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L^p SOBOLEV MAPPING PROPERTIES OF THE BERGMAN PROJECTIONS ON n-DIMENSIONAL GENERALIZED HARTOGS TRIANGLES

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Abstract. The n-dimensional generalized Hartogs triangles $\mathbb{H}^n_{\mathbf{p}}$ with $n \geq 2$ and $\mathbf{p} := (p_1, \ldots, p_n) \in (\mathbb{R}^+)^n$ are the domains defined by

$$\mathbb{H}^n_{\mathbf{p}} := \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n : |z_1|^{p_1} < \cdots < |z_n|^{p_n} < 1\}.$$

In this paper, we study the L^p Sobolev mapping properties for the Bergman projections on the n-dimensional generalized Hartogs triangles $\mathbb{H}^n_{\mathbf{p}}$, which can be viewed as a continuation of the work by S. Zhang in [25] and a higher-dimensional generalization of the work by L. D. Edholm and J. D. McNeal in [16].

1. Introduction

Let Ω be a bounded domain in $\mathbb{C}^n$. As usual, for $p > 0$, the space $L^p(\Omega)$ consists of all Lebesgue measurable functions f on Ω such that

$$\|f\|_p := \left(\int_\Omega |f(z)|^p dV(z) \right)^{\frac{1}{p}} < \infty,$$

where V denotes the Lebesgue measure. The Bergman space $A^p(\Omega)$ consists of holomorphic functions f in $L^p(\Omega)$. In complex analysis, the most important orthogonal projection is the Bergman projection of $L^2(\Omega)$ onto $A^2(\Omega)$. The Bergman projection $\mathbf{B}$ is an integral operator of the form

$$\mathbf{B}_\Omega f(z) = \int_\Omega B_\Omega(z, w) f(w) dV(w), \quad f \in L^2(\Omega), \tag{1}$$

where $B_\Omega : \Omega \times \Omega \to \mathbb{C}$ is the Bergman kernel of Ω. It should be remarked that when the integral in (1) converges, it is taken as the definition of $\mathbf{B}_\Omega f$, even if $f \notin L^2(\Omega)$. If $\{\phi_\alpha\}$ is an orthonormal basis for the Bergman space $A^2(\Omega)$,

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then the Bergman kernel has the following formula:

$$B_\Omega(z,w) = \sum_\alpha \phi_\alpha(z)\overline{\phi_\alpha(w)}. \tag{2}$$

We refer the readers to [17, Section 1.1] for more on this topic.

It is of interest to study the mapping properties for the Bergman projections of various domains on the associated L^p Sobolev spaces. We recall that for any given bounded domain $\Omega \subset \mathbb{C}^n$, the L^p Sobolev space of order k with $1 < p < \infty$ and $k \in \mathbb{N}$ is the function space defined by

$$L^p_k(\Omega) := \left\{ f \in L^1_{\mathrm{loc}}(\Omega) : \|f\|_{p,k} := \left(\int_\Omega \sum_{|\alpha|\le k} |D^\alpha_{z,\bar z}(f)|^p dV(z) \right)^{\frac{1}{p}} < \infty \right\}, \tag{3}$$

where the summation is running over all the multi-indices $\alpha = (\alpha_1, \dots, \alpha_{2n})$ with $|\alpha| := \alpha_1 + \cdots + \alpha_{2n} \le k$, and $D^\alpha_{z,\bar z} := \frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} \partial \bar z_1^{\alpha_2} \cdots \partial z_n^{\alpha_{2n-1}} \partial \bar z_n^{\alpha_{2n}}}$. The derivatives in (3) should be interpreted in the distributional sense. $\|\cdot\|_{p,k}$ defined in (3) is said to be the associated Sobolev norm. It is clear that the L^p Sobolev spaces of order k reduce to the ordinary L^p spaces when $k = 0$, i.e., $L^p_0(\Omega) = L^p(\Omega)$. There are many papers concerning the mapping properties for the Bergman projections on L^p Sobolev spaces in different settings. For many classes of pseudoconvex domains, various authors have shown that the Bergman projection is bounded on L^p Sobolev spaces for all $1 < p < \infty$ and $k \in \mathbb{Z}^+$. See [7, 18–21, 23], for instance. Some regularity results on L^2 Sobolev spaces was shown in [5, 6]. For some L^p_k-irregular results, we refer the readers to [1, 3, 4] and reference therein for more information.

Recently, the L^p and L^p Sobolev mapping properties of the Bergman projections of the various kinds of generalized Hartogs triangles have been investigated by many authors (see e.g. [2, 10, 11, 13–15, 24–26]). In [12], L. Chen studied the weighted Sovolev regularity of the Bergman projection on the classical Hartogs triangle and a class of its n-dimensional generalizations. Later in [16], L. D. Edholm and J. D. McNeal investigated the L^p Sobolev mapping properties of the Bergman projection on the power-generalized Hartogs triangle $\mathbb{H}_\gamma$ defined by

$$\mathbb{H}_\gamma := \{z = (z_1, z_2) \in \mathbb{C}^2 : |z_1|^\gamma < |z_2| < 1\}, \quad \gamma \in \mathbb{R}^+.$$

They obtained that the Bergman projection $\mathbf{B}_{\mathbb{H}_\gamma}$ is bounded on L^p Sobolev space $L^p_k(\mathbb{H}_\gamma)$ if and only if $\gamma = 1$, $k = 1$ and $p \in (\frac{4}{3}, 2)$. Inspired by their works, it is reasonable and interesting to investigate the L^p Sobolev mapping properties of the Bergman projection over n-dimensional generalized Hartogs triangles

$$\mathbb{H}^n_{\mathbf{p}} := \{z = (z_1, \dots, z_n) \in \mathbb{C}^n : |z_1|^{p_1} < \cdots < |z_n|^{p_n} < 1\},$$

$$\mathbf{p} := (p_1, \dots, p_n) \in (\mathbb{R}^+)^n, \ n \ge 2,$$

which can be viewed as a natural generalization of $\mathbb{H}_\gamma$ to dimension n. In [25], the author explicitly calculated the Bergman kernel of $\mathbb{H}^n_{\mathbf{p}}$ with $\mathbf{p} \in (\mathbb{Z}^+)^n$ and obtained an optimal estimate for it, then he used the estimate to study the L^p mapping properties for the associated Bergman projection $\mathbf{B}_{\mathbb{H}^n_{\mathbf{p}}}$. Thanks to the results obtained in [25], in this paper we study the L^p Sobolev mapping properties for the Bergman projection of $\mathbb{H}^n_{\mathbf{p}}$. Our strategy combine the methods of [12], [16], [25] and some new ingredients. For convenience, we denote $B_{\mathbf{p}} := B_{\mathbb{H}^n_{\mathbf{p}}}$ and $\mathbf{B}_{\mathbf{p}} := \mathbf{B}_{\mathbb{H}^n_{\mathbf{p}}}$.

For any $\mathbb{H}^n_{\mathbf{p}}$ with $\mathbf{p} = (p_1, \ldots, p_n) \in (\mathbb{Z}^+)^n$, we set $k_1, \ldots, k_n \in \mathbb{Z}^+$ satisfying $k_1p_1 = \cdots = k_np_n$. We denote $D_n := \gcd(k_1, \ldots, k_n)$. Our first main result is the following theorem concerning the differentiated Bergman projections, which is a key tool in studying the L^p Sobolev mapping properties of the Bergman projection $\mathbf{B}_{\mathbf{p}}$.

Theorem 1.1. *For $r = 1, \ldots, n$ and $\mathbf{p} = (p_1, \ldots, p_n) \in (\mathbb{Z}^+)^n$, it holds that the differentiated Bergman projections $\frac{\partial}{\partial z_r} \circ \mathbf{B}_{\mathbf{p}}$ maps $L^p_1(\mathbb{H}^n_{\mathbf{p}}) \longrightarrow L^p(\mathbb{H}^n_{\mathbf{p}})$ boundedly for*

$$p \in \left(\frac{2\sum_{i=1}^{n} k_j}{\sum_{i=1}^{n} k_j + D_n + k_r}, \frac{2\sum_{i=1}^{n} k_j}{\sum_{i=1}^{n} k_j - D_n + k_r} \right),$$

where $D_n = \gcd(k_1, \ldots, k_n)$.

Remark. (a) When $n = 2$ and $p_1 = 1$, Theorem 1.1 reduces to the known result obtained by L. D. Edholm and J. D. McNeal in [16, Theorem 4.4] for 2-dimensional generalized Hartogs triangle $\mathbb{H}_{1/n} := \{z = (z_1, z_2) \in \mathbb{C}^2 : |z_1| < |z_2|^n < 1\}$.

(b) Our proof of Theorem 1.1 employs the technique used in [12], [16] and [25]. A key step in the proof of Theorem 1.1 is to obtain an optimal estimate for a kind of "modified" Bergman kernel $\widetilde{B_{\mathbf{p}}}$ (see (10)). It is worth noting that in the setting of 2-dimensional generalized Hartogs triangle $\mathbb{H}_{1/n}$ considered in [16, Theorem 4.16], the Bergman kernel of $\mathbb{H}_{1/n}$ can be explicitly written in a simple form and thus the optimal estimate of it is easy to get. However, in our case of the general n-dimensional setting, the Bergman kernel $B_{\mathbf{p}}$ is of a much more complicated form(see Lemma 2.5) and so does the "modified" Bergman kernel $\widetilde{B_{\mathbf{p}}}$. Therefore, we need to apply the techniques used in [25] to obtain an optimal estimate for it.

