) $\{(\lambda_{k+1} - x)^\gamma, (\lambda_{k+1} - x)^\gamma \,|\, 0 \leq \gamma \leq 2\} \in \Im_{\lambda_{k+1}}$;

(iv) $\{(\lambda_{k+1} - x)^\alpha, (\lambda_{k+1} - x)^\beta \,|\, \alpha < 0, \beta \geq 1, \alpha^2 \leq \beta\} \in \Im_{\lambda_{k+1}}$.

In [5], Ilias-Makhoul gave following general inequality.

Lemma 2.2. *Let $A : \mathcal{D} \subset \mathcal{H} \to \mathcal{H}$ be a self-adjoint operator defined on a dense domain $\mathcal{D}$, which is semibounded below and has a discrete spectrum $\lambda_1 \leq \lambda_2 \leq \cdots$. Let $\{Q_p : A(\mathcal{D}) \to \mathcal{H}\}_{p=1}^n$ be a collection of symmetric operators, leaving $\mathcal{D}$ invariant. We denote by $\{u_i\}_{i=1}^\infty$ a basis of orthonormal eigenvectors of A, u_i corresponding to λ_i. If for $k \geq 1$ we have $\lambda_{k+1} > \lambda_k$, then for any $(f,g) \in \Im_{\lambda_{k+1}}$,*

$$\left(\sum_{i=1}^k \sum_{p=1}^n f(\lambda_i)\langle [A, Q_p]u_i, Q_p u_i\rangle\right)^2 \leq \left(\sum_{i=1}^k \sum_{p=1}^n g(\lambda_i)\langle [A, Q_p]u_i, Q_p u_i\rangle\right)\left(\sum_{i=1}^k \sum_{p=1}^n \frac{f^2(\lambda_i)}{g(\lambda_i)(\lambda_{k+1}-\lambda_i)}\|[A, Q_p]u_i\|^2\right). \tag{2.1}$$

At last, we will introduce the Bochner formula and the Reilly formula of the Laplacian. For a smooth function u defined on a Riemannian manifold $(M, \langle,\rangle)$, we can give the following the Bochner-type formula:

$$\frac{1}{2}\Delta\left(|\nabla u|^2\right) = |\nabla^2 u|^2 + \mathrm{Ric}(\nabla u, \nabla u) + \langle \nabla u, \nabla(\Delta u)\rangle, \tag{2.2}$$

where $\nabla^2 u$ is the Hessian of u. Let ν be the unit outward normal vector of ∂M. The shape operator of ∂M is given by $S(X) = \nabla_X \nu$ and the second fundamental form of ∂M is defined as $II(X,Y) = \langle S(X), Y\rangle$, here $X, Y \in T\partial M$. The eigenvalues of S are called the principal curvatures of ∂M and the mean curvature H of ∂M is given by $H = \frac{1}{n-1} trS$, here trS denotes the trace of S. We can now state Reilly formula [8]. For a smooth function u defined on M, the following identity holds if $h = \frac{\partial u}{\partial \nu}|_{\partial M}, z = u|_{\partial M}$:

$$\int_M \left((\Delta u)^2 - |\nabla^2 u|^2 - \mathrm{Ric}(\nabla u, \nabla u)\right) = \int_{\partial M}\left((n-1)Hh + \overline{\Delta} z\right)h + \int_{\partial M}\left(II(\overline{\nabla} z, \overline{\nabla} z) - \langle \overline{\nabla} z, \overline{\nabla}\frac{\partial z}{\partial \nu}\rangle\right), \tag{2.3}$$

here $\overline{\Delta}$ and $\overline{\nabla}$ represent the Laplacian and the gradient on ∂M with respect to the induced metric on ∂M, respectively.

3. Proof of the main results

In this section, we will give the proof of the main results which list in Section 1.

