ON WEAKLY QUASI $n$-ABSORBING SUBMODULES

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Abstract. Let $R$ be a commutative ring with $1 \neq 0$, $n$ be a positive integer and $M$ be an $R$-module. In this paper, we introduce the concept of weakly quasi $n$-absorbing submodule which is a proper generalization of quasi $n$-absorbing submodule. We define a proper submodule $N$ of $M$ to be a weakly quasi $n$-absorbing submodule if whenever $a \in R$ and $x \in M$ with $0 \neq a^n x \in N$, then $a^n \in (N :_R M)$ or $a^{n-1} x \in N$. We study the basic properties of this notion and establish several characterizations.

1. Introduction

Throughout the whole paper, all rings are assumed to be commutative with $1 \neq 0$, all modules are considered to be unitary and $n$ is a positive integer. Let $R$ be a ring with $1 \neq 0$, $M$ be an $R$-module and $N$ be a proper submodule of $M$. In [9], the authors introduced and investigated the concept of 2-absorbing (resp., weakly 2-absorbing) submodules. They defined a submodule $N$ to be a 2-absorbing submodule (resp., weakly 2-absorbing submodule) of $M$ if whenever $a,b \in R$ and $m \in M$ with $abm \in N$ (resp., $0 \neq abm \in N$), then $ab \in (N :_R M)$ or $am \in N$ or $bm \in N$. A more general concept than 2-absorbing submodule is the concept $n$-absorbing submodule. From [10], a proper submodule $N$ of $M$ is said to be an $n$-absorbing (resp., strongly $n$-absorbing) submodule of $M$ if whenever $a_1 \cdots a_n m \in N$ for $a_1,\ldots,a_n \in R$ and $m \in M$ (resp., $I_1 \cdots I_n L \subset N$ for ideals $I_1,\ldots,I_n$ of $R$ and a submodule $L$ of $M$), then either $a_1 \cdots a_n \in (N :_R M)$ (resp., $I_1 \cdots I_n \subset (N :_R M)$) or there are $n-1$ of $a_i$’s (resp., $I_i$’s) whose product with $m$ (resp., $L$) is in $N$. Recall that a proper submodule $N$ of $M$ is called semiprime if whenever $r \in R$ and $m \in M$ with $r^2 m \in N$, then $rm \in N$. For more details about the concept of $n$-absorbing and related notions, we refer the reader to [3,4,6,7,13].

In this paper, we introduce the concept of weakly quasi $n$-absorbing submodule which is a proper generalization of quasi $n$-absorbing submodule. We define a proper submodule $N$ of $M$ to be a weakly quasi $n$-absorbing submodule if whenever $a \in R$ and $x \in M$ with $0 \neq a^n x \in N$, then $a^n \in (N :_R M)$ or...
We study the basic properties of this notion and establish several characterizations.

We denote by $\sqrt{I}$, the radical of an ideal $I$ of $R$. Let $N$ be a submodule of an $R$-module $M$. We denote by $(N :_R M)$, the residual of $N$ by $M$, that is, the set of all $r \in R$ such that $rM \subseteq N$. For $x \in M$, we denote by $\text{ann}(x)$, the annihilator of $x$, that is, the set of all $r \in R$ such that $rx = 0$.

2. Results

It is worthwhile recalling that a proper submodule $N$ of an $R$-module $M$ is a quasi $n$-absorbing submodule for some positive integer $n \geq 1$, if $a^n x \in N$ for some $a \in R$ and $x \in M$ with $a^n x \in N$, then either $a^{n-1} x \in N$ or $a^n \in (N :_R M)$. Now, we recall the concept of weakly quasi $n$-absorbing submodule defined in the introduction.

Definition. A proper submodule $N$ of an $R$-module $M$ is called a weakly quasi $n$-absorbing submodule of $M$ if $0 \neq a^n x \in N$ for some $a \in R$ and $x \in M$, then $a^n \in (N :_R M)$ or $a^{n-1} x \in N$.

Notice that from the previous definition, every quasi $n$-absorbing submodule is clearly a weakly quasi $n$-absorbing submodule. However, a weakly quasi $n$-absorbing submodule need not be a quasi $n$-absorbing submodule, as illustrated in the next example.

Example 2.1. Let $M := \mathbb{Z}/12\mathbb{Z}$ as $\mathbb{Z}$-module and $N = \{0\}$. Clearly, $N$ is a weakly quasi 2-absorbing submodule of $M$. However, $N$ is not a quasi 2-absorbing submodule of $M$ since $(N : M) = 12\mathbb{Z}$ and $2 \cdot 3 \in N$ and neither $2^2 \in (N :_R M)$ nor $2 \cdot \ldots \cdot 3 \in N$.

Now, we introduce the following definition which will be useful for studying the weakly quasi $n$-absorbing submodules.

Definition. Let $R$ be a ring, $M$ be an $R$-module and $N$ be a weakly quasi $n$-absorbing submodule of $M$. An element $a \in R$ is called an unbreakable element of $N$ if there exists an element $x \in M$ such that $a^n x = 0$ and neither $a^n \in (N :_R M)$ nor $a^{n-1} x \in N$.

It is worthwhile mentioning that if $N$ is a weakly quasi $n$-absorbing submodule of $M$ and there is no unbreakable element, then $N$ is a quasi $n$-absorbing submodule of $M$. The next lemma gives some basic facts about unbreakable elements.

Lemma 2.2. Let $R$ be a ring, $M$ be an $R$-module and $N$ be a proper weakly quasi $n$-absorbing submodule of $M$. If $a \in R$ is an unbreakable element of $N$. Then the following statements hold:

1. $a^n N = 0$.
2. $a + s$ is an unbreakable element of $N$ for every $s \in (N :_R M)$.
Proof. (1) Let \( a \) be an unbreakable element of \( N \). Then there exists \( x \in M \) with \( a^n x = 0 \) but neither \( a^n \in (N :_R M) \) nor \( a^{n-1} x \in N \). Assume by the way of contradiction that \( 0 \neq a^n N \), then \( 0 \neq a^s y \in N \) for some \( y \in N \). Since \( N \) is a weakly quasi \( n \)-absorbing submodule of \( M \) and \( a^n \notin (N :_R M) \), then \( a^{n-1} y \in N \). On the other hand, \( 0 \neq a^n (x + y) = a^s y \in N \) and \( a^n \notin (N :_R M) \) implies that \( a^{n-1} (x + y) \in N \). Thus \( a^{n-1} x \in N \), which is a contradiction. Hence, \( a^n N = 0 \).

