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THE GROWTH OF SOLUTIONS OF COMPLEX DIFFERENTIAL EQUATIONS WITH ENTIRE COEFFICIENT HAVING FINITE DEFICIENT VALUE

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Abstract. The growth of solutions of second order complex differential equations $f'' + A(z)f' + B(z)f = 0$ with transcendental entire coefficients is considered. Assuming that $A(z)$ has a finite deficient value and that $B(z)$ has either Fabry gaps or a multiply connected Fatou component, it follows that all solutions are of infinite order of growth.

1. Introduction and main results

Our main purpose is to study the growth of the solutions of the second order linear complex differential equation

$$f'' + A(z)f' + B(z)f = 0, \tag{1.1}$$

where $A(z)$ and $B(z)$ are entire functions. It's well known that every solution of this equation is also entire function. The growth of entire function f shows by the order $\rho(f)$ and lower order $\mu(f)$, which are defined respectively by

$$\rho(f) = \limsup_{r\to+\infty} \frac{\log^+ T(r,f)}{\log r} = \limsup_{r\to+\infty} \frac{\log^+ \log^+ M(r,f)}{\log r},$$

$$\mu(f) = \liminf_{r\to+\infty} \frac{\log^+ T(r,f)}{\log r} = \liminf_{r\to+\infty} \frac{\log^+ \log^+ M(r,f)}{\log r},$$

where $T(r, f)$ is the Nevanlinna characteristic function, $\log^+ x = \max\{\log x, 0\}$ and $M(r, f)$ denotes the maximum modulus of f on the circle $|z| = r$. Nevanlinna theory plays an important role in the study of complex differential equations, its basic results and standard notations, such as $T(r, f), m(r, f), N(r, f)$ and $\delta(a, f)$, can be found in [19, 29].

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It's well known that if $B(z)$ is transcendental and f_1, f_2 are two linearly independent solutions of the equation (1.1), then at least one of f_1, f_2 is of infinite order, see [17]. However, there exist some equations of form (1.1) that have a nontrivial solution of finite order. For example, $f(z) = e^z$ satisfies differential equation $f''+e^{-z}f'-(e^{-z}+1)f = 0$. A natural question is that what conditions on $A(z)$ and $B(z)$ can guarantee that every solution $f(\not\equiv 0)$ of the equation (1.1) is of infinite order? There has been many results in the literature on this subject, see [17, 19]. For example, let $A(z)$ and $B(z)$ be nonconstant entire functions, satisfying $\rho(A) < \rho(B)$ (see [13]) or $\rho(B) < \rho(A) \leq \frac{1}{2}$ (see [16]), then every nontrivial solution f of the equation (1.1) has infinite order.

In this paper, we continue to study the above question and consider the question by assuming $\max\{\rho(A), \rho(B)\} < \infty$ in the following theorems. At first, we consider the case in which one coefficient of the equation (1.1) has deficient value. Let f be a non-constant meromorphic function in the complex plane and a be any complex number, the deficiency of a with respect to $f(z)$ is defined by

$$\delta(a, f) = \liminf_{r\to\infty} \frac{m(r, \frac{1}{f-a})}{T(r, f)} = 1 - \limsup_{r\to\infty} \frac{N(r, \frac{1}{f-a})}{T(r, f)}.$$

If $\delta(a, f) > 0$, then the complex number a is named a deficient value of $f(z)$. If an entire function $f(z)$ has a finite deficient value, then $\mu(f) > 1/2$, see [30, Theorem 3.5]. Particularly, if entire coefficient $A(z)$ is of positive order and has a finite Borel exceptional value a, we have $\delta(a, A) = 1$ by [29, Theorem 2.12]. Moreover, if $B(z)$ has some other properties, there have been some results about the solution f of (1.1) with infinite order, for example see [8, 11, 21, 22]. We state one of these as following.

Theorem 1.1 ([22, Theorem 1.8])**.** *Let $A(z)$ be an entire function having a finite Borel exceptional value, and let $B(z)$ be an entire function with Fabry gaps. Then every nontrivial solution of* (1.1) *is of infinite order.*

For entire function $B(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ with Fabry gaps, it satisfies the gaps condition $\frac{\lambda_n}{n} \to \infty$ as $n \to \infty$ and it has positive order which was shown in [15, p. 651], then $B(z)$ is transcendental. In Theorem 1.1 suppose the finite Borel exceptional value of $A(z)$ is a, as mentioned above we have $\delta(a, A) = 1$ and $A(z)$ is of positive integer or infinite order by [29, Theorem 2.11]. Furthermore, we consider the general situation that assume $A(z)$ has a finite deficient value a, which implies $\delta(a, A)$ may be less than 1. In fact, there have been some results as following.

