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A NOTE ON THE MIXED VAN DER WAERDEN NUMBER

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Abstract. Let $r \geq 2$, and let $k_i \geq 2$ for $1 \leq i \leq r$. Mixed van der Waerden's theorem states that there exists a least positive integer $w = w(k_1, k_2, k_3, \ldots, k_r; r)$ such that for any $n \geq w$, every r-colouring of $[1, n]$ admits a k_i-term arithmetic progression with colour i for some $i \in [1, r]$. For $k \geq 3$ and $r \geq 2$, the mixed van der Waerden number $w(k, 2, 2, \ldots, 2; r)$ is denoted by $w_2(k; r)$. B. Landman and A. Robertson [9] showed that for $k < r < \frac{3}{2}(k-1)$ and $r \geq 2k+2$, the inequality $w_2(k; r) \leq r(k-1)$ holds. In this note, we establish some results on $w_2(k; r)$ for $2 \leq r \leq k$.

1. Introduction

For an arithmetic progression $A = \{a + ld : 0 \leq l \leq k-1\}$, we say that A is a *k-term a.p. with difference d*, or we write $\{a + ld\}_{0 \leq l \leq k-1}$. For a positive integer t, we denote the set $\{1, 2, 3, \ldots, t\}$ by $[1, t]$. An *r-colouring* of a set S is a function $\chi : S \to [1, r]$. A *monochromatic k-term a.p.* refers to a k-term a.p. such that all of its elements are of the same colour.

Consider arithmetic progressions of length three. We wish to find the least positive integer w such that regardless of how the integers $[1, w]$ are coloured using two colours, there will be a monochromatic 3-term a.p. This number w, denoted by $w(3; 2)$ is called a *van der Waerden number*. It is straightforward to deduce that $w(3; 2) = 9$ (see, for example, [9]). Hence it is natural to ask whether the analogous least positive integer exists if we use more than two colours to require a monochromatic arithmetic progression. The answer is affirmative by the van der Waerden's theorem. Van der Waerden's theorem is arguably the most fundamental Ramsey-type theorem on the integers concerning arithmetic progressions and it has been extensively studied in [6] and [9].

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Theorem 1.1 (Van der Waerden's Theorem [9]). *Let $k, r \geq 2$ be integers. There exists a least positive integer $w = w(k; r)$ such that for any $n \geq w$, every r-colouring of $[1, n]$ admits a monochromatic k-term arithmetic progression.*

For example, if the colours are red and blue, then $w(5; 2)$ represents the least positive integer such that, for every 2-colouring of $[1, w(5; 2)]$, there is a red 5-term a.p. or a blue 5-term a.p. However, instead of requiring a k-term a.p. to be of one of the colours, the mixed van der Waerden numbers allow the required length to vary with the colour. What if we "mix" the lengths so that we want the least positive integer n such that for every 2-colouring of $[1, n]$, there is a red k-term a.p. or a blue l-term a.p., where $k \neq l$? Generally, we have the following as the mixed van der Waerden's theorem.

Corollary 1.2 (Mixed van der Waerden's Theorem [9]). *Let $r \geq 2$, and let $k_i \geq 2$ for $1 \leq i \leq r$. Then there exists a least positive integer $w = w(k_1, k_2, k_3, \ldots, k_r; r)$ such that for any $n \geq w$, every r-colouring of $[1, n]$ admits a k_i-term arithmetic progression with colour i for some $i \in [1, r]$.*

It is worth noting that very few exact values of van der Waerden numbers $w(k; r)$ are known, even for small values of r and k (see, for example, [7] and [8] for the computational verification of the exact values of $w(4; 3)$ and $w(6; 2)$, respectively). For the case of mixed van der Waerden numbers $w(k_1, k_2, k_3, \ldots, k_r; r)$, some exact values have also been computed (see, for example, [1–5] for some exact values of $w(k_1, k_2, k_3, \ldots, k_r; r)$ verified by computational proofs using SAT solvers). Some general results on mixed van der Waerden numbers have also been investigated in [1, 3–6, 9].

Definition ([9]). For $k \geq 1$ and $r \geq 2$, the mixed van der Waerden number $w(k, 2, 2, \ldots, 2; r)$ is denoted by $w_2(k; r)$.

This implies that in every r-colouring of $[1, w_2(k; r)]$, there exists a monochromatic 2-term or k-term a.p. Some known results on $w_2(k; r)$ can be found in [6, 9]. We are interested in the van der Waerden numbers $w_2(k; r)$. To the best of our knowledge, the following are the most up to date results concerning $w_2(k; r)$.

Theorem 1.3 ([6]). *Let $k < r < \frac{3}{2}(k-1)$. Then $w_2(k; r) \leq r(k-1)$.*

Theorem 1.4 ([6]). *Let $k > 2$ and assume that $w_2(k; r) \leq r(k-1)$ for $\frac{3}{2}(k-1) \leq r \leq 2k+1$. Then $w_2(k; r) \leq r(k-1)$ for all $r > k$.*

In this note, we shall present some results on $w_2(k; r)$ for $2 \leq r \leq k$. We start with some notations. Let k, a and d be integers and $d \geq 1$. If $k - a$ is divisible by d, then we write $k \equiv a \mod d$ and we say k is congruent to a (modulo d). If $k - a$ is not divisible by d, then we write $k \not\equiv a \mod d$ and we say k and a are incongruent modulo d. A set of numbers $\{a_0, a_1, \ldots, a_{d-1}\}$ form a complete set of residues modulo d if $a_i \equiv i \mod d$ for $i = 0, 1, \ldots, d-1$. The greatest common divisor of two integers k and p, written as $\gcd(k, p)$ is the largest divisor dividing both k and p. Note that k and p are relatively prime if $gcd(k,p) = 1$.

2. Main results

We will start by establishing the exact mixed van der Waerden numbers $w_2(k;r)$ for small values of r, namely when $r = 2$ and $r = 3$, in Subsection 2.1. This will then followed by more general results of $w_2(k;r)$ for $2 \le r \le k$ in Subsection 2.2. Here is an easy upper bound for $w_2(k;r)$ in general.

Lemma 2.1. *Let $k \ge 3$ and $r \ge 2$ be integers. Then $w_2(k;r) \le rk$.*

Proof. Let $\chi : [1, rk] \to [1, r]$ be a colouring. We may assume that for each $i \in [2, r]$, at most one element is coloured with i. So, there are at most $r-1$ elements not coloured with 1. For each $j \in [1, r]$, there is an element in $[(j-1)k+1, jk]$ with colour other than 1. But this is not possible as there are only at most $r-1$ elements with colour other than 1. So, there is a $j_0 \in [1, r]$ such that all the elements in $[(j_0-1)k+1, j_0k]$ are coloured with 1. Then, $\{(j_0-1)k+1+l\}_{0\le l\le k-1}$ is a k-term a.p. of colour 1. □

2.1. Exact values of $w_2(k;2)$ and $w_2(k;3)$

Theorem 2.2. *Let $k \ge 1$ be an integer. Then*

$$w_2(k;2) = \begin{cases} 2k-1 & \text{if } k \text{ is even;} \\ 2k & \text{if } k \text{ is odd.} \end{cases}$$

Proof. Note that $w_2(1;2) = 2$ and $w_2(2;2) = 3$. Suppose $k \ge 3$.