(2) Since \( a \) is an unbreakable element of \( N \), then there exists \( x \in M \) with \( a^n x = 0 \) and neither \( a^n \in (N :_R M) \) nor \( a^{n-1} x \in N \). Now let \( s \in (N :_R M) \). Assume that \( 0 \neq (a + s)^n x \). We have:

\[
(a + s)^n x = \sum_{j=0}^{n-1} \binom{n}{j} a^j s^{n-j} x \in N.
\]

The fact that \( N \) is a weakly quasi \( n \)-absorbing submodule of \( M \), gives either \((a + s)^{n-1} x \in N\) or \((a + s)^n \in (N :_R M)\). Two cases are then possible:

Case 1 : \((a + s)^{n-1} x \in N\). Then one can easily check that \( a^{n-1} x \in N \) since for all \( j = 1, \ldots, n - 1 \), \( a^j s^{n-1-j} x \in N \), the desired contradiction.

Case 2 : \((a + s)^n \in (N :_R M)\). Since \( a^j s^{n-j} \in (N :_R M) \), then \( a^n \in (N :_R M) \). Hence, \((a + s)^n x = 0\) and neither \((a + s)^{n-1} x \in N\) nor \((a + s)^n \in (N :_R M)\). Thus, it follows that \( a + s \) is an unbreakable element of \( N \).

Finally, \( a + s \) is an unbreakable element of \( N \), as desired. \(\square\)

**Theorem 2.3.** Let \( R \) be a ring, \( M \) be an \( R \)-module and \( N \) be a proper weakly quasi \( n \)-absorbing submodule which is not quasi \( n \)-absorbing submodule of \( M \). Then \((N :_R M) \subseteq \sqrt{\Ann(N)}\).

Proof. Since \( N \) is a weakly quasi \( n \)-absorbing submodule which is not quasi \( n \)-absorbing submodule of \( M \), then there exists an unbreakable element \( b \) of \( N \). By Lemma 2.2(2), for every \( a \in (N :_R M) \), we have \((b + a)^n N = 0\). So, \( a + b \in \sqrt{\Ann(N)} \). By Lemma 2.2(1), \( b \in \sqrt{\Ann(N)} \) and so \( a \in \sqrt{\Ann(N)} \). Hence, \((N :_R M) \subseteq \sqrt{\Ann(N)}\), as desired. \(\square\)

Let \( R \) be a ring and \( M \) be an \( R \)-module. Recall that \( M \) is called a multiplication module if for each submodule \( N \) of \( M, N = IM \) for some ideal \( I \) of \( R \). In this case, we can take \( I = (N : R M) \) [11]. Also, recall that for a submodule \( N \) of \( M \), if \( N = IM \) for some ideal \( I \) of \( R \), then \( I \) is called a presentation ideal of \( N \). Clearly, every submodule of \( M \) has a presentation ideal if and only if \( M \) is a multiplication module. Let \( N \) and \( K \) be submodules of a multiplication \( R \)-module \( M \) with \( N = I_1 M \) and \( K = I_2 M \) for some ideals \( I_1 \) and \( I_2 \) of \( R \), the product \( N \) and \( K \) denoted by \( NK \) is defined by \( NK = I_1 I_2 M \). From [1, Theorem 3.4], the product of \( N \) and \( K \) is independent of presentation of \( N \) and \( K \). Moreover, for \( a, b \in M \), by \( ab \), we mean the product of \( Ra \) and \( Rb \). Clearly, \( NK \) is a submodule and \( NK \subseteq N \cap K \) [1]. A submodule \( N \) of an \( R \) module \( M \) is called nilpotent if \((N :_R M)^k N = 0\) for some positive integer \( k \) [2]. The next corollary is a consequence of Theorem 2.3.
Corollary 2.4. Let $R$ be a Noetherian ring and $M$ be an $R$-module. If $N$ is a proper weakly quasi $n$-absorbing submodule which is not quasi $n$-absorbing submodule of $M$. Then:

1. $N$ is nilpotent.
2. If $M$ is a faithful multiplication module, then $N^p = 0$ for some positive integer $p$.

Proof. (1) By Theorem 2.3, we have $(N :_RM) \subseteq \sqrt{\text{ann}(N)}$. Since $R$ is Noetherian, then there exists a positive integer $k \geq 1$ such that $(N :_RM)^k \subseteq \text{ann}(N)$. So, $(N :_RM)^kN = 0$. Hence, $N$ is a nilpotent submodule of $M$.

(2) By assertion (1) above, we have $(N :_RM)^kN = 0$ for some positive integer $k \geq 1$. It follows that $(N :_RM)^{k+1} \subseteq ((N :_RM)^kN :_RM) = (0 :_RM) = 0$, as $M$ is faithful. Therefore, $(N :_RM)^{k+1} = 0$. □

Let $N$ be a proper submodule of a nonzero $R$-module $M$. Then the $M$-radical of $N$, denoted here by $M - \sqrt{N}$ is defined in [12] to be the intersection of all prime submodules of $M$ containing $N$. It is shown in [11, Theorem 2.12] that if $N$ is a proper submodule of $M$, then $M - \sqrt{N} = M - \sqrt{(N :_RM)M}$. The next corollary is an application of Theorem 2.3.

Corollary 2.5. Let $R$ be a ring, $M$ be a multiplication $R$-module and $N$ be a proper faithful weakly quasi $n$-absorbing submodule which is not quasi $n$-absorbing submodule of $M$. Then $N \subseteq M - \sqrt{0}$.

Proof. Since $M$ is a multiplication module, then $N = (N :_RM)M$. So, by Theorem 2.3, it follows that $N = (N :_RM)M \subseteq \sqrt{0}M = M - \sqrt{0}$, as $N$ is faithful.

Recall that a ring is called von Neumann regular if, for every $x \in R$ there exists $y \in R$ such that $x^2y = x$. It is well known that a commutative ring is von Neumann regular if and only if every proper ideal is radical. The next corollary is another consequence of Theorem 2.3.

