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# LOCAL WELL-POSEDNESS OF DIRAC EQUATIONS WITH NONLINEARITY DERIVED FROM HONEYCOMB STRUCTURE IN 2 DIMENSIONS

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Abstract. The aim of this paper is to show the local well-posedness of 2 dimensional Dirac equations with power type and Hartree type nonlinearity derived from honeycomb structure in $H^s$ for $s > \frac{7}{8}$ and $s > \frac{3}{8}$, respectively. We also provide the smoothness failure of flows of Dirac equations.

## 1. Introduction

In this paper we consider following two Cauchy problems for massless honeycomb lattice power type Dirac equations ($\ell = 1$) and Hartree type Dirac equations ($\ell = 2$):

$$\begin{cases} (\partial_t + \alpha \cdot D)\psi = -i\kappa \mathcal{N}_\ell(\psi, \psi)\psi, \\ \psi(0) = \psi_0, \end{cases} \tag{1.1}$$

where $\psi : \mathbb{R}^{1+2} \to \mathbb{C}^2$ is the spinor field represented by a column vector, $\kappa$ is constant, $D = -i\nabla$, and $\alpha = (P_\# \alpha^1, P_\# \alpha^2)$ are the Dirac matrices defined by

$$\alpha^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \alpha^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

with $P_\# = \begin{pmatrix} \overline{\eta_\#} & 0 \\ 0 & \eta_\# \end{pmatrix}$ for honeycomb lattice constant $\eta_\# \neq 0$ arising from nonlinear Schrödinger equations (NLS) with honeycomb lattice potentials (see the Section II in [1]). The nonlinearities $N_\ell$ are defined by

$$\mathcal{N}_1(\psi_1, \psi_2) = \begin{pmatrix} b_1\psi_{11}\overline{\psi_{21}} + 2b_2\psi_{12}\overline{\psi_{22}} & 0 \\ 0 & b_1\psi_{12}\overline{\psi_{22}} + 2b_2\psi_{11}\overline{\psi_{21}} \end{pmatrix},$$
$$\mathcal{N}_2(\psi_1, \psi_2) = \left(|x|^{-1} * (\psi_1^\dagger \psi_2)\right),$$

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where $\psi_{j1}, \psi_{j2}$ are components of $\psi_j$ and the coefficients $b_1, b_2 > 0$ which are the amplitude of Bloch waves. The symbol $*$ denotes the convolution operator in $\mathbb{R}^2$ and the $\psi^\dagger$ is the complex conjugate transpose of $\psi$.

Our main equations with the nonlinearity $\mathcal{N}_\ell$ are derived from two dimensional Schrödinger equations with honeycomb lattice potential. Its rigorous derivation appears in [1]. The honeycomb lattice structure has appeared in the fabrication of graphene, a mono-crystalline graphitic film in which electrons behave like massless Dirac fermions (see [6]). Also, the nonlinear optics which model laser beam propagators in particular types of photonic crystals, have the honeycomb structure (see [2, 14]).

The equation (1.1) for $\ell = 1$ has the scaling invariance structure in $\dot{H}^{\frac{1}{2}}$. That is, for $\psi_1$ the solution to (1.1) with $\ell = 1$, the function $\psi_{1,\lambda}$ defined by $\psi_{1,\lambda}(t,x) = \lambda^{\frac{1}{2}}\psi_1(\lambda t, \lambda x)$ is also the solution to the equation (1.1) with $\ell = 1$ and satisfies that $\|\psi_{1,\lambda}(0,\cdot)\|_{\dot{H}^{\frac{1}{2}}} = \|\psi_1(0,\cdot)\|_{\dot{H}^{\frac{1}{2}}}$. By this reason, the equation (1.1) for $\ell = 1$ is said to be mass-supercritical case. Also, since $\|\psi_{2,\lambda}(0,\cdot)\|_{L^2_x} = \|\psi_2(0,\cdot)\|_{L^2_x}$ for $\psi_{2,\lambda}(t,x) = \lambda\psi_2(\lambda t, \lambda x)$, where $\psi_2$ is the solution to (1.1) with $\ell = 2$, the equation (1.1) with $\ell = 2$ has the scaling invariance structure in $L^2_x$. The equation (1.1) for $\ell = 2$ is called to be mass-critical case.

Now we state the main theorem of this paper. For simplicity of representation, we set an index $s(\ell)$ by

$$s(\ell) = \begin{cases} \frac{1}{2} & \text{if } \ell = 1, \\ 0 & \text{if } \ell = 2. \end{cases}$$

**Theorem 1.1** (Local well-posedness for $H^s$ data)**.** *Let $s > s(\ell) + \frac{3}{8}$ for $\ell = 1, 2$. Then (1.1) is locally well-posed for initial data in $H^s(\mathbb{R}^2)$.*

Here a definition of the fractional Sobolev space $H^s(\mathbb{R}^2)$ is placed in **Notations** below. In particular, LWP result of Dirac equations which have same nonlinearity $\mathcal{N}_\ell$ has been studied in [1] for $H^s(\mathbb{R}^2)$ with $s > 1$.

We can prove the results of Theorem 1.1 for massive cases ($m > 0$) in the same way as proof of Theorem 1.1. Since the physical model comes from massless Dirac fermions, we only consider the massless case ($m = 0$) in this paper.

Lemma 3.2 is deduced from Selberg's estimates and we get a coefficient $\mu^{\frac{3}{8}-s}$ in (3) of Lemma 3.2. Then the condition $s > s(\ell) + \frac{3}{8}$ is necessary in process of proof of Theorem 1.1 and the coefficient $\mu^{\frac{3}{8}-s}$ makes the gap between scaling critical index $s(\ell)$ and our well-posedness index $s(\ell) + \frac{3}{8}$.

In this paper, we consider Dirac equations with some nonlinearity. Related equations to (1.1) are well known as semi-relativistic equations as follows:

$$iu_t + \sqrt{m^2 - \Delta}u = \lambda|u|^2 u, \tag{1.2}$$

$$iu_t + \sqrt{m^2 - \Delta}u = \lambda(|x|^{-1} * |u|^2)u. \tag{1.3}$$

The Cauchy problem for semi-relativistic equations with power type nonlinearity (1.2) has been investigated in [11, 12]. In [11, 12], Dinh ([11]) showed local well-posedness (LWP) of (1.2) with massless case ($m = 0$) for $H^s(\mathbb{R}^2)$ with $s > \frac{3}{4}$ and Fujiwara, Georgiev, and Ozawa extended LWP to global well-posedness (GWP) for $H^1(\mathbb{R}^2)$. The Cauchy problem for 3 dimensional Hartree type semi-relativistic equations (1.3) has been investigated in [16, 21]. First the result of well-posedness was obtained by [21] in $H^s(\mathbb{R}^3)$ for $s \geq \frac{1}{2}$. In [21], global well-posedness holds in $H^{\frac{1}{2}}(\mathbb{R}^3)$ for small data in $L^2_x$. Later this was improved to $s > \frac{1}{4}$ in [16]. Also they ([16]) showed ill-posedness result for $H^s(\mathbb{R}^3)$ with $s < \frac{1}{4}$. For (1.3) with $d$-dimensions ($d \geq 2$), Cho and Ozawa ([9]) have revealed the Global well-posedness result for $H^s(\mathbb{R}^d)$ with $s \geq \frac{1}{2}$. Further results for semi-relativistic equations, we refer to [10].