Let $\chi : [1, 2(k-1)] \to [1, 2]$ be such that $\chi(k) = 2$ and $\chi(x) = 1$ for all $x \in [1, 2(k-1)] \setminus \{k\}$. If there is a k-term a.p. of colour 1, then there exist positive integers a, d such that $a + jd \in [1, 2(k-1)] \setminus \{k\}$ for all $j = 0, 1, \ldots, k-1$. This implies that $a + (k-1)d \le 2k-2$. Since $a \ge 1$, we have $d(k-1) \le 2k-3$. So, $d = 1$. This implies that $a \le k-1$ and $k \in \{a + j : j = 0, 1, \ldots, k-1\}$, a contradiction. Thus, $w_2(k;2) > 2(k-1)$.

Let $\chi : [1, 2(k-1)+1] \to [1, 2]$ be a colouring. We may assume that only one element is coloured with 2. If no element in $[1, k]$ is coloured with 2, then we have k-term a.p. of colour 1. So, there is an element in $[1, k]$ coloured with 2. If no element in $[k, 2(k-1)+1]$ is coloured with 2, then we also have a k-term a.p. of colour 1. So, k must be coloured with 2. If there is a k-term a.p. of colour 1, then there exist positive integers a, d such that $a + jd \in [1, 2(k-1)+1] \setminus \{k\}$ for all $j = 0, 1, \ldots, k-1$. Note that $d \ne 1$ for otherwise $a + j = k$ for some $j \in [0, k-1]$. So, we may assume that $d \ge 2$. From $a + (k-1)d \le 2k-1$, we deduce that $a = 1$ and $d = 2$. If k is even, then $k \ne 1 + 2j$ for all $j = 0, 1, \ldots, k-1$. So, $\{1 + 2j\}_{0\le j\le k-1}$ is k-term a.p. of colour 1. Hence, $w_2(k;2) = 2(k-1)+1$. If k is odd, then $k = 1 + 2j$ for some $j \in [0, k-1]$. Thus, $w_2(k;2) > 2(k-1)+1$. By Lemma 2.1, if k is odd, then $w_2(k;2) = 2k$.

This completes the proof of the theorem. □

To determine the exact values of $w_2(k;3)$ for all $k \geq 1$, we have the following lemmas.

Lemma 2.3. *If* $k \equiv 1, 4$ *or* $5 \mod 6$*, then* $w_2(k;3) > 3(k-1)+1$*. If* $k \equiv 0, 2$ *or* $3 \mod 6$*, then* $w_2(k;3) \leq 3(k-1)+1$*.*

Proof. Let $\chi : [1, 3(k-1)+1] \to [1,2,3]$ be a colouring. We may assume that there are only two elements, one of them is coloured with 2 and the other is coloured with 3. If all the elements in $[1,k]$ or in $[k+1, 2k]$ are coloured with 1, then we have a k-term a.p. of colour 1. So, there are an element in $[1,k]$ and an element in $[k+1,2k]$ not coloured with 1. If all the elements in $[2k-1, 3(k-1)+1]$ are coloured with 1, then we have a k-term a.p. of colour 1. So, we may assume that either $2k-1$ or $2k$ is not coloured with 1.

Case 1. Suppose $2k-1$ is not coloured with 1. If $j_0 \in [1,k]$ is not coloured with 1 and $j_0 \leq k-2$, then $\{k-1+l\}_{0\leq l \leq k-1}$ is k-term a.p. of colour 1. So, we may assume that the following pairs $(2k-1, k-1)$ or $(2k-1, k)$ is not coloured with 1.

Subcase 1.1. Suppose $(2k-1, k-1)$ is not coloured with 1.

Subcase 1.1.1. Suppose k is even, i.e., $k \equiv 0 \mod 2$. Since $k-1, 2k-1$ are both odd, $\{2+2l\}_{0\leq l\leq k-1}$ is a k-term a.p. of colour 1.

Subcase 1.1.2. Suppose $k \equiv 0 \mod 3$. Then, $(2k-1, k-1) \equiv (5,2) \mod 6$. Since $k-1, 2k-1$ are both $\equiv 2 \mod 3$, $\{1+3l\}_{0\leq l\leq k-1}$ is a k-term a.p. of colour 1.

Subcase 1.1.3. Suppose $k \equiv 1 \mod 6$ or $k \equiv 5 \mod 6$. If $k \equiv 1 \mod 6$, then $(2k-1, k-1) \equiv (1,0) \mod 6$. Similarly, if $k \equiv 5 \mod 6$, then $(2k-1, k-1) \equiv (3,4) \mod 6$. In either case, if there is a k-term a.p. of colour 1, then there exist positive integers a, d such that $a+jd \in [1, 3(k-1)+1] \setminus \{k-1, 2k-1\}$ for all $j = 0, 1, \ldots, k-1$. Since $a \geq 1$, we have $d(k-1) \leq 3k-3$. So, $d \leq 3$. From $d \geq 1$, we obtain $a \leq 2k-1$. Suppose $d = 1$. If $a \in [1, k-1]$, then $a + j = k-1$ for some $j \in [0, k-1]$. If $a \in [k, 2k-1]$, then $a+j = 2k-1$ for some $j \in [0, k-1]$. All these cases are not possible. If $d = 2$, then $a \leq k$. For all $j \in [0, k-1]$, $a + 2j \equiv a \mod 2$. Note that $(2k-1, k-1) \equiv (1,0) \mod 2$. If a is odd, then $a + 2j = 2k-1$ for some $j \in [0, k-1]$. If a is even, then $a + 2j = k-1$ for some $j \in [0, k-1]$. All these cases are not possible. If $d = 3$, then $a = 1$. Since $1 + 3j \equiv 1 \mod 3$ for all $j \in [0, k-1]$, we must have $1 + 3j = 2k-1$ for some $j \in [0, k-1]$ (if $k \equiv 1 \mod 6$) or $1 + 3j = k-1$ for some $j \in [0, k-1]$ (if $k \equiv 5 \mod 6$). Thus, $w_2(k;3) > 3(k-1)+1$ when $k \equiv 1 \mod 6$ or $k \equiv 1 \mod 6$.

Subcase 1.2. Suppose $(2k-1, k)$ is not coloured with 1.

Subcase 1.2.1. Suppose k is odd, i.e., $k \equiv 1 \mod 2$. Then, $(2k-1, k) \equiv (1,1) \mod 2$. Since $k, 2k-1$ are both odd, $\{2+2l\}_{0\leq l\leq k-1}$ is a k-term a.p. of colour 1.