Corollary 2.6. Let $R$ be a von Neumann regular ring, $M$ be an $R$-module and $N$ be a proper weakly quasi $n$-absorbing submodule which is not quasi $n$-absorbing submodule of $M$. Then $(N :_RM)N = 0$.

Proof. Assume that $R$ is a von Neumann regular ring. Since $N$ is a weakly quasi $n$-absorbing submodule which is not quasi $n$-absorbing submodule of $M$, then by Theorem 2.3, $(N :_RM) \subseteq \sqrt{\text{ann}(N)}$. Using the fact that $R$ is a von Neumann regular ring, then $\sqrt{\text{ann}(N)} = \text{ann}(N)$. Thus, it follows that $(N :_RM)N = 0$. □

The next corollary is another application of Corollary 2.6.

Corollary 2.7. Let $R$ be a von Neumann regular ring, $M$ be a faithful $R$-module and $N$ be a proper weakly quasi $n$-absorbing submodule which is not quasi $n$-absorbing submodule of $M$. Then $(N :_RM)^2 = 0$. 
Theorem 2.8. Let $R$ be a ring with $2$ is unit in $R$ and $M$ be an $R$-module. If $N$ is a weakly quasi $2$-absorbing submodule which not a quasi $2$-absorbing submodule, then $(N:_R M)^2 N = 0$.

Proof. By Lemma 2.2, for every submodule, then $N$ is a weakly quasi $2$-absorbing submodule which not a quasi $2$-absorbing submodule of $M$.

In the following theorem, we establish that for a ring $R$ in which $2$ is unit of $R$ and $M$ be an $R$-module, we have $(N:_R M)^2 N = 0$ for every weakly quasi $2$-absorbing submodule $N$ which is not quasi $2$-absorbing submodule of $M$.

Theorem 2.9. Let $R$ be a ring with $2$ is unit in $R$ and $M$ be an $R$-module. If $N$ is a weakly quasi $2$-absorbing submodule which not a quasi $2$-absorbing submodule, then $(N:_R M)^2 N = 0$.

Proof. By Corollary 2.6, we have $(N:_R M)N = 0$. So, $(N:_R M)^2 \subseteq ((N:_R M)N:_R M) = (0:_R M) = \text{ann}(M) = 0$ as $M$ is faithful and so $(N:_R M)^2 = 0$, as desired. □

Let $M$ be an $R$-module and $N$ be a proper submodule of $M$. We say that $N$ is a weakly strongly quasi $n$-absorbing submodule of $M$ if whenever $0 \neq I^n L \subseteq N$ for some proper ideal $I$ of $R$ and a proper submodule of $M$, then either $I^n \subseteq (N:_R M)$ or $I^{n-1} L \subseteq N$. It is clear that a weakly strongly quasi $n$-absorbing submodule is a weakly quasi $n$-absorbing submodule. In the next theorem, we show that the notions weakly strongly quasi $n$-absorbing submodule and weakly quasi $n$-absorbing submodule collapse in the case the ring $R$ is a principal domain.

Theorem 2.10. Let $R$ be a principal domain and $N$ be a proper submodule of an $R$-module $M$. Then the following assertions are equivalent:

1. $N$ is a weakly quasi $n$-absorbing submodule of $M$.
2. $N$ is a weakly strongly quasi $n$-absorbing submodule of $M$.

Proof. (1) $\Rightarrow$ (2) Let $0 \neq I^n L \subseteq N$ for some proper ideal $I$ of $R$ and a proper submodule $L$ of $M$. Since $R$ is a principal domain, then there exists an element $a \in R$ such that $I = Ra$. So, $0 \neq a^n L \subseteq N$. Assume that $a^n \notin (N:_R M)$, we claim that $a^{n-1} L \subseteq N$. Indeed, let $x \in L$. If $0 \neq a^n x$, then $a^{n-1} x \in N$ since $N$ is a weakly quasi $n$-absorbing submodule and $a^n \notin (N:_R M)$. Now assume that $a^n x = 0$. Since $a^n L \neq 0$, then $0 \neq a^n y = a^n(x + y) \in N$ for some $y \in N$. Consequently, $a^{n-1}(x + y) \in N$ and so $a^{n-1} x \in N$ as $a^{n-1} y \in N$ which is a weakly quasi $n$-absorbing submodule. Therefore, $a^{n-1} L \subseteq N$.

(2) $\Rightarrow$ (1) Straightforward. □

Proposition 2.11. Let $N$ be a proper submodule of $M$. Then the following statements are equivalent:

1. If $0 \neq I^n L \subseteq N$ for some ideal $I$ of $R$ and submodule $L$ of $M$, then either $I^n \subseteq (N:_R M)$ or $I^{n-1} L \subseteq N$.  

Proof. By Theorem 2.9, we have $(N:_R M)^2 N = 0$. So, $(N:_R M)^2 \subseteq ((N:_R M)N:_R M) = (0:_R M) = \text{ann}(M) = 0$ as $M$ is faithful and so $(N:_R M)^2 = 0$, as desired. □
(2) If $0 \neq I^n x \subseteq N$ for some ideal $I$ of $R$ and $x \in M$, then $I^n \subseteq (N :_R M)$ or $I^{n-1} x \subseteq N$.

Proof. (1) $\Rightarrow$ (2) Straightforward.

(2) $\Rightarrow$ (1) Suppose that $0 \neq I^n L \subseteq N$ for some ideal $I$ of $R$ and submodule $L$ of $M$. Assume that $I^n \not\subseteq (N :_R M)$ and we show that $I^{n-1} L \subseteq N$. By the way of contradiction, suppose $I^{n-1} L \not\subseteq N$. Then there exists an element $x$ of $L$ with $I^{n-1} x \not\subseteq N$. Two cases are then possible:

Case 1: If $0 \neq I^n x \subseteq N$. Since $I^n \not\subseteq (N :_R M)$, from assumption it follows that $I^{n-1} x \subseteq N$, which is a contradiction.

Case 2: If $I^n x = 0$. The fact that $0 \neq I^n L \subseteq N$, there exists an element $y$ of $L$ with $0 \neq I^n y \subseteq N$. Now $0 \neq I^n (x + y) = I^n y \subseteq N$. Since $I^n \not\subseteq (N :_R M)$, then it follows that $I^{n-1} y \subseteq N$ and $I^{n-1} (x + y) \subseteq N$. Hence, $I^{n-1} x \subseteq N$, which is a contradiction again.