As a directly consequence of Theorem 1.1, we obtain the L^p Sobolev boundedness result for the Bergman projection of $\mathbb{H}^n_{\mathbf{p}}$ as follows, which can be viewed as a complement of the results obtained by S. Zhang in [25] and also a generalization of the results by L. D. Edholm and J. D. McNeal in [16] to the n-dimensional setting.

Theorem 1.2. *Let $n \geq 2$ and $\mathbf{p} = (p_1, \ldots, p_n) \in (\mathbb{R}^+)^n$. The Bergman projection associated to the n-dimensional generalized Hartogs triangle $\mathbb{H}^n_{\mathbf{p}}$ is bounded on $L^p_k(\mathbb{H}^n_{\mathbf{p}})$ with some $1 \leq p < \infty$ and $k \geq 1$ if and only if $k = 1$, $\mathbf{p} = c \cdot \mathbf{1} := (c, \ldots, c)$ ($c \in \mathbb{R}$ is a positive constant) and $p \in (\frac{2n}{n+1}, 2)$.*

Our last main result is about a substitute operator on n-dimensional generalized Hartogs triangle $\mathbb{H}^n_{\mathbf{1}}$, which is related to the Bergman projection $\mathbf{B_1}$. Moreover, it is expected to have better L^p Sobolev mapping behavior than $\mathbf{B_1}$ itself. This substitute operator is so called the L^∞ sub-Bergman projection. We recall that the L^∞ sub-Bergman kernel on $\mathbb{H}^n_{\mathbf{1}}$ is defined by

$$B^\infty_{\mathbf{1}}(z, w) := \sum_{\alpha \in \mathcal{A}^\infty(\mathbb{H}^n_{\mathbf{1}})} \frac{z^\alpha \overline{w}^\alpha}{\|z^\alpha\|_2^2}, \quad (z, w) \in \mathbb{H}^n_{\mathbf{1}} \times \mathbb{H}^n_{\mathbf{1}}, \tag{4}$$

where $\mathcal{A}^\infty(\mathbb{H}^n_{\mathbf{1}})$ is the set defined as follows:

$$\mathcal{A}^\infty(\mathbb{H}^n_{\mathbf{1}}) := \left\{ \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n : \sum_{i=1}^{j} \alpha_i \geq 0, \ \ j = 1, \ldots, n \right\}.$$

Since $\mathcal{A}^\infty(\mathbb{H}^n_{\mathbf{1}}) \subset \mathcal{A}^2(\mathbb{H}^n_{\mathbf{1}})$ (see Lemma 2.2), we know that the series in (4) is only part of the series that defines the ordinary Bergman kernel in (2), which follows that the series in (4) converges for all $(z, w) \in \mathbb{H}^n_{\mathbf{1}} \times \mathbb{H}^n_{\mathbf{1}}$. The L^∞ sub-Bergman projection $\mathbf{B}^\infty_{\mathbf{1}}$ on $\mathbb{H}^n_{\mathbf{1}}$ is the integral operator defined by

$$\mathbf{B}^\infty_{\mathbf{1}} f(z) := \int_{\mathbb{H}^n_{\mathbf{1}}} B^\infty_{\mathbf{1}}(z, w) f(w) dV(w),$$

whenever the integral converges. The notions of L^∞ sub-Bergman kernel and projection were first introduced by D. Chakrabarti, L. D. Edholm and J. D. McNeal in [9] to study the duality properties for the Bergman spaces of some domains in $\mathbb{C}^n$. In [16], the authors studied the L^p Sobolev mapping properties of the L^∞ sub-Bergman projections on power-generalized Hartogs triangles $\mathbb{H}^2_1$ and they showed that the associated L^∞ sub-Bergman projections are more regular than the ordinary Bergman projection on L^p Sobolev spaces. Our third main result is stated as follows, which generalizes the L^p Sobolev boundedness results for the L^∞ sub-Bergman projection on $\mathbb{H}^2_1$ in [16, Corollaries 5.8–5.10] to the n-dimensional setting.

Theorem 1.3. *The L^∞ sub-Bergman projection $\mathbf{B}^\infty_{\mathbf{1}}$ associated to the domain $\mathbb{H}^n_{\mathbf{1}}$ maps $L^p_k(\mathbb{H}^n_1)$ to itself for all $p \in (1, \frac{2n}{k})$ and $k = 1, 2, 3$.*

Remark. In Theorem 1.3 we obtained the L^p boundedness results for the L^∞ sub-Bergman projection $\mathbf{B}^\infty_{\mathbf{1}}$ only on L^p Sobolev spaces of order $k = 1, 2$ and 3. The reason why the cases of $k \geq 4$ is lost is that when $k \geq 4$ we cannot use Lemma 2.7, a crucial lemma of obtaining the boundedness ranges for a specific type of operators, to study the mapping properties for the differentiated operators $\frac{\partial^k}{\partial z_1^k} \circ \mathbf{B}^\infty_{\mathbf{1}}$, which is a key ingredient in the proof. However, by comparing Theorem 1.3 and Theorem 1.2, it is enough for us to see that the

L^∞ sub-Bergman projection $\mathbf{B}_\mathbf{1}^\infty$ has better L^p Sobolev mapping behavior than the ordinary Bergman projection $\mathbf{B}_\mathbf{1}$.

In Section 2, we review some known facts and establish several technical lemmas for our later use. Section 3 is devoted to the proofs of Theorem 1.1 and Theorem 1.2. In Section 4, we give a proof of Theorem 1.3. Throughout the paper, for given functions of several variables $a(z)$ and $b(z)$, we use $a(z) \lesssim b(z)$ to denote that $a(z) \le Cb(z)$ for a constant C, and we write $a(z) \approx b(z)$ if both $a(z) \lesssim b(z)$ and $b(z) \lesssim a(z)$ hold.

2. Preliminaries

We define the vector field

$$\mathcal{T}_w := \bar{w}\frac{\partial}{\partial \bar{w}} - w\frac{\partial}{\partial w}.$$

The operator $\mathcal{T}_w$ is useful in studying the Sobolev regularity and irregularity for Bergman projections. See [8,12,16], for instance. One of the crucial properties of $\mathcal{T}_w$ is that for any disc and annulus with centered at the origin with defining function ρ, we have that $\mathcal{T}_w\rho = 0$ along their boundaries. In order to make use of integration by parts, we need the following lemma.

Lemma 2.1 (See [16]). *Let $\Omega \subset \mathbb{C}$ be either a disc or an annulus centered at the origin. Then if $f,\ g \in L^1_1(\Omega) \cap C(\overline{\Omega})$,*

$$\int_\Omega \mathcal{T}_w f \cdot g dV = -\int_\Omega f \cdot \mathcal{T}_w g dV.$$

As a generalization of [25, Lemma 4.1], we have the following lemma, which tells us when the holomorphic monomials $z^\beta := z_1^{\beta_1}\cdots z_n^{\beta_n}$ are in $L^p(\mathbb{H}^n_\mathbf{p})$ and the L^p norm of them.

Lemma 2.2. *For $n \ge 2$, $\mathbf{p} = (p_1, \dots, p_n) \in (\mathbb{Z}^+)^n$ and $p \in (1, \infty)$, we denote*

$$\mathcal{A}^p(\mathbb{H}^n_\mathbf{p}) := \Bigg\{\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n : \alpha_1 \ge 0,\\ \sum_{i=1}^{j} \alpha_i k_i \ge \left[-\frac{2}{p}\sum_{i=1}^{j} k_i + D_j\right],\ j = 2, \dots, n\Bigg\},$$

where $D_j = \gcd(k_1, \dots, k_j)$ for $j = 2, \dots, n$. Let $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}^n$, then the holomorphic monomial $z^\beta := z_1^{\beta_1}\cdots z_n^{\beta_n} \in L^p(\mathbb{H}^n_\mathbf{p})$ if and only if $\beta \in \mathcal{A}^p(\mathbb{H}^n_\mathbf{p})$. Moreover, for holomorphic monomials $z^\beta \in L^p(\mathbb{H}^n_\mathbf{p})$ with $\beta \in \mathcal{A}^p(\mathbb{H}^n_\mathbf{p})$, we have that

$$\|z^\beta\|_p^p = \frac{(2\pi)^n}{\prod_{j=1}^{n} R_j(\beta; p)},$$

where $\|\cdot\|_p$ denotes the L^p-norm and $R_j(\beta;p)$ is defined by

$$R_j(\beta;p)=\begin{cases} p\beta_1+2, & j=1;\\ \sum\limits_{i=1}^{j-1}\frac{k_i}{k_{i+1}}(p\beta_i+2)+(p\beta_j+2), & 2\le j\le n.\end{cases}$$