Subcase 1.2.2. Suppose $k \not\equiv 1 \mod 3$. If $k \equiv 2 \mod 3$, then $(2k-1, k) \equiv (0,2) \mod 3$ and hence $k \equiv 2 \mod 3$ and $2k-1 \equiv 0 \mod 3$. If $k \equiv 0 \mod 3$, then $(2k-1,k) \equiv (2,0) \mod 3$ and hence $k \equiv 0 \mod 3$ and $2k-1 \equiv 2 \mod 3$. Following this, $\{1+3l\}_{0\le l\le k-1}$ is a k-term a.p. of colour 1.

Subcase 1.2.3. Suppose $k \equiv 4 \mod 6$. Then, $(2k-1,k) \equiv (1,4) \mod 6$. If there is a k-term a.p. of colour 1, then there exist positive integers a,d such that $a+jd \in [1,3(k-1)+1] \setminus \{k,2k-1\}$ for all $j=0,1,\ldots,k-1$. Since $a \ge 1$, we have $d(k-1) \le 3k-3$. So, $d \le 3$. From $d \ge 1$, we obtain $a \le 2k-1$. Suppose $d=1$. If $a \in [1,k]$, then $a+j=k$ for some $j \in [0,k-1]$. If $a \in [k+1,2k-1]$, then $a+j=2k-1$ for some $j \in [0,k-1]$. All these cases are not possible. If $d=2$, then $a \le k$. For all $j \in [0,k-1]$, $a+2j \equiv a \mod 2$. Note that $(2k-1,k) \equiv (1,0) \mod 2$. If a is odd, then $a+2j=2k-1$ for some $j \in [0,k-1]$. If a is even, then $a+2j=k$ for some $j \in [0,k-1]$. All these cases are not possible. If $d=3$, then $a=1$. Since $1+3j \equiv 1 \mod 3$ for all $j \in [0,k-1]$, we must have $1+3j=k$ for some $j \in [0,k-1]$. Thus, $w_2(k;3) > 3(k-1)+1$ when $k \equiv 4 \mod 6$.

Case 2. Suppose $2k$ is not coloured with 1. If $j_0 \in [1,k]$ is not coloured with 1 and $j_0 \le k-1$, then $\{k+l\}_{0\le l\le k-1}$ is k-term a.p. of colour 1. So, we may assume that the following pair $(2k,k)$ is not coloured with 1.

Subcase 2.1. Suppose k is even. If $k \equiv 2 \mod 6$, then $(2k,k) \equiv (4,2) \mod 6$. If $k \equiv 4 \mod 6$, then $(2k,k) \equiv (2,4) \mod 6$. If $k \equiv 0 \mod 6$, then $(2k,k) \equiv (0,0) \mod 6$. Since $k, 2k$ are both even, $\{1+2l\}_{0\le l\le k-1}$ is a k-term a.p. of colour 1.

Subcase 2.2. Suppose $k \equiv 0 \mod 3$, then $(2k,k) \equiv (0,0) \mod 3$. Since $k, 2k$ are both $\equiv 0 \mod 3$, $\{1+3l\}_{0\le l\le k-1}$ is a k-term a.p. of colour 1.

Subcase 2.3. Suppose $k \equiv \pm 1 \mod 6$. If $k \equiv 1 \mod 6$, then $(2k,k) \equiv (2,1) \mod 6$. If $k \equiv 5 \mod 6$, then $(2k,k) \equiv (4,5) \mod 6$. In either case, if there is a k-term a.p. of colour 1, then there exist positive integers a,d such that $a+jd \in [1,3(k-1)+1] \setminus \{k,2k\}$ for all $j=0,1,\ldots,k-1$. Since $a \ge 1$, we have $d(k-1) \le 3k-3$. So, $d \le 3$. From $d \ge 1$, we obtain $a \le 2k-1$. Suppose $d=1$. If $a \in [1,k]$, then $a+j=k$ for some $j \in [0,k-1]$. If $a \in [k+1,2k-1]$, then $a+j=2k$ for some $j \in [0,k-1]$. All these cases are not possible. If $d=2$, then $a \le k$. For all $j \in [0,k-1]$, $a+2j \equiv a \mod 2$. Note that $(2k,k) \equiv (0,1) \mod 2$. If a is odd, then $a+2j=k$ for some $j \in [0,k-1]$. If a is even, then $a+2j=2k$ for some $j \in [0,k-1]$. All these cases are not possible. If $d=3$, then $a=1$. Since $1+3j \equiv 1 \mod 3$ for all $j \in [0,k-1]$, we must have $1+3j=k$ (when $k \equiv 1 \mod 6$) or $1+3j=2k$ (when $k \equiv 1 \mod 6$) for some $j \in [0,k-1]$. Thus, $w_2(k;3) > 3(k-1)+1$ when $k \equiv \pm 1 \mod 6$.

Now, for $k \equiv 1, 4$ or $5 \mod 6$, we have $w_2(k;3) > 3(k-1)+1$. For other k, we have $w_2(k;3) \le 3(k-1)+1$.

This completes the proof of the lemma. □

Lemma 2.4. *If $k \equiv 1$ or $5 \mod 6$, then $w_2(k;3) = 3k$. If $k \equiv 4 \mod 6$, then $w_2(k;3) = 3(k-1)+2$.*

Proof. Let $\chi : [1,3(k-1)+2] \to [1,2,3]$ be a colouring. We may assume that there are only two elements, one of them is coloured with 2 and the other is coloured with 3. If all the elements in $[1,k]$ or in $[k+1,2k]$ are coloured with 1, then we have a k-term a.p. of colour 1. So, there are an element in $[1,k]$ and an element in $[k+1,2k]$ not coloured with 1. If all the elements in $[2k,3(k-1)+2]$ are coloured with 1, then we have a k-term a.p. of colour 1. So, we may assume that $2k$ is not coloured with 1.

If $j_0 \in [1,k]$ is not coloured with 1 and $j_0 \leq k-1$, then $\{k+l\}_{0\leq l\leq k-1}$ is k-term a.p. of colour 1. So, we may assume that the following pair $(2k,k)$ is not coloured with 1.