Finally, $I^{n-1} L \subseteq N$. □

In the next proposition, we study the stability of homomorphic image of a weakly quasi $n$-absorbing submodule.

**Proposition 2.11.** Let $N, L$ be submodules of an $R$-module $M$ with $L \subseteq N$. If $N$ is a weakly quasi $n$-absorbing submodule of $M$, then $N/L$ is a weakly quasi $n$-absorbing submodule of $M/L$. The converse holds if $L$ is a weakly quasi $n$-absorbing submodule of $M$.

Proof. Assume that $N$ is a weakly quasi $n$-absorbing submodule of $M$. Let $a \in R$ and $x + L \in M/L$ with $0_{M/L} \neq a^n (x + L) \in N/L$. If $a^n \in (N :_R M)$, then we are done. We may assume that $a^n \not\in (N :_R M)$. The fact that $0_{M/L} \neq a^n (x + L)$ implies that $a^n x \in N$ and $a^n x \not\in L$. So, $0 \neq a^n x \in N$. Since $N$ is a weakly quasi $n$-absorbing submodule of $M$ and $a^n \not\in (N :_R M)$, then $a^{n-1} x \in N$. Therefore, $a^{n-1} (x + L) \in N/L$ and so $N/L$ is a weakly quasi $n$-absorbing submodule of $M/N$. Conversely, assume that $L$ is a weakly quasi $n$-absorbing submodule of $M$ and $N/L$ is a weakly quasi $n$-absorbing submodule of $M/L$. Let $a \in R$ and $x \in M$ with $0 \neq a^n x \in N$. Then $a^n (x + L) \in N/L$. If $a^n (x + L) = 0_{M/L}$, then $0 \neq a^n x \in L$. Using the fact that $L$ is a weakly quasi $n$-absorbing submodule of $M$, then either $a^{n-1} x \in L \subseteq N$ or $a^n \in (L :_R M) \subseteq (N :_R M)$. If $a^n (x + L) \neq 0_{M/L}$, then either $a^n \in (N/L :_R M/L)$ or $a^{n-1} (x + L) \in N/L$. Hence, $a^n \in (N :_R M)$ or $a^{n-1} x \in N$. Finally, $N$ is a weakly quasi $n$-absorbing submodule of $M$, as desired. □

Recall that from [8, Definition 2.20(2)], a submodule $N$ of an $R$-module $M$ is said to be a strongly $(m, n)$-closed submodule if whenever $I$ is an ideal and $L$ is a submodule of $M$ with $I^m L \subseteq N$ implies that $I^n \subseteq (N :_R M)$ or $I^{n-1} L \subseteq N$.

**Theorem 2.12.** Let $N$ be a proper submodule of an $R$-module $M$. Then the following statements are equivalent:

(1) If $0 \neq I^n K \subseteq N$ for some ideal $I$ of $R$ and submodule $K$ of $M$, then either $I^n \subseteq (N :_R M)$ or $I^{n-1} K \subseteq N$. 

(2) For any ideal \( I \) of \( R \) and \( N \subseteq L \) a submodule of \( M \) with \( 0 \neq I^n L \subseteq N \) implies \( I^n \subseteq (N :_RM) \) or \( I^{n-1} L \subseteq N \).

Proof. (1) \( \Rightarrow \) (2) Straightforward.

(2) \( \Rightarrow \) (1) Let \( I \) be an ideal of \( R \) and \( K \) be a submodule of \( M \) with \( 0 \neq I^n K \subseteq N \). Then \( 0 \neq I^n(K + N) \subseteq N \). Since \( N \) is a strongly \((m,n)\)-closed submodule of \( M \) and \( L := K + N \supseteq N \), then \( I^n \subseteq (N :_RM) \) or \( I^{n-1} L = I^n K + N \subseteq N \) from the hypothesis (2). Thus \( I^n \subseteq (N :_RM) \) or \( I^{n-1} K \subseteq N \).

In the next theorem we show the relationship between a weakly quasi \( n \)-absorbing submodule \( N \) and the ideal \((N :_Rx)\) of \( R \), where \( x \in M \setminus N \). Recall from [5] that an ideal \( I \) of a ring \( R \) is a weakly semi \( n \)-absorbing ideal of \( R \) if \( 0 \neq x^{n+1} + I \) implies \( x^n \in I \).

**Theorem 2.13.** Let \( M \) be an \( R \)-module and \( N \) be a proper submodule of \( M \).

1. If \((N :_Rx)\) is a weakly semi \( n \)-absorbing ideal of \( R \) for every \( x \in M \setminus N \), then \( N \) is a weakly quasi \( n \)-absorbing submodule of \( M \).

2. Assume that \( N \) is a weakly quasi \( n \)-absorbing submodule of \( M \). Let \( x \) be an element of \( M \setminus N \) such that \( \text{ann}(x) \) is a quasi \( n \)-absorbing ideal of \( R \). Then \((N :_Rx)\) is a weakly quasi \( n \)-absorbing ideal of \( R \) for each \( x \in M \setminus N \).

Proof. (1) Let \( 0 \neq a^ny \in N \) for some \( a \in R \) and \( y \in M \). If \( y \in N \), then we are done. We may assume \( y \in M \setminus N \). If \( a^n \in (N :_RM) \), then we are done. So, we may assume \( a^n \notin (N :_RM) \) and so \( 0 \neq a^n \). Since \( a^n \in (N :_Ry) \) which is a weakly semi \( n \)-absorbing ideal of \( R \), then \( a^{n-1} \in (N :_Ry) \) and so \( a^{n-1}y \in N \). Hence, \( N \) is a weakly quasi \( n \)-absorbing submodule of \( M \).

(2) Let \( x \in M \setminus N \). Suppose that \( 0 \neq a^ny \in (N :_Rx) \) and \( a^n \notin (N :_Rx) \) for some \( a \in R \) and \( y \in M \). If \( 0 \neq a^nyx \in N \). Since \( N \) is a weakly quasi \( n \)-absorbing submodule of \( M \) and \( a^n \notin (N :_RM) \), then \( a^{n-1}yx \in N \). Hence, \( a^{n-1}y \in (N :_Rx) \). Now, suppose that \( a^nyx = 0 \). From assumption, it follows that \( a^{n-1}y \in \text{ann}(x) \), which implies that \( a^{n-1}y \in (N :_Rx) \). Consequently, \((N :_Rx)\) is a weakly quasi \( n \)-absorbing ideal of \( R \), as desired.