Theorem 1.2. *Let $A(z)$ be a finite order entire function with a finite deficient value and $B(z)$ be a transcendental entire function, satisfying any one of the following additional hypotheses*:

(1) $\mu(B) < 1/2$, *see* [28];

(2) $T(r,B) \sim \log M(r,B)$ *as* $r \to \infty$ *outside a set of finite logarithmic measure, see* [26, Lemma 2.7]*,*

then every nontrivial solution f *of the equation* (1.1) *has infinite order.*

Following this idea, we extend the condition of $A(z)$ in Theorem 1.1 to a general case, that is, $A(z)$ is an entire function with a finite deficient value. In fact, we get a result as follows.

Theorem 1.3. *Let* $A(z)$ *be an entire function with a finite deficient value, and let* $B(z)$ *be an entire function with Fabry gaps. Then every nontrivial solution of* (1.1) *is of infinite order.*

In the below, in order to explain the assumption of Theorem 1.4 we give brief introduction of transcendental entire function f having multiply connected Fatou component. In the complex dynamic theory, see [6] for example, the complex plane is divided into two sets, Fatou set and Julia set. Fatou set $\mathcal{F}(f)$ of f is where the n-th iteration of f, denoted by f^n, form a normal family. The complement of $\mathcal{F}(f)$ in $\mathbb{C}$ is called the Julia set $\mathcal{J}(f)$ of f. The set $\mathcal{F}(f)$ is completely invariant under f in the sense that $z \in \mathcal{F}(f)$ if and only if $f(z) \in \mathcal{F}(f)$. Therefore, if U is a component of $F(f)$, a so-called Fatou component, then there exists, for some $n = 0, 1, 2, \ldots$, a Fatou component U_n such that $f^n(U) \subset U_n$. If, for some $p \geq 1$, we have $U_p = U_0 = U$, then we say that U is a periodic component of period p, assuming p to be the minimal. If U_n is not eventually periodic, then U is a wandering domain of f. Although some entire functions with only simply connected Fatou component, such as Eremenko-Lyubich class function [9], there are many examples of entire function with multiply connected Fatou components. The first such function was constructed by Baker [1], who proved later [3] that this function has a multiply connected Fatou component that is a wandering domain. Moreover, Baker showed [2] that this is not a special property of this example: if U is any multiply connected Fatou component of a transcendental entire function f, then U is wandering domain which called Baker wandering domain. It has the following properties: (1) each U_n is bounded and multiply connected; (2) there exists $N \in \mathbb{N}$ such that U_n and 0 lie in a bounded complementary component of U_{n+1} for $n \geq N$; (3) $\mathrm{dis}(U_n, 0) \to \infty$ as $n \to \infty$. Therefore, if transcendental entire function f has a multiply connected Fatou component, then $\mathcal{J}(f)$ has only bounded component.

Theorem 1.4. *Let* $A(z)$ *be an entire function with a finite deficient value, and* $B(z)$ *be a transcendental entire function with a multiply connected Fatou component. Then every nontrivial solution of* (1.1) *is of infinite order.*

As mentioned above if $\rho(A) < \rho(B)$, then every nontrivial solution of (1.1) is of infinite order. On the contrary, there also have been some studies on assumption $\rho(B) < \rho(A) < \infty$ and some other conditions to get that every nontrivial solutions of (1.1) has infinite order, see [18, 20]. In [23] the authors replaced $\rho(B) < \rho(A) < \infty$ by $\mu(B) < \mu(A) < \infty$ and got the following result.

Theorem 1.5. *Suppose $A(z)$ and $B(z)$ are two entire functions with $\mu(B) < \mu(A) < \infty$, and $T(r,A) \sim \alpha \log M(r,A)$ as $r \to \infty$ outside a set of zero upper logarithmic density. Then every nontrivial solution of (1.1) satisfies*

$$\rho(f) \geq \frac{\mu(A) - \mu(B)}{21(\mu(A) + \mu(B))\sqrt{2\pi(1-\alpha)}} - 1, \quad \alpha \in (0,1).$$

If $\alpha = 1$, then $\rho(f) = \infty$.