Case 1. Suppose $k \equiv 1 \mod 6$. Then, $(2k,k) \equiv (2,1) \mod 6$. If there is a k-term a.p. of colour 1, then there exist positive integers a,d such that $a+jd \in [1,3(k-1)+2] \setminus \{k,2k\}$ for all $j=0,1,\dots,k-1$. Since $a \geq 1$, we have $d(k-1) \leq 3k-2$. So, $d \leq 3$. From $d=1$, we obtain $a \leq 2k$. If $a \in [1,k]$, then $a+j=k$ for some $j \in [0,k-1]$. If $a \in [k+1,2k]$, then $a+j=2k$ for some $j \in [0,k-1]$. All these cases are not possible. If $d=2$, then $a \leq k+1$. Note that if $a=k+1$, then a is even, for $k \equiv 1 \mod 6$. For all $j \in [0,k-1]$, $a+2j \equiv 0 \mod 2$. Note that $(2k,k) \equiv (0,1) \mod 2$. If a is odd, then $a \leq k$ and $a+2j=k$ for some $j \in [0,k-1]$. If a is even, then $a \leq k+1$ and $a+2j=2k$ for some $j \in [0,k-1]$. So, all these cases are not possible. If $d=3$, then $a=1$ or 2. Since $1+3j \equiv 1 \mod 3$ and $2+3j \equiv 2 \mod 3$ for all $j \in [0,k-1]$, we must have $1+3j=k$ for some $j \in [0,k-1]$ and $2+3j=2k$ for some $j \in [0,k-1]$. Thus, $w_2(k;3) > 3(k-1)+2$ when $k \equiv 1 \mod 6$. It follows Lemma 2.1 that $w_2(k;3)=3k$.

Case 2. Suppose $k \equiv 4 \mod 6$. Then, $(2k,k) \equiv (2,4) \mod 6$. Since $k,2k$ are both even, $\{1+2l\}_{0\leq l\leq k-1}$ is a k-term a.p. of colour 1. Thus, $w_2(k;3) \leq 3(k-1)+2$. It follows from Lemma 2.3 that $w_2(k;3)=3(k-1)+2$ when $k \equiv 4 \mod 6$.

Case 3. Suppose $k \equiv 5 \mod 6$. Then, $(2k,k) \equiv (4,5) \mod 6$. If there is a k-term a.p. of colour 1, then there exist positive integers a,d such that $a+jd \in [1,3(k-1)+2] \setminus \{k,2k\}$ for all $j=0,1,\dots,k-1$. Since $a \geq 1$, we have $d(k-1) \leq 3k-2$. So, $d \leq 3$. From $d=1$, we obtain $a \leq 2k$. If $a \in [1,k]$, then $a+j=k$ for some $j \in [0,k-1]$. If $a \in [k+1,2k]$, then $a+j=2k$ for some $j \in [0,k-1]$. All these cases are not possible. If $d=2$, then $a \leq k+1$. Note that if $a=k+1$, then a is even, for $k \equiv 5 \mod 6$. For all $j \in [0,k-1]$, $a+2j \equiv 0 \mod 2$. Note that $(2k,k) \equiv (0,1) \mod 2$. If a is odd, then $a \leq k$ and $a+2j=k$ for some $j \in [0,k-1]$. If a is even, then $a \leq k+1$ and $a+2j=2k$ for some $j \in [0,k-1]$. So, all these cases are not possible. If $d=3$, then $a=1$ or 2. Since $1+3j \equiv 1 \mod 3$ and $2+3j \equiv 2 \mod 3$ for all $j \in [0,k-1]$, we must have $1+3j=2k$ for some $j \in [0,k-1]$ and $2+3j=k$ for some $j \in [0,k-1]$. Thus, $w_2(k;3) > 3(k-1)+2$ when $k \equiv 5 \mod 6$. It follows Lemma 2.1 that $w_2(k;3)=3k$.

This completes the proof of the lemma. $\square$

Theorem 2.5. *Let $k \geq 3$ be an integer.*

$$w_2(k;3) = \begin{cases} 3k-2 & \text{if } k \equiv 0,2,3 \mod 6; \\ 3k-1 & \text{if } k \equiv 4 \mod 6; \\ 3k & \text{if } k \equiv 1,5 \mod 6. \end{cases}$$

Proof. Suppose k is odd. Let $\chi : [1, 3(k-1)] \to [1, 2, 3]$ be such that $\chi(k) = 2$, $\chi(2k) = 3$ and $\chi(x) = 1$ for all $x \in [1, 3(k-1)] \setminus \{k, 2k\}$. If there is a k-term a.p. of colour 1, then there exist positive integers a, d such that $a+jd \in [1, 3(k-1)] \setminus \{k, 2k\}$ for all $j = 0, 1, \ldots, k-1$. This implies that $a + (k-1)d \leq 3k-3$. Since $a \geq 1$, we have $d(k-1) \leq 3k-4$. So, $d \leq 2$. Suppose $d = 1$. Then, $a \leq 2k-2$. If $a \in [1, k]$, then $a+j = k$ for some $j \in [0, k-1]$. If $a \in [k+1, 2k]$, then $a+j = 2k$ for some $j \in [0, k-1]$. All these cases are not possible. Suppose $d = 2$. Then, $a \leq k-1$. Since $k \geq 3$, a can be odd or even. If a is odd, then $a + 2j$ is odd for all $j \in [0, k-1]$. So, $a + 2j = k$ for some $j \in [0, k-1]$. If a is even, then $a + 2j$ is even for all $j \in [0, k-1]$. So, $a + 2j = 2k$ for some $j \in [0, k-1]$. All these cases are not possible. Hence, $w_2(k; 3) > 3(k-1)$ when k is odd.

Suppose k is even. Let $\chi : [1, 3(k-1)] \to [1, 2, 3]$ be such that $\chi(k) = 2$, $\chi(2k-1) = 3$ and $\chi(x) = 1$ for all $x \in [1, 3(k-1)] \setminus \{k, 2k-1\}$. If there is a k-term a.p. of colour 1, then there exist positive integers a, d such that $a + jd \in [1, 3(k-1)] \setminus \{k, 2k-1\}$ for all $j = 0, 1, \ldots, k-1$. As in the previous paragraph, $d \leq 2$. Suppose $d = 1$. Then, $a \leq 2k-2$. If $a \in [1, k]$, then $a + j = k$ for some $j \in [0, k-1]$. If $a \in [k+1, 2k]$, then $a + j = 2k-1$ for some $j \in [0, k-1]$. All these cases are not possible. Suppose $d = 2$. Then, $a \leq k-1$. Since $k \geq 3$, a can be odd or even. If a is odd, then $a + 2j$ is odd for all $j \in [0, k-1]$. So, $a + 2j = 2k-1$ for some $j \in [0, k-1]$. If a is even, then $a + 2j$ is even for all $j \in [0, k-1]$. So, $a + 2j = k$ for some $j \in [0, k-1]$. All these cases are not possible. So, $w_2(k; 3) > 3(k-1)$ when k is even. Hence, $w_2(k; 3) > 3(k-1)$ for all $k \geq 3$.

The theorem then follows from Lemmas 2.3 and 2.4. □

2.2. Some results of $w_2(k; r)$ for $2 \leq r \leq k$

In this subsection, we first prove the exact values of $w_2(k; r)$ when k satisfies some divisibility conditions (see Theorems 2.7 and 2.9). We will then provide some necessary conditions on k for $w_2(k; r) \geq r(k-1)+1$ and $w_2(k; r) \leq r(k-1)$ in Theorem 2.11 and Theorem 2.13, respectively.