**Theorem 2.14.** Let \( M \) be a faithful \( R \)-module and \( N \) be a proper submodule of \( M \). If \( N \) is a weakly quasi \( n \)-absorbing submodule of \( R \), then \((N :_RM)\) is a weakly quasi \( n \)-absorbing ideal of \( R \). The converse holds if \( M \) is a cyclic faithful \( R \)-module.

The proof of the previous theorem requires the following lemma.

**Lemma 2.15.** Let \( N \) be a proper submodule of an \( R \)-module \( M \). Then the following statements are equivalent:

1. \( N \) is a weakly quasi \( n \)-absorbing submodule of \( M \).

2. For every \( a \in R \) and \( L \) a submodule of \( M \) with \( 0 \neq a^nL \subseteq N \), then \( a^{n-1}L \subseteq N \) or \( a^n \in (N :_RM) \).
Proof. (1) $\Rightarrow$ (2) Assume that $N$ is a weakly quasi $n$-absorbing submodule of $M$. Let $a \in R$ and $L$ be a submodule of $M$ such that $0 \neq a^nL \subseteq N$ and $a^n \notin (N :_RM)$. Let $x \in L$. If $0 \neq a^nx$, then $a^{n-1}x \in N$ (as $N$ is a weakly quasi $n$-absorbing submodule of $R$). We may assume that $a^nx = 0$. The fact that $0 \neq a^nL \subseteq N$ gives $0 \neq a^ny \in N$ for some $y \in L$. Since $a^n \notin (N :_RM)$, it follows that $a^{n-1}y \in N$. Set $z = y + x \in L$. So, $a^nz \neq 0$ and with similar argument as above, we get $a^{n-1}z \in L$. Therefore, $a^{n-1}x \in N$. Hence, for every $x \in L$, $a^{n-1}x \in N$. Finally, $a^{n-1}L \subseteq N$.

(2) $\Rightarrow$ (1) Assume that for every $a \in R$ and $L$ a submodule of $M$ with $0 \neq a^nL \subseteq N$, then $a^{n-1}L \subseteq N$ or $a^n \in (N :_RM)$. Let $0 \neq a^n x \in N$ for some $a \in R$ and $x \in M$. Set $L = Rx$. Then $0 \neq a^nL \subseteq N$. From assumption, we get $a^n \in (N :_RM)$ or $a^{n-1}L \subseteq N$ and so $a^{n-1}x \in N$ or $a^n \in (N :_RM)$. Hence, $N$ is weakly quasi $n$-absorbing submodule of $M$, as desired. \hfill $\Box$

Proof of Theorem 2.14. Let $0 \neq a^n b \in (N :_RM)$ for some $a, b \in R$. Since $M$ is a faithful $R$-module, then $0 \neq a^n bM = a^n(bM) \subseteq N$. By Lemma 2.15, $a^{n-1}(bM) = a^{n-1}bM \subseteq N$ or $a^n \in (N :_RM)$. Hence, $(N :_RM)$ is a weakly quasi $n$-absorbing ideal of $R$. Conversely, assume that $(N :_RM)$ is a weakly quasi $n$-absorbing ideal of $R$ and $M = Rm$ is a cyclic faithful $R$-module. Let $a \in R$ and $x \in M$ such that $0 \neq a^n x \in N$. Then there exists $b \in R$ such that $x = bm$. So, $0 \neq a^n bm \in N$. Therefore, $0 \neq a^n b \in (N :_Rm) = (N :_RM)$. The fact that $(N :_RM)$ is a weakly quasi $n$-absorbing ideal of $R$, gives either $a^n \in (N :_RM)$ or $a^{n-1}b \in (N :_RM)$. Hence, $a^n \in (N :_RM)$ or $a^{n-1}bm = a^{n-1}x \in N$, making $N$, a weakly quasi $n$-absorbing submodule of $M$. \hfill $\Box$

It is worth to mention that in Theorem 2.14 the condition “$M$ is a faithful $R$-module” is necessary. Otherwise, if $N$ is a weakly quasi $n$-absorbing submodule of $M$, then $(N :_RM)$ need not be a weakly quasi $n$-absorbing ideal of $R$, as shown in the next example.

Example 2.16. Consider the $\mathbb{Z}$-module $M := \mathbb{Z}/16\mathbb{Z}$ and $N = \{0\}$. Observe that $\text{ann}(M) = 16\mathbb{Z}$. So, $M$ is not faithful. On the other hand, $N$ is a weakly quasi $2$-absorbing submodule and $(N :_{\mathbb{Z}}M) = 16\mathbb{Z}$ is not a weakly quasi $2$-absorbing ideal of $\mathbb{Z}$ since $2 \cdot 4 \notin (N :_{\mathbb{Z}}M)$ but neither $2 \cdot 4 = 8 \in (N :_{\mathbb{Z}}M) = 16\mathbb{Z}$ nor $2 \cdot 4 \in (N :_{\mathbb{Z}}M)$.

Let $R$ be a ring. It is well known that a proper submodule $N$ of an $R$-module $M$ is said to be a weakly semiprime submodule of $M$ if $0 \neq r^2x \in N$ for some $r \in R$ and $x \in M$, then $rx \in N$. In the next theorem, we show that the class of weakly semiprime submodules is contained in the class of weakly quasi $n$-absorbing submodules for every positive integer $n \geq 2$.

Theorem 2.17. Let $R$ be a ring, $M$ be an $R$-module and $N$ be a proper submodule of $M$. If $N$ is a weakly semiprime submodule of $M$, then $N$ is a weakly quasi $n$-absorbing submodule of $M$ for every positive integer $n \geq 2$. 
Proof. Let $0 \neq a^n x \in N$ for some $a \in R$, $x \in M$ and for some positive integer $n \geq 2$. Then $0 \neq a^2(a^{n-2}x) \in N$. Since $N$ is a weakly semiprime submodule of $M$, we get $0 \neq a^{n-1}x \in N$. Hence, $N$ is a weakly quasi $n$-absorbing submodule of $M$, as desired. \hfill $\square$

The following theorem shows that the intersection of a family of weakly semiprime submodules is a weakly quasi-$n$-absorbing submodule.