Motivated by this result, we obtain the following theorem by changing the condition on coefficient $A(z)$.

Theorem 1.6. *Let $A(z)$ and $B(z)$ be two entire functions with $\mu(B) < \mu(A) < \infty$, and $A(z)$ has Fabry gaps. Then every nontrivial solution of (1.1) is of infinite order.*

2. Preliminary lemmas and auxiliary results

The Lebesgue linear measure of a set $E \subset [0,\infty)$ is $m(E) = \int_E dt$, and the logarithmic measure of a set $F \subset [1,\infty)$ is $m_l(F) = \int_F \frac{dt}{t}$. The upper and lower logarithmic densities of $F \subset [1,\infty)$ are given by

$$\overline{\log \operatorname{dens}} F = \limsup_{r\to\infty} \frac{m_l(F \cap [1,r])}{\log r}$$

and

$$\underline{\log \operatorname{dens}} F = \liminf_{r\to\infty} \frac{m_l(F \cap [1,r])}{\log r},$$

respectively. We say F has logarithmic density if $\overline{\log \operatorname{dens}}(F) = \underline{\log \operatorname{dens}}(F)$.

The proofs of our results highly rely on the estimation of logarithmic derivatives, which is due to Gundersen [12].

Lemma 2.1 ([12])**.** *Let f be a transcendental meromorphic function of finite order $\rho(f)$. Let $\varepsilon > 0$ be a given real constant, and let k and j be two integers such that $k > j \geq 0$. Then there exists a set $E \subset (1,\infty)$ with $m_l(E) < \infty$ such that for all z satisfying $|z| \notin (E \cup [0,1])$, we have*

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq |z|^{(k-j)(\rho(f)-1+\varepsilon)}. \tag{2.1}$$

The following result is a special form of [31, Corollary], in which we set $M(r,f) = \max\{|f(z)| : |z| = r\}$, $L(r,f) = \min\{|f(z)| : |z| = r\}$.

Lemma 2.2 ([31, Corollary 1])**.** *Let $f(z)$ be a transcendental meromorphic function with at most finitely many poles. If $\mathcal{J}(f)$ has only bounded components, then for any complex number a, there exist a constant $0 < d < 1$ and two sequences $\{r_n\}$ and $\{R_n\}$ of positive numbers with $r_n \to \infty$ and $R_n/r_n \to \infty (n \to \infty)$ such that*

$$M(r,f)^d \leq L(r,f), \quad r \in G, \tag{2.2}$$

where $G = \bigcup_{n=1}^{\infty} \{r : r_n < r < R_n\}$.

Obviously, in the above lemma the set G has infinite logarithmic measure, but we can not ensure the logarithmic density of G is positive. For the proof of Theorem 1.4, we need to give a modification of Lemma 2.2. In order to do this, we state two necessary results as follow.

Lemma 2.3 ([5]). *Let U be a domain in the complex plane and $f(z)$ be defined and analytic in $f^n(U)$, $(n = 0, 1, \ldots)$ inductively such that $H = \cup_{n=0}^{\infty} f^n(U)$ has at least two finite boundary points in the complex plane. If $f^{n_k}|_U \to \infty (k \to \infty)$, then for any compact subset K of H, there exists a positive constant M such that for all sufficiently large k, we have*

$$|f^{n_k}(z_1)| \leq |f^{n_k}(z_2)|^M \tag{2.3}$$

for all $z_1, z_2 \in K$.

Lemma 2.4 ([7]). *Let f be a transcendental entire function with a multiply connected wandering domain U. For each $z_0 \in U$ and each open set $D \subset U$ containing z_0, there exists $\alpha \in (0, 1)$ such that for sufficiently large $n \in \mathbb{N}$,*

$$\{z : |f^n(z_0)|^{1-\alpha} < |z| < |f^n(z_0)|^{1+\alpha}\} \subset f^n(D) \subset U_n. \tag{2.4}$$