Lemma 2.6. *Let a, d, k, r be positive integers and let $i \geq 0$ be integer such that $\gcd(k, d) = 1$, $a \in [ik+1, (i+1)k]$ and $0 \leq i \leq r-d$. Then, there exists $j_0 \in [0, k-1]$ such that $a + j_0 d \in \{(i+1)k, (i+2)k, \ldots, (i+d)k\}$.*

Proof. If $(i+s)k \equiv (i+s')k \mod d$ for some $1 \leq s, s' \leq d$, then $(i+s) \equiv (i+s') \mod d$, for $\gcd(k, d) = 1$. This implies that $s = s'$. Thus, all the elements in the d-tuple $((i+1)k, (i+2)k, \ldots, (i+d)k)$ are incongruent modulo d. Therefore, one of them must be $\equiv a \mod d$, say $(i+s_0)k$. Note that $1 \leq s_0 \leq d$. From $a + jd = (i + s_0)k$ and $a \geq ik + 1$, we have

$$j = \frac{(i+s_0)k - a}{d} \leq \frac{s_0k-1}{d} \leq k - \frac{1}{d} < k.$$

This implies that $j \leq k-1$. From $a+jd = (i+s_0)k$ and $a \leq (i+1)k$, we have

$$j = \frac{(i+s_0)k - a}{d} \geq \frac{(s_0-1)k}{d} \geq 0.$$

So, $a + j_0d = (i + s_0)k$ for some $j_0 \in [0, k-1]$. □

Theorem 2.7. *Suppose $k \geq r \geq 2$. Then $w_2(k;r) = rk$ if and only if $\gcd(k,p) = 1$ for all primes p with $2 \leq p \leq r$.*

Proof. Suppose that $\gcd(k,p) = 1$ for all primes p with $2 \leq p \leq r$. Let $\chi_0 : [1, rk-1] \to [1, 2, \ldots, r]$ be a colouring such that $\chi_0((i-1)k) = i$ for $2 \leq i \leq r$ and $\chi_0(x) = 1$ for all $x \in [1, rk-1] \setminus \{k, 2k, \ldots, (r-1)k\}$. If there is a k-term a.p. of colour 1, then there exist positive integers a, d such that $a + jd \in [1, rk-1] \setminus \{k, 2k, \ldots, (r-1)k\}$ for all $j = 0, 1, \ldots, k-1$. Note that $a + jd \equiv a \mod d$ for all $j \in [0, k-1]$. Since $a + (k-1)d \leq rk - 1$ and $a \geq 1$, we have $(k-1)d \leq rk-2$, which implies that $d \leq r$ (here, we use the fact $k \geq r$). Next, $a \leq (r-d)k - 1 + d$. Let $a \in [i_0k+1, (i_0+1)k]$. Then, $i_0 \leq r-d$. Consider the d-tuple $((i_0+1)k, (i_0+2)k, \ldots, (i_0+d)k)$.

Since $d \leq r$ and $\gcd(k,p) = 1$ for all primes p with $2 \leq p \leq r$, we have $\gcd(k,d) = 1$. By Lemma 2.6, $a + j_0d = (i_0 + s_0)k$ for some $j_0 \in [0, k-1]$ and $s_0 \in [1, d]$.

If $i_0 \leq r-d-1$, then $(i_0+s_0)k \leq (r-1)k$. So, $a+j_0d \in \{k, 2k, \ldots, (r-1)k\}$, a contradiction. If $i_0 = r-d$ and $s_0 \leq d-1$, then $(i_0+s_0)k \leq (r-1)k$. So, $a + j_0d \in \{k, 2k, \ldots, (r-1)k\}$, a contradiction. If $i_0 = r-d$ and $s_0 = d$, then $(i_0+s_0)k = rk$. So, $rk = a + j_0d \in [1, rk-1]$, a contradiction. Hence, there is no a k-term a.p. of colour 1 and $w_2(k;r) > rk-1$. It follows from Lemma 2.1 that $w_2(k;r) = rk$.

Now, suppose that $w_2(k;r) = rk$. Then, there is a colouring $\chi : [1, rk-1] \to [1, 2, \ldots, r]$ such that there is no k-term a.p. of colour 1 and for each $i \geq 2$, there is no 2-term a.p. of colour i. We may assume that there are only $r-1$ elements in which each of them is assigned with a colour in $\{2, 3, \ldots, r\}$ and none of them are assigned with the same colour. For each $j \in [1, r-1]$, if all the elements in $[(j-1)k+1, jk]$ are coloured with 1, then we have a k-term a.p. of colour 1. So, for each $j \in [1, r-1]$, there is an element in $[(j-1)k+1, jk]$ not coloured with 1. If all the elements in $[(r-1)k, rk-1]$ are coloured with 1, then we have a k-term a.p. of colour 1. So, we may assume that $(r-1)k$ is not coloured with 1.

If $j_0 \in [(r-3)k+1, (r-2)k]$ is not coloured with 1 and $j_0 \leq (r-2)k-1$, then $\{(r-2)k+l\}_{0 \leq l \leq k-1}$ is k-term a.p. of colour 1. So, we may assume that $(r-2)k$ is not coloured with 1. Next, if $j_1 \in [(r-4)k+1, (r-3)k]$ is not coloured with 1 and $j_1 \leq (r-3)k-1$, then $\{(r-3)k+l\}_{0 \leq l \leq k-1}$ is k-term a.p. of colour

1. So, we may assume that $(r-3)k$ is not coloured with 1. By continuing this process, we may assume that the $(r-1)$-tuple $(k, 2k, \ldots, (r-1)k)$ is not coloured with 1.

Suppose that $\gcd(k, p_0) = p_0$ for some prime p_0 with $2 \le p_0 \le r$. Note that

$$1 + (k-1)p_0 \le 1 + rk - r \le rk - 1.$$

Since $1 + jp_0 \equiv 1 \mod p_0$ for all $j \in [0, k-1]$, $1 + jp_0 \notin \{k, 2k, \ldots, (r-1)k\}$. Thus, $\{1 + lp_0\}_{0 \le l \le k-1}$ is k-term a.p. of colour 1, a contradiction. Hence, $\gcd(k, p) = 1$ for all primes p with $2 \le p \le r$. □