**Theorem 2.18.** Let $R$ be a ring, $M$ be an $R$-module. Let $(N_i)_{i \in I}$ be a family of weakly semiprime submodules of $M$. Then $\bigcap_{i \in I} N_i$ is a weakly quasi $n$-absorbing submodule of $M$ for all positive integer $n \geq 2$.

Proof. Suppose that $0 \neq a^n x \in N := \bigcap_{i \in I} N_i$ for some $a \in R$ and $x \in M$. Then $0 \neq a^n x \in N_i$ for all $i \in I$. Since $N_i$ is a weakly semiprime module, then $ax \in N_i$ for all $i \in I$. Therefore, $a^{n-1}x = a^{n-2}(ax) \in N_i$ for all $i \in I$ and so $a^{n-1}x \in N$. Hence, $\bigcap_{i \in I} N_i$ is a weakly quasi $n$-absorbing submodule of $M$ for all positive integer $n \geq 2$. \hfill $\square$

**Theorem 2.19.** Let $M_1, M_2$ be $R$-modules with $M = M_1 \oplus M_2$, $n$ be a positive integer and $N_1$ (resp., $N_2$) be a proper submodule of $M_1$ (resp., $M_2$). Then the following statements are equivalent:

1. $N_1 \oplus M_2$ (resp., $M_1 \oplus N_2$) is a weakly quasi $n$-absorbing submodule of $M$ which is not a quasi $n$-absorbing submodule.

2. If $N_1$ (resp., $N_2$) is a weakly quasi $n$-absorbing submodule of $M_1$ (resp., $M_2$) which is not a quasi $n$-absorbing submodule of $M_1$ (resp., $M_2$) and $a^n M_2 = 0$ (resp., $a^n M_1 = 0$) for every unbreakable element $a$ of $N_1$ (resp., $N_2$).

The proof of the previous theorem needs the following lemma.

**Lemma 2.20.** Let $M_1, M_2$ be $R$-modules with $M = M_1 \oplus M_2$, $n$ be a positive integer and $N_1$ (resp., $N_2$) be proper weakly quasi $n$-absorbing submodule of $M_1$ (resp., $M_2$). Let $a \in R$. Then the following statements are equivalent:

1. $a$ is an unbreakable element of $N_1$ (resp., $N_2$).

2. $a$ is an unbreakable element of $N_1 \oplus M_2$ (resp., $M_1 \oplus N_2$).

Proof. Assume that $a$ is an unbreakable element of $N_1$. Then there exists $x \in M_1$ with $a^nx = 0$ and neither $a^n \in (N_1 :_RM_1)$ nor $a^{n-1}x \in N_1$. Then $a^n(x, 0) = (0, 0)$ and neither $a^n \in (N_1 \oplus M_2 :_RM_1 \oplus M_2)$ nor $a^{n-1}(x, 0) \in N_1 \oplus M_2$. Hence, $a$ is an unbreakable element of $N_1 \oplus M_2$. Conversely, assume that $a \in R$ is an unbreakable element of $N_1 \oplus M_2$. Then there exists $(x, y) \in M_1 \oplus M_2$ with $a^n(x, y) = (0, 0)$ and neither $a^n \in (N_1 \oplus M_2 :_RM_1 \oplus M_2)$ nor $a^{n-1}(x, y) \in N_1 \oplus M_2$. Hence, $a^n x = 0$ for $x \in M_1$ and neither $a^n \in (N_1 :_RM_1)$ nor $a^{n-1}x \in N_1$. Thus, $a$ is an unbreakable element of $N_1$.

With similar proof as above, one can easily show that $a$ is an unbreakable element of $N_2$ if and only if $a$ is an unbreakable element of $M_1 \oplus N_2$. \hfill $\square$
Proof of Theorem 2.19. (1) ⇒ (2) Assume that $N_1 \oplus M_2$ is a weakly quasi $n$-absorbing submodule of $M$ which is not a quasi $n$-absorbing submodule. Then by Proposition 2.11, $N_1 \simeq N_1 \oplus M_2$ is a weakly $n$-absorbing submodule of $M_1$. Now, by Lemma 2.20, it follows that $N_1$ is not a quasi $n$-absorbing submodule of $M_1$ since $N_1$ admits an element which is unbreakable, $a \in R$, as $a$ is an unbreakable element of $N_1 \oplus M_2$. It remains to show that if $a$ is an unbreakable element of $N_1$, then $a^n M_2 = 0$. Assume by way of contradiction that $a$ is an unbreakable element of $N_1$ and $a^n M_2 \neq 0$. Then $a^n y \neq 0$ for some $y \in M_2$. Since $a$ is an unbreakable element of $N_1$, then there exists $x \in M_1$ with $a^n x = 0$ and neither $a^n \in (N_1 :_R M_1)$ nor $a^{n-1} \in N_1$. Since $0 \neq a^n (y, x) \in N_1 \oplus M_2$, then the fact that $N_1 \oplus M_2$ is a weakly quasi $n$-absorbing submodule of $M_1 \oplus M_2$ and $a^n \notin (N_1 \oplus M_2 : N_1 \oplus M_2)$ give that $a^{n-1} x \in N_1$, which is a contradiction. Hence, $a^n M_2 = 0.$

(2) ⇒ (1) Assume that $N_1$ is a weakly quasi $n$-absorbing which is not quasi $n$-absorbing submodule of $M_1$ and $a^n M_2 = 0$ for every unbreakable $a$ element of $N_1$. Let $b \in R$ and $(x, y) \in M_1 \oplus M_2$ with $0 \neq b^n (x, y) \in N_1 \oplus M_2$. If $0 \neq b^n x \in N_1$, then either $b^n \in (N_1 \oplus M_2 :_R M)$ or $b^{n-1} (x, y) \in N_1 \oplus M_2$. Now, suppose that $b^n = 0$ and neither $b^n \in (N_1 :_R M_1)$ nor $b^{n-1} \in N_1$, then $b$ is an unbreakable element of $N_1$. From assumption, we have $b^n M_2 = 0$, and so $b^n (x, y) = 0$, which is a contradiction. Therefore, either $b^n \in (N_1 \oplus M_2 :_R M_1 \oplus M_2)$ or $b^{n-1} (x, y) \in N_1 \oplus M_2$. Finally, we conclude that $N_1 \oplus M_2$ is a weakly quasi $n$-absorbing submodule of $M$. Now the fact $N_1 \oplus M_2$ is not a quasi $n$-absorbing submodule of $M$ follows from Lemma 2.20. The proof is complete. □

Now we establish some facts for $N_1 \oplus N_2$ to be a quasi $n$-absorbing submodule of $M_1 \oplus M_2$ for some positive integer $0 < n$.