Proof. From the definition of $\mathbb{H}^n_{\mathbf{p}}$ we know that z_1 may equal to 0 for $z=(z_1,\dots,z_n)\in\mathbb{H}^n_{\mathbf{p}}$. Therefore we have $\beta_1\ge 0$ if $z^\beta:=z_1^{\beta_1}\cdots z_n^{\beta_n}\in L^p(\mathbb{H}^n_{\mathbf{p}})$. We assume that $z^\beta\in L^p(\mathbb{H}^n_{\mathbf{p}})$ and consider the integral $\int_{\mathbb{H}^n_{\mathbf{p}}}|z^\beta|^p dV(z)$. Using the transformation $\xi_1=z_1^{p_1},\dots,\xi_n=z_n^{p_n}$, we obtain that

$$\int_{\mathbb{H}^n_{\mathbf{p}}}|z^\beta|^p dV(z)\approx\int_{\mathbb{H}^n_{\mathbf{1}}}\prod_{j=1}^n|\xi_j|^{b_j}dV(\xi),$$

where $b_j:=\frac{2(1-p_j)+p\beta_j}{p_j}$ for $j=1,\dots,n$ and $\mathbb{H}^n_{\mathbf{1}}$ is the domain defined by

$$\mathbb{H}^n_{\mathbf{1}}:=\{(z_1,\dots,z_n)\in\mathbb{C}^n:|z_1|<\cdots<|z_n|<1\}.$$

It is easy for one to check that under the biholomorphic mapping

$$\eta=F(\xi):=\left(\frac{\xi_1}{\xi_2},\ \frac{\xi_2}{\xi_3},\dots,\ \frac{\xi_{n-1}}{\xi_n}\right),$$

the domain $\mathbb{H}^n_{\mathbf{1}}$ is biholomorphic to the product domain $\mathbb{D}\times(\mathbb{D}^*)^{n-1}$, where $\mathbb{D}$ and $\mathbb{D}^*$ denote the unit disc and the punctured unit disc in $\mathbb{C}$, respectively. Therefore, we have that

$$\int_{\mathbb{H}^n_{\mathbf{p}}}|z^\beta|^p dV(z)\approx\int_{\mathbb{D}}|\eta_1|^{B_1}dV(\eta_1)\cdot\prod_{j=2}^n\int_{\mathbb{D}^*}|\eta_j|^{B_j+2(j-1)}dV(\eta_j), \tag{5}$$

where $B_j:=\sum_{i=1}^j b_i$, $j=1,\dots,n$. Then it follows that $\int_{\mathbb{H}^n_{\mathbf{p}}}|z^\beta|^p dV(z)<\infty$ if and only if $B_j+2(j-1)>-2$ for $j=2,\dots,n$, which are equivalent to

$$\sum_{i=1}^j(\beta_i p+2)k_i>0,\quad j=2,\dots,n.$$

Note that $\frac{\beta_i k_i}{D_j}\in\mathbb{Z}$ for $j=2,\dots,n$ and $i=1,\dots,j$, then it is easy for one to check that

$$\begin{aligned}\sum_{i=1}^j(\beta_i p+2)k_i>0&\Longleftrightarrow\sum_{i=1}^j\frac{\beta_i k_i}{D_j}>-\frac{2}{p}\sum_{i=1}^j\frac{k_i}{D_j}\\ &\Longleftrightarrow\sum_{i=1}^j\frac{\beta_i k_i}{D_j}\ge\left[-\frac{2}{p}\sum_{i=1}^j\frac{k_i}{D_j}+1\right]\\ &\Longleftrightarrow\sum_{i=1}^j\beta_i k_i\ge\left[-\frac{2}{p}\sum_{i=1}^j k_i+D_j\right]\end{aligned}$$

for each $j = 2, \ldots, n$. This proves the first part of the lemma. The second part of the lemma follows directly from (5) and some simple calculations. The proof is complete. $\square$

Denote $\mathbf{1} := (1, \ldots, 1)$ and $c \cdot \mathbf{1} := (c, \ldots, c)$, where $c > 0$ is a real constant. From the definition of $\mathbb{H}^n_{\mathbf{p}}$, it is easily seen that the domain $\mathbb{H}^n_{c\cdot\mathbf{1}}$ coincides with $\mathbb{H}^n_{\mathbf{1}}$ for any $c > 0$. Therefore, in the following context, we identify the multi-indices $\mathbf{1}$ with $c \cdot \mathbf{1}$ for simplicity and convenience.

As a direct application of Lemma 2.2, the irregularity results for the Bergman projection $\mathbf{B}_{\mathbf{p}}$ on L^p Sobolev spaces are investigated in the following two lemmas:

Lemma 2.3. *Let $p \in (1, \infty)$. If $\mathbf{p} \neq \mathbf{1} = (1, \ldots, 1)$, then $\mathbf{B}_{\mathbf{p}}$, the Bergman projection of $\mathbb{H}^n_{\mathbf{p}}$, fails to map $L^p_k(\mathbb{H}^n_{\mathbf{p}}) \to L^p_k(\mathbb{H}^n_{\mathbf{p}})$ for any $k \in \mathbb{Z}^+$.*

Proof. Firstly we note that by repeating the same argument as in [15, Theorem 1.2], it follows easily that the Bergman projection is bounded on $L^p(\mathbb{H}^n_{\mathbf{p}})$ if and only if $p = 2$ when the reals numbers $\frac{p_1}{p_2}, \frac{p_2}{p_3}, \ldots, \frac{p_{n-1}}{p_n}$ are not all rational numbers. From this and the same method used in [16, Corollary 3.5], we can obtain that if $\frac{p_1}{p_2}, \frac{p_2}{p_3}, \ldots, \frac{p_{n-1}}{p_n}$ are not all rational numbers then $\mathbf{B}_{\mathbf{p}}$ cannot map $L^p_k(\mathbb{H}^n_{\mathbf{p}})$ to itself for any $p \in (1, \infty)$ and $k \in \mathbb{Z}^+$. Therefore, we may assume that $\mathbf{p} = (p_1, \ldots, p_n) \in (\mathbb{Z}^+)^n$.

Now we divide the rest of the proof into two steps.

Step 1. We first prove that if $p \notin I_{\mathbf{p}} := \left(\frac{2\sum_{i=1}^n k_i}{\sum_{i=1}^n k_i + D_n}, \frac{2\sum_{i=1}^n k_i}{\sum_{i=1}^n k_i - D_n}\right) := (a_{\mathbf{p}}, b_{\mathbf{p}})$, then the Bergman projection $\mathbf{B}_{\mathbf{p}}$ fails to map $L^p_k(\mathbb{H}^n_{\mathbf{p}})$ to itself. Indeed, from the proof of [25, Proposition 4.2] we know that there exists a function $f(w) := w_1^{\beta_1} \overline{w}_2^{-\beta_2} \cdots \overline{w}_n^{-\beta_n}$ with $\beta_1 \geq 0$, $\beta_2, \ldots, \beta_n < 0$ and $\sum_{i=1}^n (\beta_i + 1) k_i = D_n$ which satisfies

$$\mathbf{B}_{\mathbf{p}} f(w) = g(w) := w^\beta = w_1^{\beta_1} \cdots w_n^{\beta_n}.$$

It is obvious that $f \in \mathcal{C}^\infty(\overline{\mathbb{H}^n_{\mathbf{p}}})$, therefore we have that $f \in L^p_k(\mathbb{H}^n_{\mathbf{p}})$ for any $1 < p < \infty$ and $k \in \mathbb{N}$. If $p \notin (a_{\mathbf{p}}, b_{\mathbf{p}})$, then by Lemma 2.2 we know that $g \notin L^p(\mathbb{H}^n_{\mathbf{p}})$, which follows that $g \notin L^p_k(\mathbb{H}^n_{\mathbf{p}})$ for any $k \in \mathbb{N}$. This means that $\mathbf{B}_{\mathbf{p}}$ fails to map $L^p_k(\mathbb{H}^n_{\mathbf{p}})$ to itself.