Lemma 2.8. *For each $i \in [1, r-1]$, let $a_i \in [(r-i-1)k+1, (r-i)k]$ and $j_i = (r-i)k - a_i$. Suppose there is no k-term a.p. with common difference 1 in $[1, (r-1)k] \setminus \{a_1, a_2, \ldots, a_{r-1}\}$. Then,*

$$0 \le j_{r-1} \le j_{r-2} \le \cdots \le j_2 \le j_1.$$

Proof. Note that $a_1 \in [(r-2)k+1, (r-1)k]$. If $a_2 \le (r-2)k - j_1 - 1$, then $\{(r-2)k - j_1 + l\}_{0 \le l \le k-1}$ is k-term a.p. with common difference 1, because $(r-2)k - j_1 + (k-1) = (r-1)k - j_1 - 1 = a_1 - 1 < a_1$. So, we may assume that $a_2 \ge (r-2)k - j_1$. Since $j_2 = (r-2)k - a_2$, we have $0 \le j_2 \le j_1$. Next, if $a_3 \le (r-3)k - j_2 - 1$, then $\{(r-3)k - j_2 + l\}_{0 \le l \le k-1}$ is k-term a.p. with common difference 1, because $(r-3)k - j_2 + (k-1) = (r-2)k - j_2 - 1 = a_2 - 1 < a_2$. So, we may assume that $a_3 \ge (r-3)k - j_2$. Since $j_3 = (r-3)k - a_3$, we have $0 \le j_3 \le j_2$. By continuing this process, we see that $0 \le j_{r-1} \le j_{r-2} \le \cdots \le j_2 \le j_1$. □

Theorem 2.9. *Suppose $k \ge r \ge 2$ and r is a prime. If $\gcd(k, r) = r$ and $\gcd(k, p) = 1$ for all primes p with $2 \le p \le r-1$, then $w_2(k; r) = r(k-1) + 1$.*

Proof. Let $\chi_0 : [1, r(k-1)] \to [1, 2, \ldots, r]$ be a colouring such that $\chi_0((i-1)k) = i$ for $2 \le i \le r$ and $\chi_0(x) = 1$ for all $x \in [1, r(k-1)] \setminus \{k, 2k, \ldots, (r-1)k\}$. If there is a k-term a.p. of colour 1, then there exist positive integers a, d such that $a + jd \in [1, r(k-1)] \setminus \{k, 2k, \ldots, (r-1)k\}$ for all $j = 0, 1, \ldots, k-1$. Note that $a + jd \equiv a \mod d$ for all $j \in [0, k-1]$. Since $a + (k-1)d \le r(k-1)$ and $a \ge 1$, we have $(k-1)d \le r(k-1) - 1$, which implies that $d \le r-1$. Next, $a \le (r-d)(k-1)$. Let $a \in [i_0k+1, (i_0+1)k]$. Then, $i_0 \le r-d-1$. Consider the d-tuple $((i_0+1)k, (i_0+2)k, \ldots, (i_0+d)k)$.

Since $d \le r-1$ and $\gcd(k, p) = 1$ for all primes p with $2 \le p \le r-1$, we have $\gcd(k, d) = 1$. By Lemma 2.6, $a + j_0d = (i_0 + s_0)k$ for some $j_0 \in [0, k-1]$ and $s_0 \in [1, d]$. Since $(i_0 + s_0)k \le (r-1)k$, we have $a + j_0d \in \{k, 2k, \ldots, (r-1)k\}$, a contradiction. Thus, $w_2(k; r) > r(k-1)$.

Assume that $w_2(k; r) > r(k-1) + 1$. Then, there is a colouring $\chi : [1, r(k-1)+1] \to [1, 2, \ldots, r]$ such that there is no k-term a.p. of colour 1 and for each $i \ge 2$, there is no 2-term a.p. of colour i. We may assume that there are only $r-1$ elements in which each of them is assigned with a colour in $\{2, 3, \ldots, r\}$ and none of them are assigned with the same colour. For each $j \in [1, r-1]$, if all the elements in $[(j-1)k+1, jk]$ are coloured with 1, then we have a k-term a.p. of colour 1. So, for each $i \in [1, r-1]$, let $a_i \in [(r-i-1)k+1, (r-i)k]$ be the element not coloured with 1, and let $j_i = (r-i)k - a_i$.

If all the elements in $[(r-1)k-r+2, r(k-1)+1]$ are coloured with 1, then we have a k-term a.p. of colour 1. So, we may assume that the element a_1 not coloured with 1 in $[(r-2)k+1, (r-1)k]$ satisfies the inequality $(r-1)k-r+2 \leq a_1 \leq (r-1)k$. Since $j_1 = (r-1)k - a_1$, we have $0 \leq j_1 \leq r-2$.

It follows from Lemma 2.8 that

$$0 \leq j_{r-1} \leq j_{r-2} \leq \cdots \leq j_2 \leq j_1 \leq r-2.$$

Note that $\{1+lr\}_{0 \leq l \leq k-1}$ is k-term a.p. and $1+lr \equiv 1 \mod r$ for all l. Since there is no k-term a.p. of colour 1, $a_{i_0} = 1+lr$ for some l and i_0. This implies that $1 \equiv a_{i_0} \equiv (r-i_0)k - j_{i_0} \equiv -j_{i_0} \mod r$ (here, we use the fact that $k \equiv 0 \mod r$). Now, $j_{i_0} = r-z$ for some integer $z \in [2, r]$. Therefore, $1 \equiv z \mod r$, a contradiction. Hence, $w_2(k; r) = r(k-1)+1$. □

Lemma 2.10. *For each $i \in [1, r-1]$, let $a_i \in [(r-i-1)k+1, (r-i)k]$ and $j_i = (r-i)k - a_i$. Let p be a prime such that $2 \leq p \leq r-1$ and $\gcd(k, p) = p$. Suppose there is no k-term a.p. with common difference 1 or p in $[1, (r-1)k] \setminus \{a_1, a_2, \ldots, a_{r-1}\}$. Then,*

$$0 \leq j_{r-1} < j_{r-2} < \cdots < j_2 < j_1.$$

Furthermore, either $j_i = j_{i+1}+1$ for all $1 \leq i \leq r-2$ or $j_1 \geq r$.

Proof. By Lemma 2.8, $0 \leq j_{r-1} \leq j_{r-2} \leq \cdots \leq j_2 \leq j_1$. Suppose that $p = r-1$. If $\{a_1, a_2, \ldots, a_{r-1}\}$ is not a complete set of residue modulo p, then there is an $a \in [1, p]$ such that $a \not\equiv a_i \mod p$ for all $1 \leq i \leq r-1$. Since $a + (k-1)p \leq pk = (r-1)k$ and $a + lp \equiv a \mod p$ for all l, $\{a+lp\}_{0 \leq l \leq k-1}$ is a k-term a.p. with common difference p in $[1, (r-1)k] \setminus \{a_1, a_2, \ldots, a_{r-1}\}$, a contradiction. Thus, $\{a_1, a_2, \ldots, a_{r-1}\}$ is a complete set of residue modulo p. Note that $a_i = (r-i)k - j_i \equiv -j_i \mod p$. Therefore, $\{j_1, j_2, \ldots, j_{r-1}\}$ is also a complete set of residue modulo p. If $j_1 \geq r$, we are done. So, we may assume that $j_1 \leq r-1$. There are two possibilities, either $(j_1, j_2, \ldots, j_{r-1}) = (r-2, r-3, \ldots, 0)$ or $(j_1, j_2, \ldots, j_{r-1}) = (r-1, r-2, \ldots, 1)$. Hence, $j_i = j_{i+1}+1$ for all $1 \leq i \leq r-2$.