Theorem 2.21. Let $M_1, M_2$ be $R$-modules and $N_1$ (resp., $N_2$) be a submodule of $M_1$ (resp., $M_2$). If $N_1 \oplus N_2$ is a weakly quasi $n$-absorbing submodule of $M = M_1 \oplus M_2$ that is not quasi $n$-absorbing submodule for some positive integer $n > 0$, then one of the following two assertions hold:

1. $N_1$ and $N_2$ are weakly quasi $n$-absorbing submodules and if there exists an unbreakable element $a$ of $N_1$, then $a^n N_2 = 0$.
2. $N_1$ and $N_2$ are weakly quasi $n$-absorbing submodules and if there exists an unbreakable element $b$ of $N_2$, then $b^n N_1 = 0$.

Proof. (1) Suppose that $N_1 \oplus N_2$ is a weakly quasi $n$-absorbing submodule that is not quasi $n$-absorbing submodule of $M$. Let $a \in R$ and $x \in M_1$ with $0 \neq a^n x \in N_1$. Then $0 \neq a^n (x, 0) \in N_1 \oplus N_2$ which is a weakly quasi $n$-absorbing submodule of $M$. It follows that $a^{n-1} x \in N_1$ or $a^n \in (N_1 :_R M_1)$. Hence, $N_1$ is a weakly quasi $n$-absorbing submodule of $M_1$. The same argument shows that $N_2$ is a weakly quasi $n$-absorbing submodule of $M_2$. Now, suppose that $N_1$ admits an unbreakable element $a \in R$. Then $a^n x = 0$ but neither $a^n \in (N_1 :_R M_1)$ nor $a^{n-1} x \in N_1$ for some $x \in M_1$. Assume that $a^n N_2 \neq 0$. If
Then there exists $z \in N_2$ such that $0 \neq a^n z \in N_2$, so $0 \neq a^n(x, z) = (0, a^n z) \in \mathcal{N}_1 \oplus \mathcal{N}_2$ which is a weakly quasi $n$-absorbing submodule of $M$. So, either $a^{n-1}(x, z) \in \mathcal{N}_1 \oplus \mathcal{N}_2$ or $a^n \in (\mathcal{N}_1 \oplus \mathcal{N}_2 :_R M)$. Therefore, $a^{n-1}x \in \mathcal{N}_1$ or $a^n \in (\mathcal{N}_1 :_R M_1)$, which is a contradiction. Hence, $a^n N_2 = 0$.

(2) Similar proof as assertion (1) above.

Remark 2.22. Let $\mathcal{N}_1$ (resp., $\mathcal{N}_2$) be a submodule of $M_1$ (resp., $M_2$). If $\mathcal{N}_1$ and $\mathcal{N}_2$ are weakly quasi $n$-absorbing submodules, then $\mathcal{N}_1 \oplus \mathcal{N}_2$ need not be a weakly quasi $n$-absorbing submodule of $M_1 \oplus M_2$. For instance, take $M_1 = M_2 = \mathbb{Z}$ and $\mathcal{N}_1 = 2^2 \mathbb{Z}$, $\mathcal{N}_2 = 3 \mathbb{Z}$. It is clear that $\mathcal{N}_1$ and $\mathcal{N}_2$ are weakly quasi 2-absorbing submodules of $\mathbb{Z}$ since they are quasi 2-absorbing submodules.

However, $\mathcal{N}_1 \oplus \mathcal{N}_2$ is not a weakly quasi 2-absorbing submodule of $M_1 \oplus M_2$ since $2^2(3, 3) \in 2^2 \mathbb{Z} \oplus 3 \mathbb{Z}$, but neither $2^2 = 4 \not\in (2^2 \mathbb{Z} \oplus 3 \mathbb{Z} :_2 \mathbb{Z} \oplus \mathbb{Z}) = 12 \mathbb{Z}$ nor $2(3, 3) = (6, 6) \not\in 2^2 \mathbb{Z} \oplus 3 \mathbb{Z}$.

The next theorem establishes about when the submodule $\mathcal{N}_1 \oplus \mathcal{N}_2$ is a weakly quasi $(n+1)$-absorbing submodule.

Theorem 2.23. Let $M_1, M_2$ be $R$-modules and $\mathcal{N}_1$ (resp., $\mathcal{N}_2$) be a submodule of $M_1$ (resp., $M_2$). Consider the following assertions:

(1) $\mathcal{N}_1$ is a weakly quasi $n$-absorbing submodule of $M_1$, $\mathcal{N}_2$ is a quasi $n$-absorbing submodule of $M_2$ and $a^n y = 0$ whenever $a^n y \in \mathcal{N}_2$, for some $a \in R$ and $y \in M_2$.

(2) $\mathcal{N}_2$ is a weakly quasi $n$-absorbing submodule of $M_2$, $\mathcal{N}_1$ is a quasi $n$-absorbing submodule of $M_1$ and $a^n x = 0$ whenever $a^n x \in \mathcal{N}_1$, for some $a \in R$ and $x \in M_1$.

If (1) or (2) holds, then $\mathcal{N}_1 \oplus \mathcal{N}_2$ is a weakly quasi $(n+1)$-absorbing submodule of $M$.

Proof. Suppose that (1) holds. Let $0 \neq a^{n+1}(x, y) \in \mathcal{N}_1 \oplus \mathcal{N}_2$ for some $a \in R$ and $(x, y) \in M$. From assumption $a^n(ay) = 0$ and $0 \neq a^n(ax) \in \mathcal{N}_1$ and $\mathcal{N}_2$ is a quasi $n$-absorbing submodule of $M_2$. Since $\mathcal{N}_1$ is a weakly quasi $n$-absorbing submodule of $M_1$, it follows that $a^n x \in \mathcal{N}_1$. On the other hand $a^n(ay) = 0 \in \mathcal{N}_2$ and the fact that $\mathcal{N}_2$ is a quasi $n$-absorbing submodule of $M_2$ gives $a^n y \in \mathcal{N}_2$. Finally, $a^n(x, y) \in \mathcal{N}_1 \oplus \mathcal{N}_2$. Hence, $\mathcal{N}_1 \oplus \mathcal{N}_2$ is a weakly quasi $n+1$-absorbing submodule of $M$. The same argument if assertion (2) holds.