Step 2. We prove that if $\mathbf{p} \neq \mathbf{1} = (1, \ldots, 1)$, then $\mathbf{B}_{\mathbf{p}}$ fails to map $L^p_k(\mathbb{H}^n_{\mathbf{p}})$ to itself for any $p \in (1, \infty)$ and $k \in \mathbb{Z}^+$. It is obvious that $\frac{\partial g}{\partial w_j} = \beta_j w_1^{\beta_1} \cdots w_j^{\beta_j - 1} \cdots w_n^{\beta_n}$ for $j = 1, \ldots, n$. We claim that $\frac{\partial g}{\partial w_j} (j = 1, \ldots, n)$ can not be all in $L^{a_{\mathbf{p}}}(\mathbb{H}^n_{\mathbf{p}})$. If not, that is, $\frac{\partial g}{\partial w_j} \in L^{a_{\mathbf{p}}}(\mathbb{H}^n_{\mathbf{p}})$ holds for all $j = 1, \ldots, n$, then by Lemma 2.2 we have that

$$\sum_{i=1}^n \beta_i k_i - k_j \geq \left[-\frac{2}{a_{\mathbf{p}}} \sum_{i=1}^n k_i + D_n \right], \quad j = 1, \ldots, n. \tag{6}$$

Note that $\beta_1,\dots,\beta_n$ satisfy $\sum_{i=1}^n(\beta_i+1)k_i = D_n$. Then by a direct calculation we know that (6) is equivalent to $k_j \le D_n$ for all $j=1,\dots,n$. Since $D_n = \gcd(k_1,\dots,k_n)$, we can conclude $k_1 = \cdots = k_n = D_n$, which means that $\mathbf{p} = \mathbf{1} = (1,\dots,1)$. This contradicts to the assumption that $\mathbf{p} \neq \mathbf{1}$. Thus we can suppose that $\frac{\partial g}{\partial w_{j_0}} \notin L^{a_{\mathbf{p}}}(\mathbb{H}^n_{\mathbf{p}})$ for some $j_0 \in \mathbb{N}$ and $1 \le j_0 \le n$. From this we know that $\frac{\partial g}{\partial w_{j_0}} \notin L^p(\mathbb{H}^n_{\mathbf{p}})$ for any $p \in I_{\mathbf{p}}$, which follows directly that $g \notin L^p_k(\mathbb{H}^n_{\mathbf{p}})$ for any $p \in I_{\mathbf{p}}$ and $k \in \mathbb{Z}^+$. Since $g = \mathbf{B}_{\mathbf{p}}(f)$ and $f \in \mathcal{C}^\infty(\overline{\mathbb{H}^n_{\mathbf{p}}})$, by Step 1 we know that $\mathbf{B}_{\mathbf{p}}$ fails to map $L^p_k(\mathbb{H}^n_{\mathbf{p}})$ to itself for any $p \in (1,\infty)$ and $k \in \mathbb{Z}^+$ if $\mathbf{p} \neq \mathbf{1} = (1,\dots,1)$. The proof is complete. □

Lemma 2.4. *For $\mathbb{H}^n_{\mathbf{1}}$, we have that $\mathbf{B}_{\mathbf{1}}$ fails to map $L^p_k(\mathbb{H}^n_{\mathbf{1}})$ to itself for any $k \ge 2$ and $k \in \mathbb{N}$.*

Proof. Consider two order derivatives of g

$$\frac{\partial^2 g}{\partial w_s \partial w_t} = \beta_s\beta_t w_1^{\beta_1}\cdots w_s^{\beta_s-1}\cdots w_t^{\beta_t-1}\cdots w_n^{\beta_n},$$

where $1 \le s,t \le n$, $\beta_1,\dots,\beta_n \in \mathbb{Z}$ satisfy that $\beta_1 \ge 0$, $\beta_2,\dots,\beta_n < 0$ and $\sum_{i=1}^n(\beta_i+1)k_i = D_n$. We claim that for any positive numbers $1 \le s,t \le n$, $\frac{\partial^2 g}{\partial w_s\partial w_t} \notin L^{a_{\mathbf{1}}}(\mathbb{H}^n_{\mathbf{1}})$. If not, we assume that $\frac{\partial^2 g}{\partial w_s\partial w_t} \in L^{a_{\mathbf{1}}}(\mathbb{H}^n_{\mathbf{1}})$ for some positive numbers $1 \le s,t \le n$. By Lemma 2.2 we know that $\sum_{i=1}^n \beta_i k_i - k_s - k_t \ge -\sum_{i=1}^n k_i$. By combining this with $\sum_{i=1}^n(\beta_i+1)k_i = D_n$, we obtain that $k_s + k_t \le D_n$, which is contradicted to $D_n = \gcd(k_1,\dots,k_n)$. Thus we obtain that $\frac{\partial^2 g}{\partial w_s\partial w_t} \notin L^{a_{\mathbf{1}}}(\mathbb{H}^n_{\mathbf{1}})$ for any $s,t \in \mathbb{N}$ and $1 \le s,t \le n$, which leads that $g \notin L^p_k(\mathbb{H}^n_{\mathbf{1}})$ for any $k \ge 2$ and $p \in I_{\mathbf{p}} = (a_{\mathbf{p}}, b_{\mathbf{p}})$. Therefore, from the proof of Step 1 of Lemma 2.3, we know that $\mathbf{B}_{\mathbf{1}}$ fails to map $L^p_k(\mathbb{H}^n_{\mathbf{p}})$ to itself for any $k \ge 2$ and $k \in \mathbb{N}$. The proof is complete. □

It is well known that the Bergman kernel functions play a critical role in studying the mapping properties for the Bergman projections. For the n-dimensional generalized Hartogs triangles $\mathbb{H}^n_{\mathbf{p}}$ with $\mathbf{p} \in (\mathbb{Z}^+)^n$, J.-D. Park in [22] showed that the Bergman kernel of $\mathbb{H}^n_{\mathbf{p}}$ can be represented in a rational form. Later in [25], S. Zhang explicitly calculated the Bergman kernel of $\mathbb{H}^n_{\mathbf{p}}$ as follows, which is an important tool in our proofs of the main results.

Lemma 2.5 (See [25, Theorem 3.1])**.** *Let $n \ge 2$ and denote $m_{j,j+1} := \operatorname{lcm}(k_j, k_{j+1})$ for $j = 1,\dots,n-1$ and $m_{n,n+1} := k_n$. We also denote $k_j^{(j)} := \frac{k_j}{d_{j,j+1}}$ and $k_{j+1}^{(j)} := \frac{k_{j+1}}{d_{j,j+1}}$ for $j = 1,\dots,n-1$, where $d_{j,j+1} := \gcd(k_j,k_{j+1})$ for $j = 1,\dots,n-1$ and $d_{n,n+1} := k_n$. Then*

$$B_{\boldsymbol{p}}(z,w) = \frac{\sum\limits_{\alpha_1=0}^{N_1}\cdots\sum\limits_{\alpha_n=0}^{N_n}\nu(P_1)\cdots\nu(P_n) r_1^{\alpha_1}\cdots r_n^{\alpha_n}}{\pi^n K\cdot(1-r_n)^2\prod\limits_{j=1}^{n-1}\left(r_j^{k_{j+1}^{(j)}} - r_{j+1}^{k_j^{(j)}}\right)^2}, \tag{7}$$

where $r_j := z_j\overline{w}_j$, $j = 1, \dots, n$, $K := k_1 \cdots k_n$,

$$N_1 := \left[\frac{2m_{1,2} - 1 - k_1}{k_1}\right], \quad N_l := \left[\frac{2m_{l-1,l} + 2m_{l,l+1} - k_l - 2}{k_l}\right], \quad l = 2, \dots, n$$

and $P_1, \dots, P_n$ *are defined by*

$$P_1 := 2m_{1,2} - k_1 + 1 - k_1\alpha_1, \quad P_l := 2m_{l,l+1} - k_l - k_l\alpha_l + P_{l-1}, \quad l = 2, \dots, n.$$

The functions $\nu(P_l)(l = 1, \dots, n)$ *in* (7) *are defined by*

$$\nu(P_l) := \begin{cases} P_l - 1, & 2 \leqslant P_l \leqslant m_{l,l+1} + 1, \\ 2m_{l,l+1} - P_l + 1, & m_{l,l+1} + 1 < P_l \leqslant 2m_{l,l+1}, \\ 0, & P_l < 2 \text{ or } P_l > 2m_{l,l+1}. \end{cases}$$

Now we need to define a class of integral operators on $\mathbb{H}^n_{\mathbf{p}}$ with $\mathbf{p} \in (\mathbb{Z}^+)^n$, which contains the Bergman projections on $\mathbb{H}^n_{\mathbf{p}}$ and the differentiated Bergman projections.

Definition. Let T be an integral operator over $\mathbb{H}^n_{\mathbf{p}}$ with kernel $k(z,w)$, that is,

$$Tf(z) = \int_{\mathbb{H}^n_{\mathbf{p}}} k(z,w)f(w)dV(w), \quad z \in \mathbb{H}^n_{\mathbf{p}}.$$

If there exist real numbers $a_2, \dots, a_n$ and $b_2, \dots, b_n$ such that

$$|k(z,w)| \lesssim \frac{\prod_{j=2}^{n} |z_j|^{a_j}|w_j|^{b_j}}{\prod_{j=1}^{n-1} \left|(z_{j+1}\overline{w}_{j+1})^{k_j^{(j)}} - (z_j\overline{w}_j)^{k_{j+1}^{(j)}}\right|^2 |1 - z_n\overline{w}_n|^2}$$

holds for any $(z,w) \in \mathbb{H}^n_{\mathbf{p}} \times \mathbb{H}^n_{\mathbf{p}}$, then we say that T is an integral operator over $\mathbb{H}^n_{\mathbf{p}}$ of general type-$(a_2, \dots, a_n; b_2, \dots, b_n)$.

A comparison between the above definition and [25, Definition 2.1] shows that the class of integral operators defined as above is more general than that defined in [25, Definition 2.1], which required $a_j = b_j$ for $j = 2, \dots, n$.

The following lemma gives an optimal estimate for the Bergman kernels $B_{\mathbf{p}}$ for $\mathbb{H}^n_{\mathbf{p}}$ and determines the general type of the associated Bergman projections.