Suppose that $2 \leq p < r-1$. Note that for each $i \in [p, r-1]$,

$$\{j_i, j_{i-1}, \ldots, j_{i-p+1}\},$$

is a complete set of residue modulo p, for otherwise there is a k-term a.p. with common difference p in $[(r-i-1)k+1, (r-i+p-1)k] \setminus \{a_i, a_{i-1}, \ldots, a_{i-p+1}\}$. This implies that $j_i \geq j_{i+1}+1$ for all i and whence, $0 \leq j_{r-1} < j_{r-2} < \cdots < j_2 < j_1$. If $j_i = j_{i+1}+1$ for all $1 \leq i \leq r-2$, we are done. Suppose that $j_{i_0} \geq j_{i_0+1}+2$ for some i_0. Since $p < r-1$, we can find an integer $i \in [p, r-1]$ such that $i-p+1 \leq i_0 \leq i$. Now, $\{j_i, j_{i-1}, \ldots, j_{i-p+1}\}$ and $\{j_{i-1}, j_{i-2}, \ldots, j_{i-p}\}$, both are a complete set of residue modulo p. This implies that $j_i \equiv j_{i-p} \mod p$. If $j_{i-p} = j_i + p$, then from $j_{i-p+1} > \cdots > j_{i-1} > j_i$, we must have

$$(j_i, j_{i-1}, \ldots, j_{i-p+1}) = (j_i, j_i + 1, \ldots, j_i + p - 1),$$

and in particular, $j_{i_0} = j_{i_0+1} + 1$, a contradiction. Hence, $j_{i-p} = j_i + zp$ for some integer $z \geq 2$. From $0 \leq j_{r-1} < j_{r-2} < \cdots < j_2 < j_1$, we have

$$j_1 \geq j_2 + 1 \geq j_3 + 2 \geq \cdots \geq j_{i-p} + i - p - 1,$$

and

$$j_i \geq j_{i+1} + 1 \geq j_{i+2} + 2 \geq \cdots \geq j_{r-1} + r - 1 - i \geq r - 1 - i.$$

Hence, $j_1 \geq j_{i-p} + i - p - 1 = j_i + zp + i - p - 1 \geq r - 1 - i + zp + i - p - 1 = r - 2 + (z-1)p \geq r$. □

Theorem 2.11. *Suppose $k \geq r \geq 2$ and $\gcd(k, p) = p$ for some prime p with $2 \leq p \leq r - 1$. If $w_2(k; r) \geq r(k-1) + 1$, then $k \not\equiv 1 \mod q$ for all primes q with $2 \leq q \leq r - 1$.*

Proof. Suppose $w_2(k; r) \geq r(k-1) + 1$. Let $\chi : [1, r(k-1)] \to [1, 2, \ldots, r]$ be a colouring such that there is no k-term a.p. of colour 1 and for each $i \geq 2$, there is no 2-term a.p. of colour i. We may assume that there are only $r - 1$ elements in which each of them is assigned with a colour in $\{2, 3, \ldots, r\}$ and none of them are assigned with the same colour. For each $j \in [1, r-1]$, if all the elements in $[(j-1)k + 1, jk]$ are coloured with 1, then we have a k-term a.p. of colour 1. So, for each $i \in [1, r-1]$, let $a_i \in [(r-i-1)k + 1, (r-i)k]$ be the element not coloured with 1, and let $j_i = (r-i)k - a_i$.

If all the elements in $[(r-1)k - r + 1, r(k-1)]$ are coloured with 1, then we have a k-term a.p. of colour 1. So, we may assume that the element a_1 not coloured with 1 in $[(r-2)k+1, (r-1)k]$, satisfies the inequality $(r-1)k - r + 1 \leq a_1 \leq (r-1)k$. Since $j_1 = (r-1)k - a_1$, we have $0 \leq j_1 \leq r - 1$.

It follows from Lemma 2.10 that

(i) $0 \leq j_{r-1} < j_{r-2} < \cdots < j_2 < j_1 \leq r - 1$;
(ii) $j_i = j_{i+1} + 1$ for all i.

Thus, there are two possibilities, either $(j_1, j_2, \ldots, j_{r-1}) = (r-2, r-3, \ldots, 0)$ or $(j_1, j_2, \ldots, j_{r-1}) = (r-1, r-2, \ldots, 1)$. This implies that either $a_i = (r-i)k - j_i = (r-i)k - (r-i)$ or $a_i = (r-i)k - j_i = (r-i)k - (r-i-1)$. In either case, $a_x - a_y = (y-x)k + j_y - j_x = (y-x)k + y - x$ where the last equality follows from $j_x = j_y + y - x$.

Suppose there is a prime $q \leq r - 1$ such that $k \equiv 1 \mod q$. For all x, y with $1 \leq x < y \leq q$, we have

$$a_x - a_y = (y-x)k + y - x = (y-x)(k-1).$$

So, $a_x \equiv a_y \mod q$. Choose an $a \in [(r-q-1)k+1, (r-q)k]$ such that $a \not\equiv a_q \mod q$. We may choose $a = (r-q-1)k + 1$ or $(r-q-1)k + 2$ as one of them is not congruent to a_q modulo q. Note that $\{a + lq\}_{0 \leq l \leq k-1}$ is k-term a.p.,

$a+(k-1)q \leq (r-q-1)k+2+(k-1)q = rk-k-q+2 \leq rk-k \leq r(k-1)$ and $a+lq \equiv a \mod q$ for all l. So, $\{a+lq\}_{0\leq l\leq k-1}$ is k-term a.p. of colour 1 in $[(r-q-1)k+1, r(k-1)] \setminus \{a_q, a_{q-1}, \ldots, a_1\}$, a contradiction. Hence, there is no prime $q \leq r-1$ such that $k \equiv 1 \mod q$. □

Lemma 2.12. *Suppose $k \geq r$. Let $a_i = (r-i)k-(r-i-1)$ for all $1 \leq i \leq r-1$. If $r-1 \geq d \geq 2$, $a_{r-i_0-d} \leq r(k-1)$ and $\{a_{r-i_0-1}, a_{r-i_0-2}, \ldots, a_{r-i_0-d}\}$ is not a complete set of residue modulo d for some $1 \leq i_0 \leq r-1$, then*

(a) $k \equiv 1 \mod q$ *for some prime q with $2 \leq q \leq r-1$.*

(b) *there exist positive integers a, d such that $a+ld \in [i_0k+1, (i_0+d)k] \setminus \{a_{r-i_0-1}, a_{r-i_0-2}, \ldots, a_{r-i_0-d}\}$ for all $l = 0, 1, \ldots, k-1$.*

Proof. (a) Note that $a_{r-i_0-j_1} \equiv a_{r-i_0-j_2} \mod d$ for some $1 \leq j_1 < j_2 \leq d$. This implies that

$$(i_0+j_1)k-(i_0+j_1-1) \equiv (i_0+j_2)k-(i_0+j_2-1) \mod d.$$

So,

$$(j_1-j_2)(k-1) \equiv 0 \mod d.$$

If $\gcd(k-1, d) = 1$, then $j_1 \equiv j_2 \mod d$. This is not possible as $0 < j_2-j_1 \leq d-1$. So, there is a prime $q \leq r-1$ such that $k \equiv 1 \mod q$.