Proof of Theorem 1.1. Using the method given by Levitin-Parnovski in [6], we insert q balls $\{B_p = B(x_p, r_p) : p = 1, \ldots, q\}$ of radii $r_1 \geq r_2 \geq \cdots \geq r_q$ inside Ω such that these balls do not intersect each other. Let $R(x)$ be the first nonconstant radial eigenfunction of the Neumann Laplacian in a unit ball $B(0,1)$ normalized in such a way that it is equal to 1 on the boundary of the ball, and let ξ be the corresponding eigenvalue of $R(x)$. Then the function

$$G(x) := \begin{cases} R(r_p^{-1}(x-x_p)), & x_p \in B_p, \\ 1, & \text{otherwise}, \end{cases} \tag{3.1}$$

satisfies Neumann condition on $\partial\Omega$, i.e., $\frac{\partial g}{\partial \nu}\Big|_{\partial\Omega} = 0$.

Taking $A = -\Delta, Q_p = G$ in (2.1) and summing over p from 1 to q, we have

$$\begin{aligned} &\left(\sum_{i=1}^{k}\sum_{p=1}^{n} f(\mu_i)\langle[-\Delta, R_p]u_i, R_p u_i\rangle\right)^2 \\ &\le \left(\sum_{i=1}^{k}\sum_{p=1}^{n} g(\mu_i)\langle[-\Delta, R_p]u_i, R_p u_i\rangle\right)\left(\sum_{i=1}^{k}\sum_{p=1}^{n} \frac{f^2(\mu_i)}{g(\mu_i)(\mu_{k+1}-\mu_i)}\|[-\Delta, R_p]u_i\|^2\right). \end{aligned} \tag{3.2}$$

By direct computations, we have

$$\langle[-\Delta, R_p]u_i, B_p u_i\rangle = \int_{B_p} |\nabla R_p|^2 u_i^2, \tag{3.3}$$

and

$$\|[-\Delta, R_p]u_i\|^2 = \int_{B_p} |2\langle\nabla R_p, \nabla u_i\rangle + u_i \Delta R_p|^2 . \tag{3.4}$$

Since ξ is the eigenvalue corresponding to R_p, and by $R_p^2 \le 1$ and $|\nabla R_p|^2 \le \frac{1}{q} r_q^{-2}$, we have

$$\begin{aligned} &\sum_{i=1}^{k}\sum_{p=1}^{q} \frac{f^2(\mu_i)}{g(\mu_i)(\mu_{k+1}-\mu_i)} \int_{B_p} |2\langle\nabla R_p, \nabla u_i\rangle + u_i\Delta R_p|^2 \\ &= \sum_{i=1}^{k}\sum_{p=1}^{q} \frac{f^2(\mu_i)}{g(\mu_i)(\mu_{k+1}-\mu_i)} \int_{B_p} \left|2\langle\nabla R_p, \nabla u_i\rangle + r_p^{-2}\xi u_i R_p\right|^2 \\ &\le \sum_{i=1}^{k}\sum_{p=1}^{q} \frac{f^2(\mu_i)}{g(\mu_i)(\mu_{k+1}-\mu_i)} \int_{B_p} \left(8|\nabla R_p|^2|\nabla u_i|^2 + 2r_p^{-4}\xi^2 u_i^2 R_p^2\right) \\ &\le \sum_{i=1}^{k} \frac{f^2(\mu_i)}{g(\mu_i)(\mu_{k+1}-\mu_i)} \left(8r_q^{-2}\mu_i + \sum_{p=1}^{q} 2\xi^2 r_p^{-4}\right), \end{aligned} \tag{3.5}$$

and

$$\sum_{i=1}^{k}\sum_{p=1}^{q} g(\mu_i) \int_{B_p} |\nabla R_p|^2 u_i^2 \le \sum_{i=1}^{k} g(\mu_i) r_q^{-2}. \tag{3.6}$$

On the other hand, since $u_1 \equiv \frac{1}{\sqrt{|\Omega|}}$, we have

$$\sum_{i=1}^{k}\sum_{p=1}^{q} f(\mu_i) \int_{B_p} |\nabla R_p|^2 u_i^2 \ge \sum_{i=1}^{k}\sum_{p=1}^{q} f(\mu_i)|\Omega|^{-1} \int_{B_p} |\nabla R_p|^2$$

$$= \sum_{i=1}^{k} f(\mu_i)|\Omega|^{-1}\xi r_p^{-2} \sum_{p=1}^{q} \int_{B_p} R_p^2$$

$$= \sum_{i=1}^{k} f(\mu_i)|\Omega|^{-1}\xi \sum_{p=1}^{q} r_p^{-2+n}. \tag{3.7}$$