The difficulty stems from the absence of null-structure of $\mathcal{N}_\ell$. We describe the difference between $\psi^\dagger \beta \psi$ and $|\psi|^2$ where $\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The quadratic term $\psi^\dagger \beta \psi$ has a null-structure which represents like $|\psi_1|^2 - |\psi_2|^2$. On the other hand, another term $|\psi|^2 = |\psi_1|^2 + |\psi_2|^2$ does not have the null-structure. Since this structure induces delicate bilinear estimates, Dirac equations with null-structure lead to better results than the case without null-structure. However, we do not use this structure because our nonlinearities $\mathcal{N}_1$ are essentially the same as $|\psi|^2$. For this reason, it is picky to control the nonlinear term $\mathcal{N}_\ell$. Hence we describe Lemma 3.2 used crucially in the proof of Theorem 1.1.

Also we consider the Dirac equation with Coulomb type nonlinearity which has null-structure:

$$(i\partial_t + \alpha \cdot D)\psi = \lambda \left(\psi^\dagger \beta \psi\right) \psi, \tag{1.4}$$

$$(i\partial_t + \alpha \cdot D)\psi = \lambda \Big( |x|^{-\gamma} * \left(\psi^\dagger \beta \psi\right) \Big) \psi. \tag{1.5}$$

As a known result for the equation (1.4), Bejenaru and Herr ([3]) showed the GWP in $H^{\frac{1}{2}}(\mathbb{R}^2)$. And the known results for the equation (1.5) are in [8, 20]. In [20], An author of this paper revealed the LWP in $H^s(\mathbb{R}^2)$ with $s > \frac{\gamma-1}{2}$ and $1 \leq \gamma < 2$. It was studied in [8] that global well-posedness and small data scattering holds in $H^s(\mathbb{R}^2)$ for $s > \gamma - 1$ and $1 < \gamma < 2$. As related to (1.5), there is a Dirac equation with Yukawa potential. One may find many results of the Dirac equation which has Yukawa potential nonlinearity in [7, 25–27].

In view of scaling we expect that LWP results for (1.1) is optimal in $H^{s(\ell)}$. For this expectation we introduce the following theorem which denotes the smooth failure of our main equation (1.1) for $s < s(\ell)$.

**Theorem 1.2.** *Let $s < s(\ell)$ and $T > 0$. If the flow map $\phi \mapsto u$ in (1.1) exists as a map from $H^s(\mathbb{R}^2)$ to $C([-T, T]; H^s(\mathbb{R}^2))$, it fails to be $C^3$ at the origin.*

If the equation (1.1) has well-posedness in $[-T, T]$ for some $T > 0$, the flows of (1.1) have the smoothness in $[-T, T]$. Since Theorem 1.2 implies that the smoothness of flows of (1.1) fails, This yields the ill-posedness of (1.1) for $H^s$ with $s < s(\ell)$.

The smoothness failure of some equations was studied for many authors in [4, 16, 20, 22]. Molinet, Saut, and Tzvetkov ([22]), Bejenaru and Tao ([4]), and Herr and Lenzmann ([16]) have proved the ill-posedness results similar to Theorem (1.2) for Benjamin-Ono equations, 1-d Schrödinger equations and semi-relativistic equations, respectively. For Dirac equation, ill-posedness results have been shown in [20].

It is still opened the well-posedness of (1.1) in $H^s(\mathbb{R}^2)$ for $s(\ell) \le s \le s(\ell)+\frac{3}{8}$. For filling up this gap, we have to obtain better bilinear estimates than Lemma 3.2. For this purpose we should find some structure of nonlinearity of (1.1) like null-structure. Then we may improve LWP in $H^s$ with sobolev index $s$ below $s(\ell)+\frac{3}{8}$.

The paper is organized as follows: In Section 2, we discuss projection operators. In Section 3, we introduce the function spaces and the bilinear estimates the most useful on proof of main theorem. In Section 4, we prove Theorem 1.1 via the standard contraction method. In Section 5, we establish the proof of Proposition 4.1 arising in Section 4. In the last section, we discuss Theorem 1.2 by contradiction argument.

**Notations.**

• Space and space-time Fourier transform: $\widehat{f} = \mathcal{F}_x(f)$ denotes the space variable Fourier transform of $f$ and $\mathcal{F}_\xi^{-1}(g)$ the inverse Fourier transform of $g$ such that

$$\mathcal{F}_x(f)(\xi) = \int_{\mathbb{R}^2} e^{-ix\cdot\xi} f(x)\,dx, \quad \mathcal{F}_\xi^{-1}(g)(x) = (2\pi)^{-2}\int_{\mathbb{R}^2} e^{ix\cdot\xi} g(\xi)\,d\xi.$$

$\widetilde{f} = \mathcal{F}_{t,x}(f)$ denotes the space-time variables Fourier transform of $f$ such that

$$\mathcal{F}_{t,x}(f)(\tau,\xi) = \int_{\mathbb{R}^{1+2}} e^{-it\cdot\tau - ix\cdot\xi} f(t,x)\,dtdx.$$

• Fractional derivatives and Sobolev spaces: $D^s = (-\Delta)^{\frac{s}{2}} = \mathcal{F}_x^{-1}|\xi|^s\mathcal{F}_x$, $\Lambda^s = (1-\Delta)^{\frac{s}{2}} = \mathcal{F}_x^{-1}(1+|\xi|^2)^{\frac{s}{2}}\mathcal{F}_x$ for $s > 0$. Let us denote $\dot{H}^s = D^s L_x^2$ and $H^s = \Lambda^s L_x^2$ for $s \in \mathbb{R}$.

• Mixed-normed spaces: For a Banach space $X$ and an interval $I$, $u \in L_I^q X \cap \mathbb{C}$ if and only if $u(t) \in X$ for a.e. $t \in I$ and $\|u\|_{L_I^q X} := \|\|u(t)\|_X\|_{L_I^q} < \infty$. Especially, we denote $L_I^q L_x^r = L_t^q(I; L_x^r(\mathbb{R}^2))$, $L_{I,x}^q = L_I^q L_x^q$, $L_t^q L_x^r = L_{\mathbb{R}}^q L_x^r$. For vector-valued function $\psi \in L_I^q X \cap \mathbb{C}^2$, we also denote that $\|\psi\|_{L_I^q X} := \||\psi|\|_{L_I^q X}$.

• Littlewood-Paley operators: Let us define $\beta_1 \in C_0^\infty(-2,2)$ such that $\beta_1(s) = 1$ if $|s| \le 1$ and $\beta_\lambda(s) := \beta(\frac{s}{\lambda}) - \beta(\frac{2s}{\lambda})$ for $\lambda > 1$. Then we define the frequency projection $\mathcal{F}(P_\lambda f)(\xi) = \beta_\lambda(\xi)\widehat{f}(\xi)$, $P_{\le\lambda} = \sum_{\mu=1}^{\lambda} P_\mu$ and $P_{\ge\lambda} = I - P_{\le\frac{\lambda}{2}}$. Also, for measurable set $S \subset \mathbb{R}^2, R \subset \mathbb{R}^{1+2}$, we denote that $\mathcal{F}_x(P_S f)(\xi) = \chi_S(\xi)\widehat{f}(\xi)$ and $\mathcal{F}_{t,x}(P_R f)(\tau,\xi) = \chi_R(\tau,\xi)\widetilde{f}(\tau,\xi)$.

• As usual different positive constants depending only on $\alpha, \kappa$ are denoted by the same letter $C$, if not specified. $A \lesssim B$ and $A \gtrsim B$ means that $A \le CB$

and $A \geq C^{-1}B$, respectively for some $C > 0$. $A \sim B$ means that $A \lesssim B$ and $A \gtrsim B$.

## 2. Preliminaries

In this section, for simplicity of the Cauchy problem, we define the projection operators and rewrite the equations (1.1) to integral equations.