Lemma 2.6 (See [25, Proposition 3.2]). *Let* $D_j := \gcd(k_1, \dots, k_j)$ *and* $A_j := \frac{2m_{j-1,j}}{k_j} - 1 - \frac{D_{j-1}}{k_j} + \frac{D_j}{k_j}$ *for* $j = 2, \dots, n$. *Then we have the following estimate for the Bergman kernel* $B_{\mathbf{p}}$:

$$|B_{\mathbf{p}}(z,w)| \lesssim \frac{\prod_{j=2}^{n} |r_j|^{A_j}}{|1 - r_n|^2 \prod_{j=1}^{n-1} |r_j^{k_{j+1}^{(j)}} - r_{j+1}^{k_j^{(j)}}|^2}, \qquad (z,w) \in \mathbb{H}^n_{\mathbf{p}} \times \mathbb{H}^n_{\mathbf{p}}.$$

That is, the Bergman projection of $\mathbb{H}^n_{\mathbf{p}}$ *is an integral operator over* $\mathbb{H}^n_{\mathbf{p}}$ *of general type-*$(A_2, \dots, A_n; A_2, \dots, A_n)$.

The L^p boundedness ranges for the class of integral operators defined in Definition 2 are determined in the following lemma, which extends [25, Proposition 2.4] to a wider class of integral operators.

Lemma 2.7. *Let T be an integral operator over $\mathbb{H}^n_{\mathbf{p}}$ of general type-$(a_2, \ldots, a_n; b_2, \ldots, b_n)$ with real numbers a_j and b_j $(j = 2, \ldots, n)$ satisfying the following conditions*:

$$\text{(i)}\ \ a_j \le 2k_{j-1}^{(j-1)},\ \ j = 2, \ldots, n;$$

$$\text{(ii)}\ \ b_j > 2k_{j-1}^{(j-1)} - 1 - \frac{D_{j-1}}{k_j},\ \ j = 2, \ldots, n;$$

$$\text{(iii)}\ \ a_j + b_j > 2\left(2k_{j-1}^{(j-1)} - 1 - \frac{D_{j-1}}{k_j}\right),\ \ j = 2, \ldots, n,$$

where $D_j = \gcd(k_1, \ldots, k_j)$ for $j = 1, \ldots, n$. Let $p \in (1, \infty)$ and denote $m_{j-1,j} := \operatorname{lcm}(k_{j-1}, k_j)$ for $j = 2, \ldots, n$. Then T is bounded on $L^p(\mathbb{H}^n_{\mathbf{p}})$ if p satisfies

$$\frac{2\sum\limits_{j=1}^{n} k_j}{2\sum\limits_{j=1}^{n} k_j - 2\sum\limits_{j=2}^{n} m_{j-1,j} + \sum\limits_{j=2}^{n} k_j b_j} < p < \frac{2\sum\limits_{j=1}^{n} k_j}{2\sum\limits_{j=2}^{n} m_{j-1,j} - \sum\limits_{j=2}^{n} k_j a_j}$$

whenever

$$2\sum_{j=1}^{n} k_j - 2\sum_{j=2}^{n} m_{j-1,j} + \sum_{j=2}^{n} k_j b_j > 2\sum_{j=2}^{n} m_{j-1,j} - \sum_{j=2}^{n} k_j a_j.$$

and

$$\sum_{j=2}^{n} k_j a_j > 2\sum_{j=2}^{n} m_{j-1,j} - 2\sum_{j=1}^{n} k_j.$$

Proof. The proof of the lemma is nearly the same as that in [25, Proposition 2.4]. We omit it here. $\square$

3. Proof of Theorems 1.1 and 1.2

Proof of Theorem 1.1. We divide the proof into two cases.

Case 1. $2 \le r \le n$. Let $f \in L^p_1(\mathbb{H}^n_{\mathbf{p}})$, where $1 < p < \infty$ and $\mathbf{p} = (p_1, \ldots, p_n) \in (\mathbb{Z}^+)^n$. For any $2 \le r \le n$ and $r \in \mathbb{N}$, we define the sets $E_{r,j} (j = 1, \ldots, n)$ as follows:

$$E_{r,j} := \begin{cases} \{w_1 \in \mathbb{C} : |w_1| < |w_{r+1}|^{\frac{p_{r+1}}{p_1}}\}, & j = 1; \\ \{w_j \in \mathbb{C} : |w_{j-1}|^{\frac{p_{j-1}}{p_j}} < |w_j| < |w_{j+1}|^{\frac{p_{j+1}}{p_j}}\}, & j = 2, \ldots, r; \\ \{w_j \in \mathbb{C} : 0 < |w_j| < |w_{j+1}|^{\frac{p_{j+1}}{p_j}}\}, & j = r+1, \ldots, n, \end{cases}$$

where $w_{n+1} := 1$. It is easy for one to see that the holomorphic derivatives of the Bergman kernel $B_{\mathbf{p}}(z,w)$ can be transferred to anti-holomorphic derivatives as follows:

$$\frac{\partial}{\partial z_r} B_{\mathbf{p}}(z,w) = \frac{\overline{w}_r}{z_r} \cdot \frac{\partial}{\partial \overline{w}_r} B_{\mathbf{p}}(z,w).$$

Then for any $2 \leq r \leq n$ and $r \in \mathbb{N}$, we have that

$$\begin{aligned}
&\frac{\partial}{\partial z_r} \circ \mathbf{B}_{\mathbf{p}} f(z) \\
&= \frac{\partial}{\partial z_r} \int_{\mathbb{H}^n_{\mathbf{p}}} B_{\mathbf{p}}(z,w) f(w) dV(w) \\
&= \int_{\mathbb{H}^n_{\mathbf{p}}} \frac{\partial}{\partial z_r} B_{\mathbf{p}}(z,w) f(w) dV(w) \\
&= \frac{1}{z_r} \int_{\mathbb{H}^n_{\mathbf{p}}} \overline{w}_r \frac{\partial}{\partial \overline{w}_r} (B_{\mathbf{p}}(z,w)) f(w) dV(w) \\
&= \frac{1}{z_r} \int_{\mathbb{H}^n_{\mathbf{p}}} \mathcal{T}_{w_r}(B_{\mathbf{p}}(z,w)) f(w) dV(w) \\
&= \frac{1}{z_r} \int_{E_{r,n}} \int_{E_{r,n-1}} \cdots \int_{E_{r,r+1}} \int_{E_{r,1}} \int_{E_{r,2}} \cdots \int_{E_{r,r}} \mathcal{T}_{w_r}(B_{\mathbf{p}}(z,w)) \\
&\quad \times f(w) dV(w_r) \cdots dV(w_2) dV(w_1) dV(w_{r+1}) \cdots dV(w_{n-1}) dV(w_n) \\
&= -\frac{1}{z_r} \int_{E_{r,n}} \int_{E_{r,n-1}} \cdots \int_{E_{r,r+1}} \int_{E_{r,1}} \int_{E_{r,2}} \cdots \int_{E_{r,r}} B_{\mathbf{p}}(z,w) \\
&\quad \times \mathcal{T}_{w_r} f(w) dV(w_r) \cdots dV(w_2) dV(w_1) dV(w_{r+1}) \cdots dV(w_{n-1}) dV(w_n) \\
&= -\frac{1}{z_r} \int_{\mathbb{H}^n_{\mathbf{p}}} B_{\mathbf{p}}(z,w) \mathcal{T}_{w_r} f(w) dV(w) \\
&= \int_{\mathbb{H}^n_{\mathbf{p}}} \frac{w_r}{z_r} B_{\mathbf{p}}(z,w) \frac{\partial f}{\partial w_r}(w) dV(w) - \int_{\mathbb{H}^n_{\mathbf{p}}} \frac{\overline{w}_r}{z_r} B_{\mathbf{p}}(z,w) \frac{\partial f}{\partial \overline{w}_r}(w) dV(w), \qquad (8)
\end{aligned}$$

where Lemma 2.1 can be applied since $E_{r,r}$ is an annulus centered at the origin. The derivatives $\frac{\partial f}{\partial w_r}$ and $\frac{\partial f}{\partial \overline{w}_r}$ should be interpreted distributionally. By Lemma 2.6, we know that $\frac{w_r}{z_r} B_{\mathbf{p}}(z,w)$ satisfies the following estimate:

$$\left| \frac{w_r}{z_r} B_{\mathbf{p}}(z,w) \right| \lesssim \frac{|z_2|^{A_2} \cdots |z_r|^{A_r - 1} \cdots |z_n|^{A_n} |w_2|^{A_2} \cdots |w_r|^{A_r+1} \cdots |w_n|^{A_n}}{|1-r_n|^2 \prod_{j=1}^{n-1} \left| r_j^{k_{j+1}^{(j)}} - r_{j+1}^{k_j^{(j)}} \right|^2},$$

where $A_j := \frac{2m_{j-1,j}}{k_j} - 1 - \frac{D_{j-1}}{k_j} + \frac{D_j}{k_j}$ for $j = 2, \ldots, n$. From this and Definition 2, we can conclude that the integral operators with kernel $\frac{w_r}{z_r} B_{\mathbf{p}}(z,w)$ or $\frac{\overline{w}_r}{z_r} B_{\mathbf{p}}(z,w)$ are of general type-$(A_2, \ldots, A_r - 1, \ldots, A_n; A_2, \ldots, A_r + 1, \ldots, A_n)$, from which and Lemma 2.7 we can know that they are bounded on $L^p(\mathbb{H}^n_{\mathbf{p}})$ if $p \in$