Lemma 2.5. *Let $f(z)$ be a transcendental entire function. If $\mathcal{F}(f)$ has a multiply connected component, then there exist a constant $0 < d < 1$ and a set $G \subset (1, +\infty)$ with positive lower logarithmic density such that*

$$M(r, f)^d \leq L(r, f), \quad r \in G. \tag{2.5}$$

Proof. It was shown in [4] that for a transcendental entire function, every multiply connected component of its Fatou set must be Baker wandering domain. Then there exists a Baker wandering domain U containing a set D in Lemma 2.4. We have a sequence $D_n = f^n(D)$ in $\cup_{n=0}^{\infty} f^n(D)$. Set $G = \cup_{n=n_0}^{\infty}\{r : |f^n(z_0)|^{1-\alpha} < r < |f^n(z_0)|^{1+\alpha}\}$ and $C(r) = \{z : |z| = r\}$. Obviously, G has positive lower logarithmic density 2α. For every $r \in G$, we have $C(r) \subset D_n$ for some n and we have a curve $\gamma(r)$ in D and a positive integer $\widehat{n}$ such that $C(r) = f^{\widehat{n}}(\gamma(r))$. Therefore, we have two points w_1 and w_2 in $\gamma(r) \subset D$ such that

$$M(r, f) = \max_{z \in C(r)} |f(z)| = |f(f^{\widehat{n}}(w_1))|, \tag{2.6}$$

$$L(r, f) = \min_{z \in C(r)} |f(z)| = |f(f^{\widehat{n}}(w_2))|. \tag{2.7}$$

Since $\overline{D}$ is a compact subset of U and $f^n(U) \to \infty (n \to \infty)$, by Lemma 2.3 we have for some constant $M > 1$

$$|f^{\widehat{n}+1}(w_1)| \leq |f^{\widehat{n}+1}(w_2)|^M. \tag{2.8}$$

Combining (2.6), (2.7), (2.8), we obtain (2.5) with $d = 1/M$. $\square$

Lemma 2.6 ([10, Lemma 1]). *Let $h(z)$ be a meromorphic function of finite order ρ. Given $\zeta > 0$ and $0 < l < 1/2$, there exist a positive constant $K(\rho, \zeta)$ and a set $F_\zeta \subset [0, +\infty)$ of lower logarithmic density*

$$\underline{\log \mathrm{dens}} F_\zeta := \liminf_{r\to\infty} \frac{\int_{F_\zeta \cap [1,\infty)} t^{-1} dt}{\log r} \geq 1 - \zeta$$

such that

$$r \int_J \left| \frac{h'(re^{i\theta})}{h(re^{i\theta})} \right| d\theta < K(\rho, \zeta)(l \log \frac{1}{l}) T(r, h) \tag{2.9}$$

for all $r \in F_\zeta$ and every interval $J \subset [0, 2\pi)$ of length l.

We state the following lemma in regard to entire function with Fabry gaps. It can be found in [10, Theorem 1] and [14, Lemma 2.2].

Lemma 2.7 ([10, Theorem 1]). *Let $B(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ be an entire function of finite order with Fabry gaps. Then, for any given $\varepsilon > 0$,*

$$\log L(r, B) > (1 - \varepsilon) \log M(r, B) \tag{2.10}$$

holds outside a set of logarithmic density 0, here $L(r, B), M(r, B)$ are as mentioned in Lemma 2.2.

For entire function with a finite deficient value, the following lemma is an important estimation of its growth in a small sector.

Lemma 2.8. *Let $A(z)$ be an entire function with a finite deficient value a. Then there exist a small interval $I \in [0, 2\pi)$ and a set $F_\zeta \subset (1, \infty)$ with lower logarithmic density at least $1 - \zeta$ such that*

$$|A(re^{i\varphi})| \leq |a| + 1 \tag{2.11}$$

for $r \in F_\zeta$ and $\varphi \in I$.

Proof. Assume that $A(z)$ has deficiency $\delta(a, A) = 2\delta$ at $a \in \mathbb{C}$. Then from the definition of deficiency, the proximate function of $\frac{1}{A-a}$ satisfying $m(r, \frac{1}{A-a}) > \delta T(r, A)$, so there exists a point $z_r = re^{i\theta_r}$ such that

$$\log |A(z_r) - a| \leq -\delta T(r, A). \tag{2.12}$$

Set $z = re^{i\theta}$ and $w = re^{it}$, by the integral we obtain

$$\begin{aligned} \log(A(z) - a) - \log(A(z_r) - a) &= \int_{z_r}^{z} \frac{(A(w) - a)'}{A(w) - a} dw \\ &= r \int_{\theta_r}^{\theta} \frac{(A(re^{it}) - a)'}{A(re^{it}) - a} ie^{it} dt. \end{aligned} \tag{2.13}$$