### 2.1. Projection operator

We first define the projections about (1.1) as follows:

$$\Pi^{\pm}(D) := \frac{1}{2}\left(I \pm \frac{\alpha \cdot D}{|\eta_{\#}||\nabla|}\right).$$

Then we get

$$\alpha \cdot D = |\eta_{\#}||\nabla| \left(\Pi^{+}(D) - \Pi^{-}(D)\right).$$

Using these projection operators, we decompose

$$\psi = \psi_{+} + \psi_{-},$$

where $\psi_{\pm} := \Pi^{\pm}(D)\psi$. Also, these projection operators satisfy that

$$\Pi^{\pm}(D)\Pi^{\pm}(D) = \Pi^{\pm}(D), \qquad \Pi^{\pm}(D)\Pi^{\mp}(D) = 0.$$

Applying these operator to (1.1) we see that

$$(\partial_t \pm |\eta_{\#}||\nabla|)\psi_{\pm} = -i\kappa\Pi^{\pm}(D)\mathcal{N}_{\ell}(\psi, \psi)\psi \tag{2.1}$$

for $\ell = 1, 2$ with initial data

$$\psi_{\pm}(0) =: \psi_{0,\pm} \in H^s.$$

To simplify the representation of (2.1), we set the spinner

$$\phi_{\pm}(t, x) = \psi_{\pm}\left(\frac{t}{|\eta_{\#}|}, x\right).$$

Hence $\phi$ satisfies that

$$(i\partial_t \pm |\nabla|)\phi_{\pm} = \frac{1}{|\eta_{\#}|}(i\partial_t \pm |\eta_{\#}||\nabla|)\psi_{\pm} = -\frac{i\kappa}{|\eta_{\#}|}\Pi^{\pm}(D)\mathcal{N}_{\ell}(\phi, \phi)\phi$$

for $\ell = 1, 2$. We still call the spinner to $\psi$. Then we finally get the second main equation

$$(i\partial_t \pm |\nabla|)\psi_{\pm} = -i\kappa_{\#}\Pi^{\pm}(D)\mathcal{N}_{\ell}(\psi, \psi)\psi, \tag{2.2}$$

where $\kappa_{\#} = \frac{\kappa}{|\eta_{\#}|}$.

By Duhamel's formula, we can represent the equation (2.2) written as an integral equation

$$\psi_{\pm}(t) = S_{\pm}(t)\psi_{0,\pm} + \kappa_{\#}\int_0^t S_{\pm}(t-t')\Pi^{\pm}(D)\left[\mathcal{N}_{\ell}\Big(\psi(t'), \psi(t')\Big)\psi(t')\right] dt' \tag{2.3}$$

for $\ell = 1, 2$. Here we define the linear propagator $S_\pm(t)$ as follows:

$$S_\pm(t)f = e^{\mp it|\nabla|}f. \tag{2.4}$$

### 2.2. Fractional Leibniz rule

The following lemma called fractional Leibniz rule is useful in the proof of LWP.

**Lemma 2.1** ([15, 17, 18]). *Let $0 < s < 1,\ 1 < p < \infty$. Then*

$$\|D^s(fg) - fD^s g - gD^s f\|_{L^p} \lesssim \|D^{s_1}f\|_{L^{p_1}}\|D^{s_2}g\|_{L^{p_2}}$$

*provided $s = s_1 + s_2$ with $0 \le s_1, s_2 \le 1$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$.*

The proof of Lemma 2.1 is in [15, 17, 18].

## 3. Function spaces and bilinear estimates

### 3.1. Functions spaces: $X^{s,b}$-space

We first introduce $X_\pm^{s,b}$ space which will be useful in local theories. (See e.g. [5, 19, 24].) Let us define the norm for $s, b \in \mathbb{R}$ as follows:

$$X_\pm^{s,b}(T) := \left\{\psi : \left\|\chi_{[-T,T]}\psi\right\|_{X_\pm^{s,b}} < \infty\right\}$$

with a norm

$$\|\psi\|_{X_\pm^{s,b}} := \left(\int_{\mathbb{R}^{1+2}} \left|\langle\xi\rangle^s \langle\tau \mp |\xi|\rangle^b\, \widetilde{\psi}(\tau,\xi)\right|^2 d\tau d\xi\right)^{\frac{1}{2}}.$$

In particular, we denote that $X_{\pm_j}^{s,b}$ is $X_+^{s,b}$ for $\pm_j = +$ and $X_{\pm_j}^{s,b}$ is $X_-^{s,b}$ for $\pm_j = -$. These function spaces satisfy the embedding for $b > \frac{1}{2}$

$$X_\pm^{s,b}(T) \hookrightarrow C([-T,T]; H^s).$$

### 3.2. Bilinear estimates

Theorem 2.1 of [23] leads us the following lemma used in the proof of Lemma 3.2.

**Lemma 3.1** (Theorem 2.1 of [23]). *Let $\lambda > 0$ and $L \ge 1$. Let us define the thickened cones*

$$K_{\lambda,L}^\pm = \left\{(\tau,\xi) : |\xi| \lesssim \lambda,\ \tau \mp |\xi| = O(L)\right\}.$$

*Then*

$$\|P_{K_{\lambda,L}^\pm \cap(\mathbb{R}\times B_\mu)}u\|_{L_{t,x}^4} \lesssim \mu^{\frac{1}{4}}\lambda^{\frac{1}{8}}L^{\frac{3}{8}}\|P_{K_{\lambda,L}^\pm \cap(\mathbb{R}\times B_\mu)}u\|_{L_{t,x}^2}$$

*for $u : \mathbb{R}^{1+2} \to \mathbb{C}$ and any ball $B_\mu \subset \mathbb{R}^2$ with radius $\mu > 0$.*

The following lemma is crucial in the proof of Theorem 1.1.

**Lemma 3.2.** *Let $s > \frac{3}{8}$, $b > \frac{1}{2}$, and $u : \mathbb{R}^{1+2} \to \mathbb{C}$. Then the following holds:*

(1) $\|P_{B_\mu} u\|_{L^4_{t,x}} \lesssim \mu^{\frac14} \|P_{B_\mu} u\|_{X_\pm^{\frac18,b}}$ *for* $u \in X_\pm^{\frac18,b}$ *and any ball* $B_\mu$ *with radius* $\mu > 0$*,*

(2) $\|u\|_{L^4_{t,x}} \lesssim \|u\|_{X_\pm^{\frac38,b}}$ *for* $u \in X_\pm^{\frac38,b}$*,*

(3) $\|P_\mu(u_1\overline{u_2})\|_{L^2_{t,x}} \lesssim \mu^{\frac38 - s}\|u_1\|_{X^{s,b}_{\pm_1}}\|u_2\|_{X^{s,b}_{\pm_2}}$ *for* $\mu > 0$*,* $u_j \in X^{s,b}_{\pm_j}$*.*

*Proof.* We first prove (1). Lemma 3.1 yields that, for $\lambda \geq 1$,

$$
\begin{aligned}
\|P_{B_\mu} P_\lambda u\|_{L^4_{t,x}} &\lesssim \sum_{L\geq 1} \mu^{\frac14}\lambda^{\frac18} L^{\frac38} \|P_{K^\pm_{\lambda,L}\cap(\mathbb{R}\times B_\mu)} u\|_{L^2_{t,x}} \\
&\lesssim \sum_{L\geq 1} \mu^{\frac14}\lambda^{\frac18} L^{\frac38} \|P_{K^\pm_{\lambda,L}\cap(\mathbb{R}\times B_\mu)} u\|_{L^2_{t,x}} \\
&\lesssim \sum_{L\geq 1} \mu^{\frac14}\lambda^{\frac18} L^{\frac38 - b} \|L^b \chi_{K^\pm_{\lambda,L}\cap(\mathbb{R}\times B_\mu)} \widetilde{u}\|_{L^2_{\tau,\xi}} \\
&\lesssim \sum_{L\geq 1} \mu^{\frac14}\lambda^{\frac18} L^{\frac38 - b} \| \langle \tau \mp |\xi| \rangle^b \chi_{K^\pm_{\lambda,L}\cap(\mathbb{R}\times B_\mu)} \widetilde{u}\|_{L^2_{\tau,\xi}} \\
&\lesssim \mu^{\frac14}\lambda^{\frac18}\|P_{B_\mu} P_\lambda u\|_{X^{0,b}_\pm}.
\end{aligned}
$$