Substituting (3.3)-(3.7) into (3.2), we have

$$\left(\sum_{i=1}^{k} f(\mu_i)\right)^2 \leq \frac{|\Omega|^2}{\xi^2 \left(\sum_{p=1}^{q} r_p^{-2+n}\right)^2} \left(\sum_{i=1}^{k} g(\mu_i) r_q^{-2}\right) \times \left(\sum_{i=1}^{k} \frac{f^2(\mu_i)}{g(\mu_i)(\mu_{k+1}-\mu_i)} \left(8r_q^{-2}\mu_i + \sum_{p=1}^{q} 2\xi^2 r_p^{-4}\right)\right).$$

If $f \equiv g$, we infer from (3.2) that

$$\sum_{i=1}^{k}\sum_{p=1}^{n} f(\mu_i)\langle[-\Delta, R_p]u_i, R_p u_i\rangle \leq \sum_{i=1}^{k}\sum_{p=1}^{n} \frac{f(\mu_i)}{(\mu_{k+1}-\mu_i)}\|[-\Delta, R_p]u_i\|^2, \tag{3.8}$$

then

$$\sum_{i=1}^{k} f(\mu_i) \leq \frac{2|\Omega|}{\xi \sum_{p=1}^{q} r_p^{-2+n}} \sum_{i=1}^{k} \frac{f(\mu_i)}{(\mu_{k+1}-\mu_i)} \left(4r_q^{-2}\mu_i + \sum_{p=1}^{q} \xi^2 r_p^{-4}\right).$$

This completes the proof of Theorem 1.1. □

In what follows, we give the proofs of Theorems 1.4 and 1.5, some ideas come from the paper of Chen-Cheng-Wang-Xia in [1, 10].

Proof of Theorem 1.4. Let f be the eigenfunction corresponding to the first eigenvalue Λ_1 of the problem (1.18), which means

$$\begin{cases} \Delta^2 f - \tau\Delta f = \Lambda_1 f, & \text{in } \Omega, \\ f = (1-\sigma)\frac{\partial^2 f}{\partial^2 \nu} + \sigma\Delta f = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.9}$$

Then, it follows from the divergence theorem that

$$\Lambda_1 \int_M f^2 = \int_M f\left(\Delta^2 f - \tau\Delta f\right) = \int_M (\Delta f)^2 + \tau\int_M |\nabla f|^2 - \int_{\partial M} h\Delta f, \tag{3.10}$$

where $h = \frac{\partial f}{\partial \nu}\Big|_{\partial M}$. Since

$$\Delta f\Big|_{\partial M} = \overline{\Delta} z + (n-1)Hh + \frac{\partial^2 f}{\partial^2 \nu}\Big|_{\partial M},$$

where $z = f\big|_{\partial M}$, $\overline{\Delta}$ is the Laplacian on ∂M. It follows from the boundary condition $f\big|_{\partial M} = (1-\sigma)\frac{\partial^2 f}{\partial^2 \nu}\Big|_{\partial M} + \sigma \Delta f\Big|_{\partial M} = 0$ that

$$\Delta f\Big|_{\partial M} = (1-\sigma)(n-1)Hh. \tag{3.11}$$

Substituting (3.11) into (3.10), we have

$$\Lambda_1 = \frac{\int_M (\Delta f)^2 + \tau \int_M |\nabla f|^2 - \int_{\partial M}(1-\sigma)(n-1)Hh^2}{\int_M f^2}. \tag{3.12}$$

Taking f into the Reilly formula [8], we have

$$\begin{aligned}\int_M \left((\Delta f)^2 - |\nabla^2 f|^2\right) &= \int_M \mathrm{Ric}(\nabla f, \nabla f) + (n-1)\int_{\partial M} Hh^2 \\ &\geq (n-1)\int_M |\nabla f|^2 + (n-1)\int_{\partial M} Hh^2.\end{aligned}$$

Substituting $|\nabla^2 f|^2 \geq \frac{1}{n}(\Delta f)^2$ into above inequality, we have

$$\int_M (\Delta f)^2 \geq n\int_M |\nabla f|^2 + n\int_{\partial M} Hh^2. \tag{3.13}$$