$\left(\frac{2\sum_{i=1}^{n} k_j}{\sum_{i=1}^{n} k_j+D_n+k_r}, \frac{2\sum_{i=1}^{n} k_j}{\sum_{i=1}^{n} k_j-D_n+k_r}\right)$. Since $f \in L_1^p(\mathbb{H}_{\mathbf{p}}^n)$, it follows that $\frac{\partial f}{\partial w_r}, \frac{\partial f}{\partial \overline{w}_r} \in L^p(\mathbb{H}_{\mathbf{p}}^n)$. From the above analysis and (8) we obtain that

$$\begin{aligned}\left\|\frac{\partial}{\partial z_r} \circ \mathbf{B}_{\mathbf{p}}(f)\right\|_p &\leq \left\|\int_{\mathbb{H}_{\mathbf{p}}^n} \frac{w_r}{z_r} B_{\mathbf{p}}(z,w) \frac{\partial f}{\partial w_r}(w) dV(w)\right\|_p \\ &\quad + \left\|\int_{\mathbb{H}_{\mathbf{p}}^n} \frac{\overline{w}_r}{\overline{z}_r} B_{\mathbf{p}}(z,w) \frac{\partial f}{\partial \overline{w}_r}(w) dV(w)\right\|_p \\ &\lesssim \left\|\frac{\partial f}{\partial w_r}\right\|_p + \left\|\frac{\partial f}{\partial \overline{w}_r}\right\|_p \leq \|f\|_{1,p}\end{aligned}$$

holds for all $p \in \left(\frac{2\sum_{i=1}^{n} k_j}{\sum_{i=1}^{n} k_j+D_n+k_r}, \frac{2\sum_{i=1}^{n} k_j}{\sum_{i=1}^{n} k_j-D_n+k_r}\right)$, which means that $\frac{\partial}{\partial z_r} \circ \mathbf{B}_{\mathbf{p}}$ maps $L_1^p(\mathbb{H}_{\mathbf{p}}^n) \longrightarrow L^p(\mathbb{H}_{\mathbf{p}}^n)$ boundedly for all $p \in \left(\frac{2\sum_{i=1}^{n} k_j}{\sum_{i=1}^{n} k_j+D_n+k_r}, \frac{2\sum_{i=1}^{n} k_j}{\sum_{i=1}^{n} k_j-D_n+k_r}\right)$ and $r = 2, \ldots, n$.

Case 2. $r = 1$. In this case, firstly we need to do some tricky things with the kernel of $\frac{\partial}{\partial z_1} \circ \mathbf{B}_{\mathbf{p}}$. Then the key step of the proof is to obtain an optimal estimate for the newly defined kernel.

As in Case 1, for any $f \in L_1^p(\mathbb{H}_{\mathbf{p}}^n)$ with $p \in (1, \infty)$, we have that

$$\frac{\partial}{\partial z_1} \circ \mathbf{B}_{\mathbf{p}} f(z) = \frac{1}{z_1} \int_{E_{1,n}} \cdots \int_{E_{1,1}} \mathcal{T}_{w_1}(B_{\mathbf{p}}(z,w)) f(w) dV(w_1) \cdots dV(w_n). \tag{9}$$

Denote the "modified" kernel $\widetilde{B_{\mathbf{p}}}(z,w)$ by

$$\widetilde{B_{\mathbf{p}}}(z,w) := B_{\mathbf{p}}(z,w) - \sum_{\alpha \in \mathcal{A}^2(\mathbb{H}_{\mathbf{p}}^n), \alpha_1=0} \frac{z^\alpha \overline{w}^\alpha}{C_\alpha^2}, \qquad (z,w) \in \mathbb{H}_{\mathbf{p}}^n \times \mathbb{H}_{\mathbf{p}}^n, \tag{10}$$

where $\mathcal{A}^2(\mathbb{H}_{\mathbf{p}}^n)$ is defined in Lemma 6 and $C_\alpha := \|z^\alpha\|_2$.

Now we need the following claim, which concerns the estimate for the new kernel $\widetilde{B_{\mathbf{p}}}(z,w)$.

Claim. The integral operator on $\mathbb{H}_{\mathbf{p}}^n$ with kernel $\frac{w_1}{z_1} \widetilde{B_{\mathbf{p}}}(z,w)$ is of general type-$(A_2 - \frac{k_1}{k_2}, A_3, \ldots, A_n; A_2 + \frac{k_1}{k_2}, A_3, \ldots, A_n)$.

Proof of Claim. Indeed, from Lemma 2.5, we know that

$$\sum_{\alpha \in \mathcal{A}^2(\mathbb{H}_{\mathbf{p}}^n), \alpha_1=0} \frac{z^\alpha \overline{w}^\alpha}{C_\alpha^2} = \frac{\sum_{\alpha \in \mathcal{G}, \alpha_1=0} \nu(P_1) \cdots \nu(P_n) r_1^{\alpha_1} \cdots r_n^{\alpha_n}}{\pi^n K \cdot r_2^{2k_1^{(1)}} (1-r_n)^2 \prod_{j=2}^{n-1} \left(r_j^{k_{j+1}^{(j)}} - r_{j+1}^{k_j^{(j)}}\right)^2}$$

$$:= \frac{\sum\limits_{\alpha\in\mathcal{G},\alpha_1=0} C(\alpha) r^\alpha}{\pi^n K \cdot r_2^{2k_1^{(1)}}(1-r_n)^2 \prod\limits_{j=2}^{n-1}\left(r_j^{k_{j+1}^{(j)}} - r_{j+1}^{k_j^{(j)}}\right)^2},$$

where $C(\alpha) := \nu(P_1)\cdots\nu(P_n)$, $r^\alpha := r_1^{\alpha_1}\cdots r_n^{\alpha_n}$ and $\mathcal{G} := \{\alpha = (\alpha_1,\dots,\alpha_n) \in (\mathbb{Z}^+)^n : 0 \le \alpha_i \le N_i, i = 1,\dots,n\}$. Then it follows that

$$\begin{aligned}
\widetilde{B_{\mathbf{p}}}(z,w) &= B_{\mathbf{p}}(z,w) - \sum_{\alpha\in\mathcal{A}^2(\mathbb{H}^n_{\mathbf{p}}),\alpha_1=0} \frac{z^\alpha \overline{w}^\alpha}{C_\alpha^2} \\
&= \frac{\sum\limits_{\alpha\in\mathcal{G}} C(\alpha) r^\alpha}{\pi^n K \cdot (1-r_n)^2 \prod\limits_{j=1}^{n-1}\left(r_j^{k_{j+1}^{(j)}} - r_{j+1}^{k_j^{(j)}}\right)^2} \\
&\quad - \frac{\sum\limits_{\alpha\in\mathcal{G},\alpha_1=0} C(\alpha) r^\alpha}{\pi^n K \cdot r_2^{2k_1^{(1)}}(1-r_n)^2 \prod\limits_{j=2}^{n-1}\left(r_j^{k_{j+1}^{(j)}} - r_{j+1}^{k_j^{(j)}}\right)^2} \\
&= \frac{r_2^{2k_1^{(1)}} \sum\limits_{\alpha\in\mathcal{G}} C(\alpha) r^\alpha - (r_1^{k_2^{(1)}} - r_2^{k_1^{(1)}})^2 \sum\limits_{\alpha\in\mathcal{G},\alpha_1=0} C(\alpha) r^\alpha}{\pi^n K \cdot r_2^{2k_1^{(1)}}(1-r_n)^2 \prod\limits_{j=1}^{n-1}\left(r_j^{k_{j+1}^{(j)}} - r_{j+1}^{k_j^{(j)}}\right)^2} \\
&= \frac{\sum\limits_{\alpha\in\mathcal{G},\alpha_1\neq 0} C(\alpha) r^\alpha}{\pi^n K \cdot (1-r_n)^2 \prod\limits_{j=1}^{n-1}\left(r_j^{k_{j+1}^{(j)}} - r_{j+1}^{k_j^{(j)}}\right)^2} \\
&\quad - \frac{r_1^{2k_2^{(1)}} \sum\limits_{\alpha\in\mathcal{G},\alpha_1=0} C(\alpha) r^\alpha}{\pi^n K \cdot r_2^{2k_1^{(1)}}(1-r_n)^2 \prod\limits_{j=1}^{n-1}\left(r_j^{k_{j+1}^{(j)}} - r_{j+1}^{k_j^{(j)}}\right)^2} \\
&\quad + \frac{2 r_1^{k_2^{(1)}} \sum\limits_{\alpha\in\mathcal{G},\alpha_1=0} C(\alpha) r^\alpha}{\pi^n K \cdot r_2^{k_1^{(1)}}(1-r_n)^2 \prod\limits_{j=1}^{n-1}\left(r_j^{k_{j+1}^{(j)}} - r_{j+1}^{k_j^{(j)}}\right)^2}.
\end{aligned}$$