Then we have

$$
\|P_{B_\mu} u\|_{L^4_{t,x}} \lesssim \sum_{\lambda\geq 1} \|P_{B_\mu} P_\lambda u\|_{L^4_{t,x}} \lesssim \sum_{\lambda \geq 1} \mu^{\frac14}\lambda^{\frac18}\|P_{B_\mu} P_\lambda u\|_{X^{0,b}_\pm} \lesssim \mu^{\frac14} \|P_{B_\mu} u\|_{X_\pm^{\frac18,b}}.
$$

For (2), by (1) we obtain

$$
\|u\|_{L^4_{t,x}} \lesssim \left( \sum_{\mu \geq 1} \|P_\mu u\|^2_{L^4_{t,x}} \right)^{\frac12} \lesssim \left( \sum_{\mu\geq 1} \mu^{\frac14} \|P_\mu u\|^2_{X_\pm^{\frac18,b}} \right)^{\frac12} \lesssim \|u\|_{X_\pm^{\frac38,b}}.
$$

Let us now prove (3). Using frequency localization and (2), we see that

$$
\begin{aligned}
&\|P_\mu(u_1\overline{u_2})\|_{L^2_{t,x}} \\
&\lesssim \sum_{\substack{\lambda_1,\lambda_2\geq 1\\ \mu\lesssim\lambda_1\sim\lambda_2}} \|P_\mu(u_{1,\lambda_1}\overline{u_{2,\lambda_2}})\|_{L^2_{t,x}} + \sum_{\substack{\lambda_1,\lambda_2\geq 1\\ \lambda_{\min}\lesssim\lambda_{\max}\sim\mu}} \|P_\mu(u_{1,\lambda_1}\overline{u_{2,\lambda_2}})\|_{L^2_{t,x}} \\
&\lesssim \sum_{\substack{\lambda_1,\lambda_2\geq 1\\ \mu\lesssim\lambda_1\sim\lambda_2}} \|u_{1,\lambda_1}\|_{L^4_{t,x}}\|u_{2,\lambda_2}\|_{L^4_{t,x}} + \sum_{\substack{\lambda_1,\lambda_2\geq 1\\ \lambda_{\min}\lesssim\lambda_{\max}\sim\mu}} \|u_{1,\lambda_1}\|_{L^4_{t,x}}\|u_{2,\lambda_2}\|_{L^4_{t,x}} \\
&\lesssim \sum_{\substack{\lambda_1,\lambda_2\geq 1\\ \mu\lesssim\lambda_1\sim\lambda_2}} \lambda_1^{\frac38 - s}\lambda_2^{\frac38 - s}\|u_{1,\lambda_1}\|_{X^{s,b}_{\pm_1}}\|u_{2,\lambda_2}\|_{X^{s,b}_{\pm_2}} \\
&\quad + \sum_{\substack{\lambda_1,\lambda_2\geq 1\\ \lambda_{\min}\lesssim\lambda_{\max}\sim\mu}} \lambda_{\max}^{\frac38 - s}\|u_{1,\lambda_1}\|_{X^{s,b}_{\pm_1}}\|u_{2,\lambda_2}\|_{X^{s,b}_{\pm_2}}
\end{aligned}
$$

$$\lesssim \mu^{\frac{3}{8}-s}\|u_1\|_{X^{s,b}_{\pm_1}}\|u_2\|_{X^{s,b}_{\pm_2}}.$$

Here we used $\lambda_{\max} = \max(\lambda_1, \lambda_2)$, $\lambda_{\min} = \min(\lambda_1, \lambda_2)$. □

## 4. Local well-posedness: Proof of Theorem 1.1

Let us define a complete Banach metric space $(\mathcal{M}^{s,b}(T,\delta), d)$ as follows:

$$\begin{aligned}
\mathcal{M}^{s,b}(T,\delta) &:= \left\{\psi \in C\left([-T,T] : L^2_x\right) \cap X^{s,b}_{\pm}(T) : \|\psi\|_{\mathcal{M}^{s,b}} < \delta\right\},\\
\|\psi\|_{\mathcal{M}^{s,b}} &:= \|\psi_+\|_{X^{s,b}_+} + \|\psi_-\|_{X^{s,b}_+},\\
d(\psi,\phi) &:= \|\psi-\phi\|_{\mathcal{M}^{s,b}}.
\end{aligned}$$

We now consider a map $\mathcal{D}$ on $\mathcal{M}^{s,b}(T,\delta)$ by

$$\begin{aligned}
\mathcal{D}(\psi) = &\sum_{\pm_0\in\{\pm\}} S_{\pm_0}(t)\psi_{0,\pm_0}\\
&+ \sum_{\substack{\pm_j\in\{\pm\}\\ j=0,1,2,3}} \kappa_{\#}\int_0^t S_{\pm_0}(t-t')\Pi^{\pm_0}(D)\left[\mathcal{N}_\ell(\psi_{\pm_1},\psi_{\pm_2})\psi_{\pm_3}\right]dt',
\end{aligned}$$

where $\sum_{\pm_0\in\{\pm\}} F_{\pm_0} = F_+ + F_-$. Then we first show the map $\mathcal{D}$ is a self-mapping on $\mathcal{M}^{s,b}(T,\delta)$. By Lemma 2.1 of [13] we see that

$$\left\|\chi_{[-T,T]}S_{\pm}(t)\psi_{0,\pm}\right\|_{X^{s,b}_{\pm}} \lesssim T^{\frac{1}{2}-b}\|\psi_0\|_{H^s}$$

and

$$\left\|\chi_{[-T,T]}\int_0^t S_{\pm}(t-t')f(t')dt'\right\|_{X^{s,b}_{\pm}} \lesssim T^{1-b+b'}\|f\|_{X^{s,b'}}$$

for $-\frac{1}{2} < b' < 0 < \frac{1}{2} < b \le b'+1$.

**Proposition 4.1.** *Let $s > s(\ell) + \frac{3}{8}$ for $\ell = 1,2$. Then there exist $-\frac{1}{2} < b' < -\frac{1}{4} < \frac{1}{2} < b \le b'+1$ and $\varepsilon > 0$ such that*

$$\left\|\mathcal{N}_\ell(\psi_1,\psi_2)\psi_3\right\|_{X^{s,b'}_{\pm}} \le T^{\varepsilon}\prod_{j=1}^{3}\|\psi_j\|_{X^{s,b}_{\pm_j}}$$

*for all $\psi : \mathbb{R}^{1+2} \to \mathbb{C}^2$ and $\psi_j \in X^{s,b}_{\pm_j}$ with supp$(\psi_j) \subset \{(t,x) : |t| \le T\}$.*

Proposition 4.1 will be proved in the next section. We now assume the validity of Proposition 4.1. Then we estimate

$$\|\mathcal{D}(\psi)\|_{\mathcal{M}^{s,b}} \le C_1\|\psi_0\|_{H^s} + C_2T^{\varepsilon}\sum_{\pm}\|\psi_\pm\|^3_{X^{s,b}_{\pm}} \le C_1\|\psi_0\|_{H^s} + C_2T^{\varepsilon}\delta^3.$$