Combining (3.12) and (3.13), we have

$$\Lambda_1 \geq \frac{(n+\tau)\int_M |\nabla f|^2 + (n-(1-\sigma)(n-1))\int_{\partial M} Hh^2}{\int_M f^2}.$$

Since $H > 0, -\frac{1}{n-1} < \sigma < 1$, the above inequality implies that

$$\Lambda_1 \geq \frac{(n+\tau)\int_M |\nabla f|^2}{\int_M f^2}.$$

From the Poincaré inequality $\lambda_1 \leq \frac{\int_M |\nabla f|^2}{\int_M f^2}$, where λ_1 is the first eigenvalue of the Dirichlet Laplacian. Then we have

$$\Lambda_1 \geq (n+\tau)\lambda_1. \tag{3.14}$$

If the equality holds in (3.14), that is $\Lambda_1 = (n+\tau)\lambda_1$. Then the Poincaré inequality $\lambda_1 \leq \frac{\int_M |\nabla f|^2}{\int_M f^2}$ becomes the equality $\lambda_1 = \frac{\int_M |\nabla f|^2}{\int_M f^2}$, which means that f is a first eigenfunction corresponding to the first Dirichlet eigenvalue λ_1 of M, then we have $\Delta f = -\lambda_1 f$. It follows from (3.9) that

$$\Lambda_1 f^2 = f\left(\Delta^2 f - \tau\Delta f\right) = \lambda_1^2 f^2 + \tau\lambda_1 f^2. \tag{3.15}$$

Then we can get $\lambda_1 = n$. By similar discussion in the proof of [1, Theorem 1.7], we infer from the Reilly's theorem that M is isometric to an n-dimensional unit semi-sphere. Let $S^n_+(1)$ be an n-dimensional unit semi-sphere given by

$$S^n_+(1) = \left\{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \,\Big|\, \sum_{i=1}^{n+1} x_i^2 = 1,\ x_{n+1} \geq 0\right\}.$$

It is easy to check that the function x_{n+1} on $S^n_+(1)$ is an eigenfunction of the problem (1.18) corresponding to the eigenvalue $n^2 + n\tau$ and the first Dirichlet eigenvalue λ_1 of $S^n_+(1)$. Then we can conclude that the first eigenvalue of problem (1.18) of $S^n_+(1)$ is $(n+\tau)\lambda_1$. This completes the proof of Theorem 1.4. □

Proof of Theorem 1.5. Let g be the eigenfunction corresponding to the first eigenvalue Γ_1 of the problem (1.20), which means

$$\begin{cases} \Delta^2 g = \Gamma_1 \Delta g, & \text{in } \Omega, \\ g = (1-\sigma)\frac{\partial^2 g}{\partial^2 \nu} + \sigma \Delta g = 0, & \text{on } \partial\Omega. \end{cases}$$

Then, it follows from the divergence theorem that

$$\Gamma_1 \int_M |\nabla g|^2 = \int_M (\Delta g)^2 - \int_{\partial M} s\Delta g, \tag{3.16}$$

where $s = \frac{\partial g}{\partial \nu}\Big|_{\partial M}$. Similar discussion as in the proof of Theorem 1.4, we have

$$\Gamma_1 = \frac{\int_M (\Delta g)^2 - \int_{\partial M}(1-\sigma)(n-1)Hs^2}{\int_M |\nabla g|^2}. \tag{3.17}$$

Taking g into the Reilly formula and noticing $|\nabla^2 g|^2 \geq \frac{1}{n}(\Delta g)^2$, we have

$$\int_M (\Delta g)^2 \geq n\int_M |\nabla g|^2 + n\int_{\partial M} Hs^2. \tag{3.18}$$

Combining (3.17) and (3.18), we have

$$\Gamma_1 \geq \frac{n\int_M |\nabla g|^2 + (n-(1-\sigma)(n-1))\int_{\partial M} Hs^2}{\int_M |\nabla g|^2}.$$

Since $H > 0, -\frac{1}{n-1} < \sigma < 1$, the above inequality implies that

$$\Gamma_1 \geq n.$$

If the equality holds, by similar discussion in the proof of [1, Theorem 1.8] and in the proof Theorem 1.4, we know that M is isometric to an n-dimensional unit semi-sphere. This completes the proof of Theorem 1.5. □

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