Taking modulus we get

$$|\log(A(z) - a)| \leq |\log(A(z_r) - a)| + r \int_{\theta_r}^{\theta} \left| \frac{(A(re^{it}) - a)'}{A(re^{it}) - a} \right| dt. \tag{2.14}$$

Applying Lemma 2.6 to $A - a$ and taking the principle branch of logarithmic, we can choose a set F_ζ of lower logarithmic density greater than $1 - \zeta$ such that

$$\begin{aligned}\log|(A(z)-a)| &\le \log|(A(z_r)-a)| + 4\pi + r\int_{\theta_r}^{\theta}\left|\frac{(A(re^{it})-a)'}{A(re^{it})-a}\right| dt\\ &\le (-\delta + K(\rho(A),\zeta) l \log\frac{1}{l}) T(r, A-a) + 4\pi\\ &\le (-\delta + K(\rho(A),\zeta) l \log\frac{1}{l} + o(1)) T(r, A) \qquad (2.15)\end{aligned}$$

holds. Therefore, set $\theta = \theta_r + l$, we have

$$\begin{aligned}\log|(A(re^{i\varphi})-a)| &\le \log|(A(z_r)-a)| + 4\pi + r\int_{\theta_r}^{\varphi}\left|\frac{(A(re^{it})-a)'}{A(re^{it})-a}\right| dt\\ &\le \log|(A(z_r)-a)| + 4\pi + r\int_{\theta_r}^{\theta}\left|\frac{(A(re^{it})-a)'}{A(re^{it})-a}\right| dt\\ &\le (-\delta + K(\rho(A),\zeta) l \log\frac{1}{l} + o(1)) T(r, A) \qquad (2.16)\end{aligned}$$

for all $z = re^{i\varphi}$ satisfying $r \in F_\zeta$ and $\varphi \in [\theta_r, \theta_r + l]$. Since $\lim_{l\to 0^+} l \log\frac{1}{l} = 0$, we determine l sufficiently small, then (2.16) implies $\log|A(re^{i\varphi}) - a| \le 0$, that is

$$|A(re^{i\varphi})| \le |a| + 1 \qquad (2.17)$$

for $r \in F_\zeta, \varphi \in [\theta_r, \theta_r + l]$. □

The last lemma is a simplified formulation of a result due to Miles and Rossi [24, Theorem 1], sufficient for our use.

Lemma 2.9. *Let f be a nonconstant entire function of finite order. For $\beta \in (0, 1)$ and $r > 0$, let*

$$U_r = \left\{\theta \in [0, 2\pi) : r\left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right| \ge \beta n(r, 0, f)\right\}.$$

Then for $M > 3$ there exists a set $E_M \subset [1, \infty)$ with lower logarithmic density at least $1 - \frac{3}{M}$ such that

$$m(U_r) > \left(\frac{1-\beta}{7M(\rho+1)}\right)^2, \quad r \in E_M.$$

3. Proof of theorems

3.1. Proof Theorem 1.3

Suppose there is a nontrivial solution f of (1.1) with finite order $\rho(f)$, we shall seek a contradiction. Since $A(z)$ has a finite deficient value a, by Lemma

2.8 there exist a small interval $I \in [0, 2\pi)$ and a set $F_\zeta \subset (1, \infty)$ with lower logarithmic density at least $1 - \zeta$ such that

$$|A(re^{i\varphi})| \leq |a| + 1 \tag{3.1}$$

for $r \in F_\zeta$ and $\varphi \in I$. By Lemma 2.1, there exists a set E_1 with finite logarithmic measure such that

$$\left|\frac{f^{(k)}(z)}{f(z)}\right| \leq |z|^{2\rho(f)}, \quad (k = 1, 2) \tag{3.2}$$

for all $r \notin E_1 \cup [0, 1]$. It's clear that there exists a set E with positive logarithmic density such that (2.10) and (3.1) holds simultaneously for $r \in E$ and $\varphi \in I$. Then, combining (2.10), (3.1) with (3.2), for all $z = re^{i\varphi}$ satisfying $r \in E \setminus (E_1 \cup [0, 1])$, $\varphi \in I$ and any given $\varepsilon > 0$ we have

$$\begin{aligned} M(r, B)^{1-\varepsilon} < L(r, B) \leq |B(re^{i\varphi})| &\leq \left|\frac{f''(re^{i\varphi})}{f(re^{i\varphi})}\right| + |A(re^{i\varphi})| \left|\frac{f'(re^{i\varphi})}{f(re^{i\varphi})}\right| \\ &\leq (2 + |a|) r^{2\rho(f)}. \end{aligned} \tag{3.3}$$