Set $C_1\|\psi_0\|_{H^s} < \frac{\delta}{2}$ and choose the time $T$ that satisfies $C_2T^\varepsilon\delta^3 < \frac{\delta}{2}$. Hence we see that $\|\mathcal{D}(\psi)\|_{\mathcal{M}^{s,b}} < \delta$. Therefore $\mathcal{D}$ is a self-mapping on $\mathcal{M}^{s,b}(T,\delta)$. We now describe the fact that $\mathcal{D}$ is a contraction mapping on $\mathcal{M}^{s,b}(T,\delta)$:

$$\begin{aligned} d(\mathcal{D}(\psi),\mathcal{D}(\phi)) &= \|\mathcal{D}(\psi)-\mathcal{D}(\phi)\|_{\mathcal{M}^{s,b}} \\ &\le C\left(\|\psi\|^2_{\mathcal{M}^{s,b}} + \|\phi\|^2_{\mathcal{M}^{s,b}}\right)\|\psi-\phi\|_{\mathcal{M}^{s,b}} \\ &\le 2C\delta^2\|\psi-\phi\|_{\mathcal{M}^{s,b}} < \frac{1}{2}d(\mathcal{D}(\psi),\mathcal{D}(\phi)) \end{aligned}$$

for $\delta$ satisfying that $4C\delta^2 < \frac{1}{2}$.

Therefore this completes the proof of the local existence and uniqueness of a solution to (1.1).

## 5. Proof of Proposition 4.1

### 5.1. Proof of Proposition 4.1

By duality, it suffices to prove that

$$I_\ell := \left|\iint \mathcal{N}_\ell(\psi_1,\psi_2)\psi_3\Lambda^s\psi_4^\dagger dtdx\right| \lesssim T^\varepsilon\prod_{j=1}^3\|\psi_j\|_{X^{s,b}_{\pm_j}}\|\psi_4\|_{X^{0,-b'}_{\pm_4}}$$

for $\psi_4 \in X^{0,-b'}_{\pm_4}$ and $\ell = 1,2$. We set $\psi_j = \begin{pmatrix}\psi_{j1}\\ \psi_{j2}\end{pmatrix}$ for $j = 1,2,3,4$. Then we have

$$\begin{aligned} I_1 &= \left|\iint \begin{pmatrix} \left(b_1\psi_{11}\overline{\psi_{21}} + 2b_2\psi_{12}\overline{\psi_{22}}\right)\psi_{31} \\ -\left(b_1\psi_{12}\overline{\psi_{22}} + 2b_2\psi_{11}\overline{\psi_{21}}\right)\psi_{32}\end{pmatrix}\begin{pmatrix}\Lambda^s\overline{\psi_{41}} & \Lambda^s\overline{\psi_{42}}\end{pmatrix}dtdx\right| \\ &= C\sum_{j,k,l\in\{1,2\}}\left|\iint \psi_{1j}\overline{\psi_{2j}}\psi_{3k}\Lambda^s\overline{\psi_{4l}}dtdx\right| \end{aligned}$$

and

$$\begin{aligned} I_2 &= \left|\iint |\nabla|^{-1}(\psi_1^\dagger\psi_2)\psi_3\Lambda^s\psi_4^\dagger dtdx\right| \\ &= C\sum_{j,k\in\{1,2\}}\left|\iint |\nabla|^{-1}\left(\overline{\psi_{1j}}\psi_{2j}\right)\psi_{3k}\Lambda^s\overline{\psi_{4k}}dtdx\right|. \end{aligned}$$

To compute the terms above, we introduce $\mathbb{C}$-valued version estimates below which will be proved Section 5.2.

**Lemma 5.1.** *The following two estimates hold*:

(i) *Let $s > s(1) + \frac{3}{8}$. Then there exist $-\frac{1}{2} < b' < -\frac{1}{4} < \frac{1}{2} < b \le b'+1$ and $\varepsilon > 0$ such that*

$$\left|\iint (u_1\overline{u_2})u_3\Lambda^s\overline{u_4}dtdx\right| \lesssim T^\varepsilon\prod_{j=1}^3\|u_j\|_{X^{s,b}_{\pm_j}}\|u_4\|_{X^{0,-b'}_{\pm_4}} \tag{5.1}$$

*for all $u_j : \mathbb{R}^{1+2}\to\mathbb{C}$ and $u_j \in X^{s,b}_{\pm_j}$ with supp$(u_j) \subset \{(t,x) : |t| \le T\}$.*

(ii) *Let $s > s(2) + \frac{3}{8}$. Then there exist $-\frac{1}{2} < b' < -\frac{1}{4} < \frac{1}{2} < b \le b' + 1$ and $\varepsilon > 0$ such that*

$$\left| \iint \left[ |x|^{-1} * (\overline{u_1} u_2) \right] u_3 \Lambda^s \overline{u_4} dt dx \right| \lesssim T^\varepsilon \prod_{j=1}^{3} \|u_j\|_{X^{s,b}_{\pm_j}} \|u_4\|_{X^{0,-b'}_{\pm_4}} \tag{5.2}$$

*for all $u_j : \mathbb{R}^{1+2} \to \mathbb{C}$ and $u_j \in X^{s,b}_{\pm_j}$ with supp$(u_\ell) \subset \{(t,x) : |t| \le T\}$.*

By Lemma 5.1, we get

$$\begin{aligned} I_\ell &\lesssim T^\varepsilon \sum_{j,k=1,2} \|\psi_{1j}\|_{X^{s,b}_{\pm_1}} \|\psi_{2j}\|_{X^{s,b}_{\pm_2}} \|\psi_{3k}\|_{X^{s,b}_{\pm_3}} \|\psi_{4k}\|_{X^{0,-b'}_{\pm_4}} \\ &\lesssim T^\varepsilon \prod_{j=1}^{3} \|\psi_j\|_{X^{s,b}_{\pm_j}} \|\psi_4\|_{X^{0,-b'}_{\pm_4}} \end{aligned}$$

for $\ell = 1, 2$. It completes the proof of Proposition 4.1.

## 5.2. Proof of Lemma 5.1

*Proof of* (i) *of Lemma 5.1.* We first set $\frac{7}{8} < s \le 1$. By Hölder inequality and Lemma 2.1, we can split the left-hand side of (5.1) as follows:

$$\begin{aligned} &\left| \iint (u_1 \overline{u_2}) u_3 \Lambda^s \overline{u_4} dt dx \right| \\ \le &\left| \iint \Lambda^s (\overline{u_1} u_2 u_3) \overline{u_4} dt dx - \iint \Lambda^s (\overline{u_1} u_2) u_3 \overline{u_4} dt dx - \iint \overline{u_1} u_2 \left( \Lambda^s u_3 \right) \overline{u_4} dt dx \right| \\ &+ \left| \iint \Lambda^s (\overline{u_1} u_2) u_3 \overline{u_4} dt dx \right| + \left| \iint \overline{u_1} u_2 \left( \Lambda^s u_3 \right) \overline{u_4} dt dx \right| \\ =: &\ J_1^1 + J_1^2 + J_1^3. \end{aligned}$$

We first treat the $J_1^1$. By Lemma 2.1, we estimate

$$\begin{aligned} J_1^1 &\lesssim \|\Lambda^s(\overline{u_1} u_2 u_3) - \Lambda^s(\overline{u_1} u_2) u_3 - \overline{u_1} u_2 \left( \Lambda^s u_3 \right)\|_{L_t^{\frac{4}{3}} L_x^2} \|u_4\|_{L_t^4 L_x^2} \\ &\lesssim \|\Lambda^s(u_1 \overline{u_2})\|_{L^2_{t,x}} \|u_3\|_{L_t^4 L_x^\infty} \|u_4\|_{L_t^4 L_x^2}. \end{aligned}$$