(b) Now $r(k-1) \geq a_{r-i_0-d} = (i_0+d)k-(i_0+d-1) = (i_0+d)(k-1)+1 > (i_0+d)(k-1)$ implies that $i_0 \leq r-d-1$. Note that $\{i_0k+1, i_0k+2, \ldots, i_0k+d\}$ is a complete set of residue modulo d and $a_{r-i_0-1} = (i_0+1)k-i_0 \geq (i_0+1)k-(r-d-1) = i_0k+(k-r)+d+1 \geq i_0k+d+1$. Choose an $a \in \{i_0k+1, i_0k+2, \ldots, i_0k+d\}$ such that $a \not\equiv a_{r-i_0-j} \mod d$ for all $1 \leq j \leq d$. This is possible as $\{a_{r-i_0-1}, a_{r-i_0-2}, \ldots, a_{r-i_0-d}\}$ is not a complete set of residue modulo d. Then, $i_0k+1 \leq a < a_{r-i_0-1} = (i_0+1)k-i_0$ and

$$a+(k-1)d \leq i_0k+d+(k-1)d = (i_0+d)k.$$

Hence, $a+ld \in [i_0k+1, (i_0+d)k] \setminus \{a_{r-i_0-1}, a_{r-i_0-2}, \ldots, a_{r-i_0-d}\}$ for all $l = 0, 1, \ldots, k-1$. □

Theorem 2.13. *Suppose $k \geq r \geq 2$. If $w_2(k; r) \leq r(k-1)$, then $k \equiv 1 \mod q$ for some prime q with $2 \leq q \leq r-1$.*

Proof. Let $a_i = (r-i)k-(r-i-1)$ for all $1 \leq i \leq r-1$. Let $\chi_0 : [1, r(k-1)] \to [1, 2, \ldots, r]$ be a colouring such that $\chi_0(a_i) = i+1$ for $1 \leq i \leq r-1$ and $\chi_0(x) = 1$ for all $x \in [1, r(k-1)] \setminus \{a_1, a_2, \ldots, a_{r-1}\}$. Since $w_2(k; r) \leq r(k-1)$, there is a k-term a.p. of colour 1. So, there exist positive integers a, d such that $a+ld \in [1, r(k-1)] \setminus \{a_1, a_2, \ldots, a_{r-1}\}$ for all $l = 0, 1, \ldots, k-1$. Note that $a+ld \equiv a \mod d$ for all $l \in [0, k-1]$. Since $a+(k-1)d \leq r(k-1)$ and $a \geq 1$, we have $(k-1)d \leq r(k-1)-1$, which implies that $d \leq r-1$. Next, $a \leq (r-d)(k-1)$. Let $a \in [i_0k+1, (i_0+1)k]$. Then, $i_0 \leq r-d-1$ and $a_{r-i_0-d} = (i_0+d)k-(i_0+d-1) \leq (r-1)k-(i_0+d-1) \leq (r-1)k$.

By construction, the longest interval with the same colour under χ_0 has length at most $k-1$. So it is clear that $d \geq 2$. Now, consider the d-tuple $(a_{r-i_0-1}, a_{r-i_0-2}, \ldots, a_{r-i_0-d})$. Note that for $1 \leq j \leq d$,

$$a_{r-i_0-j} = (i_0+j)k - (i_0+j-1) \in [(i_0+j-1)k+1, (i_0+j)k].$$

By part (a) of Lemma 2.12, we may assume that $\{a_{r-i_0-1}, a_{r-i_0-2}, \ldots, a_{r-i_0-d}\}$ is a complete set of residue modulo d. Therefore, $a \equiv a_{r-i_0-j} \mod d$ for some $j \in [1, d]$. From $a \geq i_0k+1$ and $j \leq d$, we have

$$\begin{aligned} l &= \frac{a_{r-i_0-j} - a}{d} = \frac{(i_0+j)k - (i_0+j-1) - a}{d} \\ &\leq \frac{(i_0+j)k - (i_0+j-1) - (i_0k+1)}{d} = \frac{j(k-1) - i_0}{d} \\ &\leq (k-1) - \frac{i_0}{d} \leq k-1. \end{aligned}$$

If $j \geq 2$, then $a_{r-i_0-j} = (i_0+j)k - (i_0+j-1) \geq (i_0+j)k - (r-2) > (i_0+1)k$. From $a \leq (i_0+1)k$, we have

$$l = \frac{a_{r-i_0-j} - a}{d} > 0.$$

This implies that there is an $l \in [0, k-1]$ such that $a + ld = a_{r-i_0-j}$, a contradiction.

Suppose $j = 1$. If $a \leq a_{r-i_0-1}$, then we have

$$l = \frac{a_{r-i_0-1} - a}{d} \geq 0.$$

This implies that there is an $l \in [0, k-1]$ such that $a + ld = a_{r-i_0-1}$, a contradiction.

Suppose $a > a_{r-i_0-1} = (i_0+1)k - i_0$. Then,

$$\begin{aligned} r(k-1) \geq a + (k-1)d &> (i_0+1)k - i_0 + (k-1)d \\ &= (i_0+d+1)k - (i_0+d) = a_{r-i_0-d-1}. \end{aligned}$$

By part (a) of Lemma 2.12, we may assume that

$$\{a_{r-i_0-2}, a_{r-i_0-3}, \ldots, a_{r-i_0-d-1}\},$$

is a complete set of residue modulo d. This implies that $a_{r-i_0-d-1} \equiv a_{r-i_0-1} \mod d$. Since $a \equiv a_{r-i_0-1} \mod d$, we have $a \equiv a_{r-i_0-d-1} \mod d$. Since $a + (k-1)d > a_{r-i_0-d-1}$ and $a < a_{r-i_0-d-1}$, there is an $l \in [0, k-1]$ such that $a + ld = a_{r-i_0-d-1}$, a contradiction. Hence, $k \equiv 1 \mod q$ for some prime q with $2 \leq q \leq r-1$. □

3. Concluding remarks

In this note, we have investigated $w_2(k; r)$ for $2 \leq r \leq k$ in general, and computed some exact values of $w_2(k; r)$ in that range. Filling in the gaps on the exact values of $w_2(k; r)$ for $2 \leq r \leq k$ that we are unable to compute here is certainly of immediate future research interests. It will also be interesting to establish the bound of $w_2(k;r) \leq r(k-1)$ for all $\frac{3(k-1)}{2} \leq r \leq 2k+1$, as this will improve the easy bound of $w_2(k;r)$ in Lemma 2.1 for all $r > k$ (see Theorem 1.4).

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