Then,

$$(1 - \varepsilon) T(r, B) \leq (1 - \varepsilon) \log M(r, B) \leq 2\rho(f) \log r + \log(|a| + 2) \tag{3.4}$$

as $r \in E \setminus (E_1 \cup [0, 1])$, $\varphi \in I$. Since $B(z)$ is transcendental, we get a contradiction.

3.2. Proof of Theorem 1.4

We assume that there is a nontrivial solution f of (1.1) with finite order. By Lemma 2.8, there exist a small interval $I \in [0, 2\pi)$ and a set $F_\zeta \subset (1, \infty)$ with lower logarithmic density at least $1 - \zeta$ such that (3.1) holds for $r \in F_\zeta$ and $\varphi \in I$. Moreover, by Lemma 2.1 there exists a set $E \subset (1, \infty)$ with finite logarithmic measure such that for all z satisfying $|z| = r \notin E \cup [0, 1]$, (3.2) holds. Since $B(z)$ has a multiply connected Fatou component, by Lemma 2.7 there exist a constant $0 < d < 1$ and a set $G \subset (1, +\infty)$ with positive lower logarithmic density 2α such that

$$M(r, B)^d \leq L(r, B), \quad r \in G. \tag{3.5}$$

Since the characteristic function of F_ζ and G satisfy, see [27, p. 44],

$$\chi_{F_\zeta \cup G}(t) = \chi_{F_\zeta}(t) + \chi_G(t) - \chi_{F_\zeta \cap G}(t), \tag{3.6}$$

from [25, p. 121] we have

$$\begin{aligned} \overline{\log \mathrm{dens}} F_\zeta + \underline{\log \mathrm{dens}} G - \underline{\log \mathrm{dens}}(F_\zeta \cap G) &\leq \overline{\log \mathrm{dens}}(F_\zeta \cup G) \\ &\leq 1. \end{aligned} \tag{3.7}$$

Obviously, we have $\overline{\log \mathrm{dens}} F_\zeta \geq \underline{\log \mathrm{dens}} F_\zeta \geq 1 - \zeta$, which implies $\underline{\log \mathrm{dens}}(F_\zeta \cap G) > 2\alpha - \zeta$. Since the constant ζ in Lemma 2.6 ([10, Lemma 1]) can take any number in $(0, 1)$, then choose ζ sufficiently small such that $2\alpha - \zeta > 0$.

Therefore, the set $F_\zeta \cap G$ has positive lower logarithmic density, so has infinite logarithmic measure.

Combining (3.1), (3.2) with (3.5), we have, for $z = re^{i\varphi}$ satisfying $r \in F_\zeta \cap G \setminus (E \cup [0,1])$ and $\varphi \in I$,

$$\begin{aligned} M(r,B)^d \le L(r,B) \le |B(z)| &\le \left|\frac{f''(z)}{f(z)}\right| + |A(z)|\left|\frac{f'(z)}{f(z)}\right| \\ &\le (|a|+2)r^{2\rho(f)}, \end{aligned} \tag{3.8}$$

where $d \in (0,1)$. Thus,

$$dT(r,B) \le d\log M(r,B) \le 2\rho(f)\log r + \log(|a|+2) \tag{3.9}$$

as $r \in F_\zeta \cap G \setminus (E \cup [0,1])$. This is a contradiction since $B(z)$ is transcendental.