Let us consider $\|\Lambda^s(u_1 \overline{u_2})\|_{L^2_{t,x}}$. Like above estimates, Lemma 2.1 yields that

$$\begin{aligned} &\|\Lambda^s(u_1 \overline{u_2})\|_{L^2_{t,x}} \\ \lesssim &\left\| \Lambda^s(u_1 \overline{u_2}) - (\Lambda^s u_1) \overline{u_2} - u_1 (\Lambda^s \overline{u_2}) \right\|_{L^2_{t,x}} + \left\| (\Lambda^s u_1) \overline{u_2} \right\|_{L^2_{t,x}} + \left\| u_1 (\Lambda^s \overline{u_2}) \right\|_{L^2_{t,x}} \\ \lesssim &\left\| \Lambda^s u_1 \right\|_{L_t^4 L_x^2} \|u_2\|_{L_t^4 L_x^\infty} + \|u_1\|_{L_t^4 L_x^\infty} \left\| \Lambda^s u_2 \right\|_{L_t^4 L_x^2}. \end{aligned}$$

By Sobolev embedding and Lemma 3.2 we get

$$\left\| u_j \right\|_{L_t^4 L_x^\infty} \lesssim \left\| \Lambda^{s - \frac{3}{8}} u_j \right\|_{L^4_{t,x}} \lesssim \|u_j\|_{X^{s,b}_{\pm_j}} \tag{5.3}$$

for $s > \frac{7}{8}$, $b > \frac{1}{2}$, and $j = 1, 2$. By embedding $X^{0,\frac{1}{4}} \hookrightarrow L_t^4 L_x^2$, the estimate (5.3) leads us that

$$\|\Lambda^s(u_1\overline{u_2})\|_{L^2_{t,x}} \lesssim \|u_1\|_{X^{s,b}_{\pm_1}} \|u_2\|_{X^{s,b}_{\pm_2}}. \tag{5.4}$$

In particular, by (5.3), we have

$$\|u_3\|_{L_t^4 L_x^\infty} \lesssim \|u_3\|_{X^{s,b}_{\pm_3}}. \tag{5.5}$$

Using the estimates (5.4), (5.5), and embedding $X^{0,\frac{1}{4}} \hookrightarrow L_t^4 L_x^2$ , we see that

$$\begin{aligned} J_1^1 &\lesssim \|u_1\|_{X^{s,b}_{\pm_1}} \|u_2\|_{X^{s,b}_{\pm_2}} \|u_3\|_{X^{s,\frac{1}{4}}_{\pm_3}} \|u_4\|_{X^{0,\frac{1}{4}}_{\pm_4}} \\ &\lesssim T^\delta \prod_{j=1}^{3} \|u_j\|_{X^{s,b}_{\pm_j}} \|u_4\|_{X^{0,-b'}_{\pm_4}}. \end{aligned}$$

On the other hand, for $J_1^2, J_1^3$, we obtain

$$J_1^2 \lesssim \|\Lambda^s(u_1\overline{u_2})\|_{L^2_{t,x}} \|u_3\|_{L_t^4 L_x^\infty} \|u_4\|_{L_t^4 L_x^2}, \tag{5.6}$$

$$J_1^3 \lesssim \|u_1\|_{L_t^4 L_x^\infty} \|u_2\|_{L_t^4 L_x^\infty} \|\Lambda^s u_3\|_{L_t^4 L_x^2} \|u_4\|_{L_t^4 L_x^2}. \tag{5.7}$$

The estimates for (5.6) are obtained in a similar way to estimates of $J_1^1$. Hence we consider (5.7). Using (2) of Lemma 3.2 and Sobolev embedding, we estimate

$$\begin{aligned} \|u_j\|_{L_t^4 L_x^\infty} &\lesssim \|\Lambda^{s-\frac{3}{8}} u_j\|_{L^4_{t,x}} \lesssim \|u_j\|_{X^{s,b}_{\pm_j}} \quad \text{for } j = 1, 2, \\ \|\Lambda^s u_3\|_{L_t^4 L_x^2} &\lesssim \|u_3\|_{X^{s,\frac{1}{4}}} \lesssim \|u_3\|_{X^{s,b}_{\pm_3}}. \end{aligned} \tag{5.8}$$

Then the estimate (5.8) yields that

$$J_1^3 \lesssim T^\delta \prod_{j=1}^{3} \|u_j\|_{X^{s,b}_{\pm_j}} \|u_4\|_{X^{0,-b'}_{\pm_4}}.$$

Therefore this completes the proof of (5.1). □

*Proof of* (ii) *of Lemma 5.1.* The LHS of (5.2) is bounded by

$$\begin{aligned} &\left| \iint \left[|x|^{-1} * (\overline{u_1} u_2)\right] u_3 \Lambda^s \overline{u_4} dtdx \right| \\ \le &\left| \iint \Lambda^s(|\nabla|^{-1}(\overline{u_1}u_2)u_3)\overline{u_4} dtdx - \iint \Lambda^s |\nabla|^{-1}(\overline{u_1}u_2)u_3\overline{u_4} dtdx \right. \\ &\quad \left. - \iint |\nabla|^{-1}(\overline{u_1}u_2)(\Lambda^s u_3)\overline{u_4} dtdx \right| \\ &\quad + \left| \iint \Lambda^s |\nabla|^{-1}(\overline{u_1}u_2)u_3\overline{u_4} dtdx \right| + \left| \iint |\nabla|^{-1}(\overline{u_1}u_2)(\Lambda^s u_3)\overline{u_4} dtdx \right| \\ =: &\ J_2^1 + J_2^2 + J_2^3. \end{aligned}$$

We first consider the $J_2^1$. Lemma 2.1 yields that

$$
\begin{aligned}
&J_2^1 \\
&\lesssim \left\|\Lambda^s\left[|\nabla|^{-1}(\overline{u_1}u_2)u_3\right] - \Lambda^s|\nabla|^{-1}(\overline{u_1}u_2)u_3 - |\nabla|^{-1}(\overline{u_1}u_2)\left(\Lambda^s u_3\right)\right\|_{L_t^{\frac{4}{3}}L_x^2} \\
&\quad \times \|u_4\|_{L_t^4 L_x^2} \\
&\lesssim \|\Lambda^s|\nabla|^{-1}(u_1\overline{u_2})\|_{L_t^2L_x^4}\|u_3\|_{L_{t,x}^4}\|u_4\|_{L_t^4L_x^2} \\
&\lesssim \sum_\mu \|P_\mu\Lambda^s|\nabla|^{-1}(u_1\overline{u_2})\|_{L_t^2L_x^4}\|u_3\|_{L_{t,x}^4}\|u_4\|_{L_t^4L_x^2}.
\end{aligned}
$$

Using Hardy-Littlewood-Sobolev and Young's convolution inequality we see that

$$
\begin{aligned}
\|P_{\le 1}\Lambda^s|\nabla|^{-1}(u_1\overline{u_2})\|_{L_t^2L_x^4} &\lesssim \|P_{\le 1}(u_1\overline{u_2})\|_{L_t^2L_x^{\frac{4}{3}}} = \|\check{\beta}_1 * (u_1\overline{u_2})\|_{L_t^2L_x^{\frac{4}{3}}} \\
&\lesssim \|\check{\beta}_1\|_{L_x^{\frac{4}{3}}}\|u_1\|_{L_t^4L_x^2}\|u_2\|_{L_t^4L_x^2} \\
&\lesssim \|u_1\|_{X_{\pm_1}^{0,\frac{1}{4}}}\|u_2\|_{X_{\pm_2}^{0,\frac{1}{4}}}.
\end{aligned}
$$