The next proposition examines the weakly quasi $n$-absorbing submodules under localization.

Proposition 2.24. Let $N$ be a proper submodule of an $R$-module $M$ and $S$ be a multiplicative closed subset consisting entirely of nonzero divisor elements of $R$ such that $(N :_R M) \cap S = \emptyset$. If $N$ is a weakly quasi $n$-absorbing submodule of $M$, then $S^{-1}N$ is a weakly quasi $n$-absorbing submodule of $S^{-1}M$.
Proof. Let \( \frac{m}{a} \neq (\frac{m}{a})^n (\frac{m}{a}) \in S^{-1}N \). Then \( 0 \neq (u \alpha)^m \in N \) for some element \( u \) of \( S \). So, \( 0 \neq (u \alpha)^m \in N \) which is a weakly quasi \( n \)-absorbing submodule of \( M \). Therefore, \((u \alpha)^m \in N \) or \((u \alpha)^m \in (N:R \ M) \). Consequently, \((\frac{m}{a})^n (\frac{m}{a}) = (\frac{m}{a})^n (\frac{m}{a}) \in S^{-1}N \) or \((\frac{m}{a})^n = (\frac{m}{a})^n \in S^{-1}(N:R \ M) \subseteq (S^{-1}N:R S^{-1}M) \). Hence, \( S^{-1}N \) is a weakly quasi \( n \)-absorbing submodule of \( S^{-1}M \), as desired. \( \square \)

The following proposition studies the weakly quasi \( n \)-absorbing property under homomorphism.

**Proposition 2.25.** Let \( f : M \to M' \) be a homomorphism of \( R \)-modules.

1. Assume that \( f \) is a monomorphism. If \( N' \) is a weakly quasi \( n \)-absorbing submodule of \( M' \), then \( f^{-1}(N') \) is a weakly quasi \( n \)-absorbing submodule of \( M \).

2. Assume that \( f \) is an epimorphism and \( \ker(f) \subseteq N \). If \( N \) is a weakly quasi \( n \)-absorbing submodule of \( M \), then \( f(N) \) is a weakly quasi \( n \)-absorbing submodule of \( M' \).

Proof. (1) Assume that \( f \) is a monomorphism of \( R \)-modules and \( N' \) is a weakly quasi \( n \)-absorbing submodule of \( M' \). Let \( 0 \neq a^n f(x) \in f^{-1}(N') \) for some \( a \in R \) and \( x \in M \). Then \( 0 \neq a^n f(x) \in N' \) which is a weakly quasi \( n \)-absorbing submodule of \( M' \). So, \( a^n \in (N':R \ M') \) or \( a^n f(x) \in N' \). Therefore, \( a^n M' \subseteq N' \) or \( f(a^n f(x)) \subseteq N' \). Hence, it follows that \( a^n M \subseteq f^{-1}(N') \) or \( a^n f(x) \subseteq f^{-1}(N') \). Thus, \( a^n \in (f^{-1}(N') : R \ M) \) or \( a^n x \subseteq f^{-1}(N') \), making \( f^{-1}(N') \), a weakly quasi \( n \)-absorbing submodule of \( M \).

(2) Assume that \( f \) is an epimorphism, \( \ker(f) \subseteq N \) and \( N \) is a weakly quasi \( n \)-absorbing submodule of \( M \). Let \( a \in R \), \( x' \in M' \) such that \( 0 \neq a^n x' \in f(N) \). Then there exists \( x \in M \) such \( x' = f(x) \). Since \( 0 \neq a^n x' = a^n f(x) = f(a^n x) \in f(N) \) and \( \ker(f) \subseteq N \), then \( 0 \neq a^n x \in N \) which is a weakly quasi \( n \)-absorbing submodule of \( M \). Therefore, \( a^n \in (N : R \ M) \) or \( a^n x \in N \). And so \( a^n M \subseteq N \) or \( a^n x \in N \). It follows that \( a^n M' \subseteq f(N) \) or \( a^n f(x) \subseteq f(N) \). Hence, \( a^n \in (f(N) : R \ M') \) or \( a^n x' \in f(N) \). Finally, \( f(N) \) is a weakly quasi \( n \)-absorbing submodule of \( M' \), as desired. \( \square \)

We close this paper by studying about when the intersection family of \((N_\alpha)_{\alpha \in I} \) is a weakly quasi \( n \)-absorbing submodule.

**Theorem 2.26.** Consider \((N_\alpha)_{\alpha \in I} \) a chain of weakly quasi \( n \)-absorbing submodules of an \( R \)-module \( M \). Then \( N = \bigcap_{\alpha \in I} N_\alpha \) is a weakly quasi \( n \)-absorbing submodule of \( M \).

Proof. Let \( 0 \neq a^n x \in N \) for some \( a \in R \) and \( x \in M \). Clearly \( 0 \neq a^n x \in N_\alpha \) for each \( \alpha \in I \). Two cases are then possible:

Case 1: If \( a^n \in (N_\alpha : R \ M) \) for all \( \alpha \in I \), then \( a^n \in \bigcap (N_\alpha : R \ M) = (\bigcap N_\alpha : R \ M) = (N : R \ M) \).
Case 2: Assume that $a^n \notin (N_{\alpha'} :_{R} M)$ for some $\alpha' \in I$. Then $a^n \notin (N_{\alpha} :_{R} M)$ for all $N_{\alpha} \subseteq N_{\alpha'}$. Using the fact that $N_{\alpha}$ is a weakly quasi $n$-absorbing submodule of $M$ for each $\alpha \in I$, then $a^{n-1}x \in N_{\alpha}$ for all $N_{\alpha} \subseteq N_{\alpha'}$.

Consequently, it follows that $a^{n-1}x \in N = \bigcap_{\alpha \in I} N_{\alpha}$.

Finally, $N = \bigcap_{\alpha \in I} N_{\alpha}$ is a weakly quasi $n$-absorbing submodule of $M$, as desired.

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