3.3. Proof Theorem 1.6

Suppose on the contrary to the assertion that there is a nontrivial solution f of (1.1) with finite order $\rho(f) = \rho$. We aim for a contradiction. In view of (1.1), we have

$$|A(z)|\left|\frac{f'}{f}\right| \le |B(z)| + \left|\frac{f''}{f}\right|. \tag{3.10}$$

By the definition of lower order, there exists an $r_0 > 1$ such that for all $r > r_0$ and any given $\varepsilon \in (0, \frac{\mu(A)-\mu(B)}{2})$, we have

$$\log M(r,A) > r^{\mu(A)-\frac{\varepsilon}{2}}. \tag{3.11}$$

Moreover, there exists a sequence $\{r_n\}$ with $r_n \to \infty$ as $n \to \infty$ such that for sufficiently large n, we have

$$\log M(r_n,B) < r_n^{\mu(B)+\varepsilon}. \tag{3.12}$$

Set

$$F_1 = \bigcup_n [r_n^{\frac{\mu(B)+\varepsilon}{\mu(A)-\varepsilon}}, r_n],$$

then $\overline{\log\text{dens}}(F_1) \ge \frac{\mu(A)-\mu(B)-2\varepsilon}{\mu(A)-\varepsilon}$. Thus, for all $r \in F_1$, combining the above two inequalities yields

$$\begin{aligned} \log M(r,B) \le \log M(r_n,B) < r_n^{\mu(B)+\varepsilon} &= \left(r_n^{\frac{\mu(B)+\varepsilon}{\mu(A)-\varepsilon}}\right)^{\mu(A)-\varepsilon} \\ &< r^{\mu(A)-\varepsilon}. \end{aligned} \tag{3.13}$$

By the argument in [13, p. 426], we know that for any nontrivial solution of (1.1) it has at least one zero, then $n(r,0,f) \ge 1$ for some $r > r_0$. Therefore, by Lemma 2.9, there exist a set $F_2 \subset [1,\infty)$ with lower logarithmic density at least $1-\frac{3}{M}$ and a set $U_r \subset [0,2\pi)$ with $m(U_r) > \left(\frac{1-\beta}{7M(\rho+1)}\right)^2$ such that

$$r\left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right| \geq \beta n(r,0,f) > \beta \tag{3.14}$$

for $r \in F_2$ and $\theta \in U_r$, where $\beta \in (0,1)$. In addition, by Lemma 2.7 there exists a set $F_3 \in (1,\infty)$ with logarithmic density 1 such that

$$\log L(r,A) > (1-\varepsilon)\log M(r,A) \tag{3.15}$$

for any given $\varepsilon > 0$ and $r \in F_3$.

Finally, by Lemma 2.1, there exists a set $F_4 \subset (1,\infty)$ with finite logarithmic measure such that (3.2) holds for all z satisfying $|z| = r \notin F_4 \cup [0,1]$. Denote F_2^c and F_3^c the complement set of F_2 and F_3 in $(0,\infty)$, respectively, the set $F_0 = F_1 \cap F_2 \cap F_3$ satisfies

$$\begin{aligned}\overline{\log\mathrm{dens}}F_0 &\geq \overline{\log\mathrm{dens}}F_1 - \overline{\log\mathrm{dens}}F_2^c - \overline{\log\mathrm{dens}}F_3^c\\ &= \overline{\log\mathrm{dens}}F_1 - (1-\underline{\log\mathrm{dens}}F_2)\\ &\geq \frac{\mu(A)-\mu(B)-2\varepsilon}{\mu(A)-\varepsilon} - \frac{3}{M} > 0 \end{aligned}\tag{3.16}$$

if we take $M = \frac{3(\mu(A)+\mu(B))}{\mu(A)-\mu(B)-2\varepsilon} > 3$ in Lemma 2.9. For $r \in F_0 \setminus (F_4 \cup [0,1]$ and $\theta \in U_r$, substituting (3.11), (3.13), (3.14), and (3.15) into (3.10), we have

$$\begin{aligned}\frac{\beta\exp\{(1-\varepsilon)r^{\mu(A)-\frac{\varepsilon}{2}}\}}{r} &\leq \frac{\beta M(r,A)^{1-\varepsilon}}{r} \leq \frac{\beta L(r,A)}{r} \leq |A(re^{i\theta})|\left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right|\\ &\leq |B(re^{i\theta})| + \left|\frac{f''(re^{i\theta})}{f(re^{i\theta})}\right| \leq M(r,B) + r^{2\rho(f)}\\ &\leq \exp\{r^{\mu(A)-\varepsilon}\} + r^{2\rho(f)}.\end{aligned}\tag{3.17}$$

Obviously, this is a contradiction for sufficiently large r. Then we complete the proof.

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