In the third inequality, we used the $\|\check{\beta}_1\|_{L_x^p} \lesssim 1$ for $p > 1$. And, by Lemma 3.2, we estimate

$$
\begin{aligned}
&\sum_{\mu\ge 2}\|P_\mu\Lambda^s|\nabla|^{-1}(u_1\overline{u_2})\|_{L_t^2L_x^4} \\
&\lesssim \sum_{\mu\ge 2}\mu^{s-1}\|P_\mu(\overline{u_1}u_2)\|_{L_t^2L_x^4} \lesssim \sum_{\mu\ge 2}\mu^{s-\frac{1}{2}}\|P_\mu(\overline{u_1}u_2)\|_{L_{t,x}^2} \\
&\lesssim \sum_{\mu\ge 2}\mu^{-\frac{1}{8}}\|u_1\|_{X_{\pm_1}^{s,b}}\|u_2\|_{X_{\pm_2}^{s,b}} \lesssim \|u_1\|_{X_{\pm_1}^{s,b}}\|u_2\|_{X_{\pm_2}^{s,b}}.
\end{aligned}
$$

Also, by the second estimate of (5.8), we get

$$
J_2^1 \lesssim \prod_{j=1}^3\|u_j\|_{X_{\pm_j}^{s,b}}\|u_4\|_{L_t^4L_x^2} \lesssim T^\delta\prod_{j=1}^3\|u_j\|_{X_{\pm_j}^{s,b}}\|u_4\|_{X_{\pm_4}^{0,-b'}}.
$$

Estimates for $J_2^2$ are obtained in almost the same way as estimates for $J_2^1$. Hence it is left to deal with $J_2^3$. By Hardy-Littlewood-Sobolev and Young's convolution inequality, we have

$$
\begin{aligned}
&\left|\iint |\nabla|^{-1}P_{\le 2}\left(\overline{u_1}u_2\right)\left(\Lambda^s u_3\right)\overline{u_4}\,dtdx\right| \\
&\lesssim \||\nabla|^{-1}P_{\le 2}(\overline{u_1}u_2)\|_{L_t^2L_x^4}\left\|P_{\le 2}\left[(\Lambda^s u_3)\overline{u_4}\right]\right\|_{L_t^2L_x^{\frac{4}{3}}} \\
&\lesssim \|P_{\le 2}(\overline{u_1}u_2)\|_{L_t^2L_x^{\frac{4}{3}}}\|(\Lambda^s u_3)\overline{u_4}\|_{L_t^2L_x^1} \\
&\lesssim \|u_1\|_{L_t^4L_x^2}\|u_2\|_{L_t^4L_x^2}\|\Lambda^s u_3\|_{L_t^4L_x^2}\|u_4\|_{L_t^4L_x^2} \\
&\lesssim \|u_1\|_{X_{\pm_1}^{0,\frac{1}{4}}}\|u_2\|_{X_{\pm_2}^{0,\frac{1}{4}}}\|u_3\|_{X_{\pm_3}^{s,\frac{1}{4}}}\|u_4\|_{X_{\pm_4}^{0,\frac{1}{4}}}.
\end{aligned}
$$

Since there is no contribution of $P_{\leq 1}(\overline{u_1}u_2)$, we assume that $P_{\leq 1}(\overline{u_1}u_2) = 0$. Let us consider the high-frequency part of $J_2^3$. By Lemma 3.2 and Bernstein's inequality we estimate

$$\begin{aligned}
J_2^3 &\lesssim \sum_{\mu \geq 2} \left| \iint |\nabla|^{-1} P_\mu(\overline{u_1}u_2)(\Lambda^s u_3)\overline{u_4} dtdx \right| \\
&\lesssim \sum_{\mu \geq 2} \mu^{-1} \left\| P_\mu(\overline{u_{\lambda_1}}u_{\lambda_2}) \right\|_{L_t^2 L_x^\infty} \left\| (\Lambda^s u_3)\overline{u_4} \right\|_{L_t^2 L_x^1} \\
&\lesssim \sum_{\mu \geq 2} \left\| P_\mu(\overline{u_{\lambda_1}}u_{\lambda_2}) \right\|_{L_{t,x}^2} \left\| \Lambda^s u_3 \right\|_{L_t^4 L_x^2} \left\| u_4 \right\|_{L_t^4 L_x^2} \\
&\lesssim \sum_{\mu \geq 2} \mu^{\frac{3}{8}-s} \|u_1\|_{X_{\pm_1}^{x,b}} \|u_{\lambda_2}\|_{X_{\pm_2}^{s,b}} \left\| \Lambda^s u_3 \right\|_{L_t^4 L_x^2} \left\| u_4 \right\|_{L_t^4 L_x^2} \\
&\lesssim T^\delta \prod_{j=1} \|u_j\|_{X_{\pm_j}^{s,b}} \|u_4\|_{X_{\pm_4}^{0,-b'}}.
\end{aligned}$$

Here we used the assumption $s > \frac{3}{8}$ and $b' < -\frac{1}{4}$. Therefore this completes the proof of the (5.2). $\square$

## 6. The proof of Theorem 1.2

This section aims to show Theorem 1.2. It adopts the argument [16, 20, 22] to prove of smoothness failure of flows of (1.1) with cubic and Hartree type nonlinearity. As in the proof of [16,22], if the flow map $\psi \mapsto u$ is $C^3$ at the origin from $H^s$ to $C([0,T);H^s)$, we have (6.4). In [16, 22], they showed smoothness failure of flows of Benjamin-Ono, semi-relativistic equations, respectively. For the results about Dirac equation, we refer to [20]. Let us consider the system of equation ($\ell = 1, 2$):

$$\begin{cases} (\partial_t + \alpha \cdot D)\psi = -i\kappa \mathcal{N}_\ell(\psi,\psi)\psi, \\ \psi(0) = \delta\psi_0 \in H^s(\mathbb{R}^2). \end{cases} \tag{6.1}$$

If the flow is $C^3$ at the origin in $H^s$, then it follows that

$$\begin{aligned}
\partial_\delta^3 \psi(0,t,\cdot) = 6C \sum_{\pm_j, j=1,2,3,4} &\int_0^t S_{\pm_1}(t-t')\Pi^{\pm_1}(D) \\
&\times \Big[ \mathcal{N}_\ell\Big(S_{\pm_2}(t')\psi_0, S_{\pm_3}(t')\psi_0\Big) S_{\pm_4}(t')\psi_0 \Big](t')dt',
\end{aligned} \tag{6.2}$$

where $S_\pm(t) = e^{-\pm it|D|}$ for $\ell = 1, 2$. From the $C^3$ smoothness we have that

$$\begin{aligned}
\sup_{0 \leq t \leq T} \Bigg\| \sum_{\pm_j, j=1,2,3,4} &\int_0^t S_{\pm_1}(t-t')\Pi^{\pm_1}(D) \\
&\times \Big[ \mathcal{N}_\ell\Big(S_{\pm_2}(t')\psi_0, S_{\pm_3}(t')\psi_0\Big) S_{\pm_4}(t')\psi_0 \Big] dt' \Bigg\|_{H^s} \lesssim \|\psi_0\|_{H^s}^3
\end{aligned} \tag{6.3}$$

for a local existence time $T$ and $j=1,2$. However we show that (6.3) fails for $s<s(\ell)$. The explicit statement is as follows:

**Proposition 6.1.** *Let $\ell=1,2$. Assume that $s<s(\ell)$. Then the estimate*

$$\sup_{0\le t\le T}\|\mathcal{L}_\ell(\varphi)(t)\|_{H^s}\lesssim\|\varphi\|_{H^s}^3 \tag{6.4}$$

*fails to hold for all $\varphi\in H^s$, where $\mathcal{L}_\ell(\varphi)(t)=\sum_{\pm_j,j=1,\dots,4}\mathcal{L}_\ell^{1,\dots,4}(\varphi)(t)$ with*

$$\mathcal{L}_\ell^{1,\dots,4}(\varphi)(t)=\int_0^t S_{\pm_1}(t-t')\Pi^{\pm_1}(D)\mathcal{N}_\ell\Big(S_{\pm_2}(t')\varphi,S_{\pm_3}(t')\varphi,\beta S_{\pm_4}(t')\varphi\Big)dt'.$$

*Proof.* Proposition 6.1 is proven by contradiction. For this purpose, let us assume that the (6.4) holds. Fix $\lambda\gg1$. We first choose $\mu=\lambda^{1-\varepsilon}$ for fixed $0<\varepsilon\ll1$. Let us define a box

$$B_\mu=\{\xi=(\xi_1,\xi_2):|\xi_1-\lambda|\lesssim\mu,\ |\xi_2|\lesssim\mu\}$$

and consider $\varphi=\begin{pmatrix}\mathcal{F}_\xi^{-1}\chi_{B_\mu}\\0\end{pmatrix}$. Then we have $\|\varphi\|_{H^s}\sim\mu\lambda^s$.