From this and the triangle inequality, we know that the kernel $\frac{w_1}{z_1}\widetilde{B_{\mathbf{p}}}(z,w)$ satisfies

$$\left|\frac{w_1}{z_1}\widetilde{B_{\mathbf{p}}}(z,w)\right| \lesssim \left|\frac{w_1}{z_1}\right| \cdot \frac{\sum\limits_{\alpha\in\mathcal{G},\alpha_1\neq 0} C(\alpha)|r^\alpha|}{\pi^n K \cdot |1-r_n|^2 \prod\limits_{j=1}^{n-1}\left|r_j^{k_{j+1}^{(j)}} - r_{j+1}^{k_j^{(j)}}\right|^2}$$

$$+\left|\frac{w_1}{z_1}\right|\cdot\frac{|r_1|^{k_2^{(1)}}\sum\limits_{\alpha\in\mathcal{G},\alpha_1=0}C(\alpha)|r^\alpha|}{\pi^n K\cdot|r_2|^{k_1^{(1)}}|1-r_n|^2\prod\limits_{j=1}^{n-1}\left|r_j^{k_{j+1}^{(j)}}-r_{j+1}^{k_j^{(j)}}\right|^2}$$

$$:= M(z,w)+N(z,w). \tag{11}$$

We first estimate $M(z,w)$. We use the notations in Lemma 2.5. Take $\alpha_1,\ldots,\alpha_n$ which satisfy $\alpha=(\alpha_1,\ldots,\alpha_n)\in\mathcal{G}$ and $C(\alpha)=\nu(P_1)\cdots\nu(P_n)\neq 0$. Then by Lemma 2.5 we know that $P_l\leq 2m_{l,l+1}$ for $l=1,\ldots,n$. Again by Lemma 2.5 we obtain that

$$\sum_{i=1}^{l}(2m_{j,j+1}-k_j-k_j\alpha_j)+1\leq 2m_{l,l+1},\quad l=1,\ldots,n,$$

which is equivalent to

$$\sum_{i=1}^{l}(2m_{j,j+1}-k_j-k_j\alpha_j)+D_l\leq 2m_{l,l+1},\quad l=1,\ldots,n. \tag{12}$$

Similar to the proof of [25, Proposition 3.2], we define the functions

$$\varphi_j(x)=(x-A_j^{'})\frac{k_j}{k_{j+1}}+\alpha_{j+1},\qquad j=1,\ldots,n-1,$$

where $A_1^{'}:=0$, $A_2^{'}:=A_2-\frac{k_1}{k_2}$ and $A_j^{'}=A_j$ for $j=3,\ldots,n$. Let $\Phi_j:=\varphi_j\circ\cdots\circ\varphi_1$ for $j=1,\ldots,n-1$. Take $l=2$ in (12) and we get that

$$k_1\alpha_1+k_2\alpha_2\geq 2m_{1,2}+D_2-k_2-k_1,$$

which follows that

$$\varphi_1(\alpha_1-1)=\frac{k_1}{k_2}(\alpha_1-1)+\alpha_2\geq\frac{2m_{1,2}}{k_2}-\frac{2k_1}{k_2}+\frac{D_2}{k_2}-1=A_2^{'}. \tag{13}$$

Take $l=3$ in (12) and we have that

$$\frac{k_1}{k_2}(\alpha_1-1)+\alpha_2\geq\frac{1}{k_2}(2m_{1,2}+2m_{2,3}-k_2-k_3-k_3\alpha_3-2k_1+D_3),$$

from which it is easy for one to check that

$$\begin{aligned}\Phi_2(\alpha_1-1)&=\frac{k_2}{k_3}\left[\frac{k_1}{k_2}(\alpha_1-1)+\alpha_2-A_2^{'}\right]+\alpha_3\\&\geq\frac{2m_{2,3}}{k_3}-\frac{D_2}{k_3}+\frac{D_3}{k_3}-1=A_3^{'}.\end{aligned}$$

From this and the same argument in the proof of [25, Proposition 3.2], we know that

$$\Phi_{l-1}(\alpha_1-1)=\varphi_{l-1}\circ\cdots\circ\varphi_1(\alpha_1-1)\geq A_l^{'},\qquad l=3,\ldots,n.$$

By noting that $C(\alpha)=\prod_{j=1}^n \nu(P_j)$ is a positive bounded function (this can be easily seen from Lemma 2.5), we have

$$\begin{aligned}
&\left|\frac{w_1}{z_1}\right|\cdot \sum_{\alpha\in\mathcal{G},\alpha_1\neq 0} C(\alpha)|r^\alpha|\\
&=|w_1|^2 \sum_{\alpha\in\mathcal{G},\alpha_1\neq 0} C(\alpha)|r_1|^{\alpha_1-1}|r_2|^{\alpha_2}\cdots|r_n|^{\alpha_n}\\
&\le |w_1|^2 \sum_{\alpha\in\mathcal{G},\alpha_1\neq 0} C(\alpha)|r_2|^{\frac{k_1}{k_2}(\alpha_1-1)+\alpha_2}|r_3|^{\alpha_3}\cdots|r_n|^{\alpha_n}\\
&=|w_1|^2 \sum_{\alpha\in\mathcal{G},\alpha_1\neq 0} C(\alpha)|r_2|^{A_2'}|r_2|^{\Phi_1(\alpha_1-1)-A_2'}|r_3|^{\alpha_3}\cdots|r_n|^{\alpha_n}\\
&\le |w_1|^2 \sum_{\alpha\in\mathcal{G},\alpha_1\neq 0} C(\alpha)|r_2|^{A_2'}|r_3|^{\Phi_2(\alpha_1-1)}\cdots|r_n|^{\alpha_n}\\
&\cdots\cdots\\
&\le |w_1|^2 \sum_{\alpha\in\mathcal{G},\alpha_1\neq 0} C(\alpha)|r_2|^{A_2'}|r_3|^{A_3'}\cdots|r_n|^{A_n'}\\
&\lesssim |w_1|^2|r_2|^{A_2'}|r_3|^{A_3'}\cdots|r_n|^{A_n'}\\
&\le |z_2|^{A_2-\frac{k_1}{k_2}}|z_3|^{A_3}\cdots|z_n|^{A_n}|w_2|^{A_2+\frac{k_1}{k_2}}|w_3|^{A_3}\cdots|w_n|^{A_n}, \quad (z,w)\in\mathbb{H}^n_{\mathbf{p}}\times\mathbb{H}^n_{\mathbf{p}}.
\end{aligned}$$

Here, from the first line to the second line we used the fact that $\alpha=(\alpha_1,\ldots,\alpha_n)\in\mathcal{G}$ and $\alpha_1\neq 0$ implies that $\alpha_1\ge 1$. From the third line to the fourth line we applied (13). The last inequality holds because $w\in\mathbb{H}^n_{\mathbf{p}}$.

Therefore, we obtain that

$$M(z,w)\le \frac{|z_2|^{A_2-\frac{k_1}{k_2}}|z_3|^{A_3}\cdots|z_n|^{A_n}|w_2|^{A_2+\frac{k_1}{k_2}}|w_3|^{A_3}\cdots|w_n|^{A_n}}{\pi^n K\cdot|1-r_n|^2\prod_{j=1}^{n-1}\left|r_j^{k_{j+1}^{(j)}}-r_{j+1}^{k_j^{(j)}}\right|^2},$$

which means that the integral operator with kernel $M(z,w)$ is of general type-$(A_2-\frac{k_1}{k_2},A_3,\ldots,A_n;A_2+\frac{k_1}{k_2},A_3,\ldots,A_n)$.

Now we come to estimate $N(z,w)$, the second term in (11). Take any $\alpha=(\alpha_1,\ldots,\alpha_n)\in\mathcal{G}$ with $C(\alpha)=\prod_{j=1}^n\nu(P_j)\neq 0$ and $\alpha_1=0$. An argument analogous to that in the proof of [25, Proposition 3.2] shows that

$$\Phi_{l-1}(0)=\varphi_{l-1}\circ\cdots\circ\varphi_1(0)\ge A_l,\qquad l=2,\ldots,n.$$