To lead a contradiction we adopt a following estimate:

$$\left|\sum_{\pm_j,j=1,\dots,4}\mathcal{F}_x\Big[\mathcal{L}_\ell^{1\cdots4}(\varphi)(t)\Big](\xi)\right|\gtrsim t\mu^{3+2s(j)}. \tag{6.5}$$

We now prove (6.5). By taking Fourier transform we see that

$$\begin{aligned}
&\mathcal{F}_x\Big[\mathcal{L}_\ell^{1\cdots4}(\varphi)(t)\Big](\xi)\\
&=\Pi^{\pm}(\xi)\int_0^t\int e^{-\pm_1 i(t-t')|\xi|}\mathcal{F}_x\left[\mathcal{N}_\ell\Big(S_{\pm_2}(t')\varphi,S_{\pm_3}(t')\varphi\Big)\right](\sigma)\\
&\qquad\qquad\times\mathcal{F}_x\left[S_{\pm_4}(t')\varphi\right](\xi-\sigma)d\sigma dt'\\
&=\ -\Pi^{\pm}(\xi)\int_{|\sigma|\lesssim\mu}\int_{-B_\mu}\mathbf{p}_{1\cdots4}(t,\xi,\sigma,\zeta)|\sigma|^{-1+2s(j)}\\
&\qquad\qquad\times\chi_{B_\mu}(-\zeta)\chi_{B_\mu}(\sigma-\zeta)\chi_{B_\mu}(\xi-\sigma)d\zeta d\sigma,
\end{aligned}$$

where $-B_\mu:=\{\xi=(\xi_1,\xi_2):(-\xi_1,-\xi_2)\in B_\mu\}$ and

$$\begin{aligned}
\mathbf{p}_{1\cdots4}(t,\xi,\sigma,\zeta)&:=\int_0^t e^{-i(\pm_1(t-t')|\xi|\pm_2t'|\zeta|\pm_3t'|\sigma-\zeta|\pm_4t'|\sigma|)}dt'\\
&=\frac{e^{-\pm_1 it|\xi|}(e^{it\omega}-1)}{i\omega}
\end{aligned}$$

with

$$\omega=\pm_1|\xi|\pm_2|\zeta|\pm_3|\sigma-\zeta|\pm_4|\sigma|.$$

From the support condition it follows that $|\sigma|\lesssim2\mu$, provided $\xi\in B_{3\mu}$. Then $|\omega|\lesssim\lambda$.

We set $t = \delta\lambda^{-1-\varepsilon}$ for fixed $0 < \delta \ll 1$. Since $|t\omega| \ll 1$ for $\lambda$ large enough, we get

$$\begin{aligned}\sum_{\pm_j, j=1,2,3,4} \mathbf{p}_{1\cdots 4}(t,\xi,\sigma,\zeta) &= \sum_{\pm_j, j=1,2,3,4} te^{-\pm_1 it|\xi|}\left(\frac{\cos(t\omega)-1}{it\omega} + i\frac{\sin(t\omega)}{it\omega}\right) \\ &= \sum_{\pm_j, j=1,2,3,4} te^{-\pm_1 it|\xi|}(O_{\pm}(\delta)+i) \\ &= \sum_{\pm_j, j=1,2,3,4} te^{-\pm_1 it|\xi|}O_{\pm}(\delta) + i\sum_{\pm_j, j=1,2,3,4} te^{-\pm_1 it|\xi|} \\ &= \sum_{\pm_j, j=1,2,3,4} te^{-\pm_1 it|\xi|}O_{\pm}(\delta) + 8it\cos(t|\xi|) \\ &= \sum_{\pm_j, j=1,2,3,4} te^{-\pm_1 it|\xi|}O_{\pm}(\delta) + 8it(1+O(\delta)).\end{aligned}$$

Hence we obtain

$$\begin{aligned}&\left|\sum_{\pm_j, j=1,2,3,4} \mathcal{F}_x\left[\mathcal{L}_\ell^{1\cdots 4}(\varphi)(t)\right](\xi)\right| \\ &\gtrsim \left|\sum_{\pm_j, j=1,2,3,4} \int_{|\sigma|\lesssim\mu}\int_{-B_\mu} \mathbf{p}_{1\cdots 4}(t,\xi,\sigma,\zeta)|\sigma|^{-1+2s(\ell)}\right. \\ &\qquad \left. \times\, \chi_{B_\mu}(-\zeta)\chi_{B_\mu}(\sigma-\zeta)\chi_{B_\mu}(\xi-\sigma)d\zeta d\sigma\right| \\ &\gtrsim t\left|\int_{|\sigma|\lesssim\mu}\int_{-B_\mu} |\sigma|^{-1+2s(\ell)}\chi_{B_\mu}(-\zeta)\chi_{B_\mu}(\sigma-\zeta)\chi_{B_\mu}(\xi-\sigma)d\zeta d\sigma\right| \\ &\gtrsim t\mu^{3+2s(\ell)}.\end{aligned}$$

Therefore we get (6.5).

We return to the main proof. Since $\mathcal{F}_x\left[\mathcal{L}_\ell^{1\cdots 4}(\varphi)(t)\right](\xi) = 0$ for $\xi \notin B_{5\mu}$, (6.5) yields that

$$\begin{aligned}\|\mathcal{L}_\ell(\varphi)(t)\|_{H^s} &= \left\|\langle\xi\rangle^s \sum_{\pm_j, j=1,2,3,4} \mathcal{F}_x\left[\mathcal{L}_\ell^{1\cdots 4}(\varphi)(t)\right](\xi)\right\|_{L^2_\xi} \\ &\gtrsim t\mu^{3+2s(\ell)}\,\|\langle\xi\rangle^s\|_{L^2_\xi(B_{5\mu})} \gtrsim t\mu^{4+2s(\ell)}\lambda^s.\end{aligned}$$

This gives us that

$$t\mu^{4+2s(\ell)}\lambda^s \lesssim \left\|\langle\xi\rangle^s \sum_{\pm_j, j=1,\cdots,4} \mathcal{F}_x\left[\mathcal{L}_j^{1\cdots 4}(\varphi)(t)\right](\xi)\right\|_{L^2_\xi} \lesssim \mu^3\lambda^{3s}. \tag{6.6}$$

Therefore, by (6.6) and $t = \delta\lambda^{-1-\varepsilon}$, we have

$$\delta \lesssim \mu^{-1-2s(\ell)}\lambda^{2s+1+\varepsilon} = \lambda^{2s+2s(\ell)+2\varepsilon\left(1+s(\ell)\right)}. \tag{6.7}$$

Then since (6.7) does not hold for $s < s(\ell)$ and $\lambda \gg 1$, we reach a contradiction. This completes the proof of Proposition 6.1. □

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