Therefore, we have

$$\begin{aligned}
\sum_{\alpha\in\mathcal{G},\alpha_1=0} C(\alpha)|r^\alpha| &= \sum_{\alpha\in\mathcal{G},\alpha_1=0} C(\alpha)|r_2|^{\Phi_1(0)}|r_3|^{\alpha_3}\cdots|r_n|^{\alpha_n}\\
&=\sum_{\alpha\in\mathcal{G},\alpha_1=0} C(\alpha)|r_2|^{A_2}|r_2|^{\Phi_1(0)-A_2}|r_3|^{\alpha_3}\cdots|r_n|^{\alpha_n}
\end{aligned}$$

$$\leq \sum_{\alpha\in\mathcal{G},\alpha_1=0} C(\alpha)|r_2|^{A_2}|r_3|^{\Phi_2(0)}\cdots|r_n|^{\alpha_n}$$

$$\cdots\cdots$$

$$\leq \sum_{\alpha\in\mathcal{G},\alpha_1=0} C(\alpha)|r_2|^{A_2}\cdots|r_n|^{\alpha_n}$$

$$\lesssim |r_2|^{A_2}\cdots|r_n|^{A_n}, \tag{14}$$

where we again used the fact that $C(\alpha)=\prod_{j=1}^n \nu(P_j)$ is a positive function for $\alpha\in\mathcal{G}$. From (14), we can find that

$$\begin{aligned}
N(z,w) &\leq \frac{|z_2|^{k_1^{(1)}-\frac{k_1}{k_2}}|w_2|^{k_1^{(1)}+\frac{k_1}{k_2}}\sum\limits_{\alpha\in\mathcal{G},\alpha_1=0} C(\alpha)|r^\alpha|}{\pi^n K\cdot|r_2|^{k_1^{(1)}}|1-r_n|^2\prod\limits_{j=1}^{n-1}\left|r_j^{k_{j+1}^{(j)}}-r_{j+1}^{k_j^{(j)}}\right|^2}\\
&= \frac{|z_2|^{-\frac{k_1}{k_2}}|w_2|^{\frac{k_1}{k_2}}\sum\limits_{\alpha\in\mathcal{G},\alpha_1=0} C(\alpha)|r^\alpha|}{\pi^n K\cdot|1-r_n|^2\prod\limits_{j=1}^{n-1}\left|r_j^{k_{j+1}^{(j)}}-r_{j+1}^{k_j^{(j)}}\right|^2}\\
&\lesssim \frac{|z_2|^{A_2-\frac{k_1}{k_2}}|z_3|^{A_3}\cdots|z_n|^{A_n}|w_2|^{A_2+\frac{k_1}{k_2}}|w_3|^{A_3}\cdots|w_n|^{A_n}}{\pi^n K\cdot|1-r_n|^2\prod\limits_{j=1}^{n-1}\left|r_j^{k_{j+1}^{(j)}}-r_{j+1}^{k_j^{(j)}}\right|^2},
\end{aligned}$$

which means that the integral operator with kernel $N(z,w)$ is of general type-$(A_2-\frac{k_1}{k_2},A_3,\ldots,A_n;A_2+\frac{k_1}{k_2},A_3,\ldots,A_n)$. Combining (11) and the above estimates for $M(z,w)$ and $N(z,w)$, the stated claim is proved. $\square$

Now we come back to the proof of Case 2. Since $\sum\limits_{\alpha\in\mathcal{A}^2(\mathbb{H}^n_{\mathbf{p}}),\alpha_1=0}\frac{z^\alpha\overline{w}^\alpha}{C_\alpha^2}$ is independent of w_1 and $\overline{w}_1$, we have that

$$\mathcal{T}_{w_1}(B_{\mathbf{p}}(z,w))=\mathcal{T}_{w_1}(\widetilde{B_{\mathbf{p}}}(z,w)).$$

Therefore, by (9) and Lemma 2.1 we have that

$$\begin{aligned}
&\frac{\partial}{\partial z_1}\circ\mathbf{B}_{\mathbf{p}}f(z)\\
&=\frac{1}{z_1}\int_{E_{1,n}}\int_{E_{1,n-1}}\cdots\int_{E_{1,1}}\mathcal{T}_{w_1}(\widetilde{B_{\mathbf{p}}}(z,w))f(w)dV(w_1)\cdots dV(w_n)\\
&=-\frac{1}{z_1}\int_{\mathbb{H}^n_{\mathbf{p}}}\widetilde{B_{\mathbf{p}}}(z,w)\mathcal{T}_{w_1}f(w)dV(w)\\
&=\int_{\mathbb{H}^n_{\mathbf{p}}}\frac{w_1}{z_1}\widetilde{B_{\mathbf{p}}}(z,w)\frac{\partial f}{\partial w_1}(w)dV(w)-\int_{\mathbb{H}^n_{\mathbf{p}}}\frac{\overline{w}_1}{z_1}\widetilde{B_{\mathbf{p}}}(z,w)\frac{\partial f}{\partial\overline{w}_1}(w)dV(w).
\end{aligned}$$

Here $\frac{\partial f}{\partial w_1}$ and $\frac{\partial f}{\partial \overline{w}_1}$ should be interpreted in the distributional sense. From the aforementioned claim and Lemma 2.7, we know that the integral operator with kernel $\frac{w_1}{z_1}\widetilde{B_{\mathbf{p}}}(z,w)$ or $\frac{\overline{w}_1}{z_1}\widetilde{B_{\mathbf{p}}}(z,w)$ is bounded on $L^p(\mathbb{H}^n_{\mathbf{p}})$ for all $p \in \left(\frac{2\sum_{i=1}^n k_j}{\sum_{i=1}^n k_j + D_n + k_1}, \frac{2\sum_{i=1}^n k_j}{\sum_{i=1}^n k_j - D_n + k_1}\right)$, which follows that

$$\begin{aligned}\left\|\frac{\partial}{\partial z_1}\circ \mathbf{B_p}(f)\right\|_p &\leq \left\|\int_{\mathbb{H}^n_{\mathbf{p}}} \frac{w_1}{z_1}\widetilde{B_{\mathbf{p}}}(z,w)\frac{\partial f}{\partial w_1}(w)dV(w)\right\|_p \\ &\quad + \left\|\int_{\mathbb{H}^n_{\mathbf{p}}} \frac{\overline{w}_1}{\overline{z}_1}\widetilde{B_{\mathbf{p}}}(z,w)\frac{\partial f}{\partial \overline{w}_1}(w)dV(w)\right\|_p \\ &\lesssim \left\|\frac{\partial f}{\partial w_1}\right\|_p + \left\|\frac{\partial f}{\partial \overline{w}_1}\right\|_p \leq \|f\|_{1,p}\end{aligned}$$

holds for all p lies in the above range. By the arbitrariness of $f \in L^p_1(\mathbb{H}^n_{\mathbf{p}})$, we conclude that $\frac{\partial}{\partial z_1}\circ \mathbf{B_p}$ is bounded from $L^p_1(\mathbb{H}^n_{\mathbf{p}})$ to $L^p(\mathbb{H}^n_{\mathbf{p}})$ for all $p \in \left(\frac{2\sum_{i=1}^n k_j}{\sum_{i=1}^n k_j + D_n + k_1}, \frac{2\sum_{i=1}^n k_j}{\sum_{i=1}^n k_j - D_n + k_1}\right)$. The proof is complete. □

Proof of Theorem 1.2. The sufficiency part of the theorem follows directly from [25, Theorem 1.1] and Theorem 1.1. Conversely, if the Bergman projection $\mathbf{B_p}$ maps $L^p_k(\mathbb{H}^n_{\mathbf{p}})$ to itself for some $p \in (1,\infty)$, then from Lemma 2.3 and Proposition 2.4 we know that $\mathbf{p} = \mathbf{1} = (1,\ldots,1)$ and $k = 1$. Thus now we only need to focus on the domain $\mathbb{H}^n_{\mathbf{1}}$. On one hand, from the conclusion of Step 1 in the proof of Lemma 2.3, we can find that if $\mathbf{B_1}$ is a bounded operator on $L^p_1(\mathbf{B_1})$ then $p \in \left(\frac{2n}{n+1}, \frac{2n}{n-1}\right)$. On the other hand, we consider the function f on $\mathbb{H}^n_{\mathbf{1}}$ which is defined in the proof of [25, Proposition 4.2]. That is, $f(w) = w_1^{\beta_1}\overline{w}_2^{-\beta_2}\cdots\overline{w}_n^{-\beta_n}$ with $\beta_1 \geq 0$, $\beta_2,\ldots,\beta_n < 0$ and $\sum_{i=1}^n(\beta_i+1) = 1$. It is obvious that $f \in L^p_1(\mathbb{H}^n_{\mathbf{1}})$ for any $p \in (1,\infty)$. Since $\mathbf{B_1}$ is bounded on $L^p_1(\mathbb{H}^n_{\mathbf{1}})$, we have that $g(w) := \mathbf{B_1}f(w) = w_1^{\beta_1}\cdots w_n^{\beta_n} \in L^p_1(\mathbb{H}^n_{\mathbf{1}})$, which means that $\frac{\partial g}{\partial w_j} = \beta_j w_1^{\beta_1}\cdots w_j^{\beta_j-1}\cdots w_n^{\beta_n} \in L^p(\mathbb{H}^n_{\mathbf{1}})$ holds for all $j = 1,\ldots,n$. By Lemma 2.2, we know that $\sum_{i=1}^n(\beta_i p + 2) - p > 0$. Note that $\beta_1,\ldots,\beta_n$ satisfy $\sum_{i=1}^n(\beta_i+1) = 1$, it follows that $p < 2$. Thus we have that $p \in \left(\frac{2n}{n+1}, 2\right)$. The proof is complete. □

4. Proof of Theorem 1.3

Proof of Theorem 1.3. Firstly we assume that $k = 1$. In this case we divide the proof into two steps.

Step 1. We first calculate the L^∞ sub-Bergman kernel of $\mathbb{H}^n_{\mathbf{1}}$ and obtain an useful estimate for it. By the definition of L^∞ sub-Bergman kernel, we have that

$$B_1^\infty(z,w) := \sum_{\alpha\in\mathcal{A}^\infty(\mathbb{H}_{\mathbf{1}}^n)} \frac{z^\alpha\overline{w}^\alpha}{\|z^\alpha\|_2^2},$$

where

$$\mathcal{A}^\infty(\mathbb{H}_{\mathbf{1}}^n) = \left\{\alpha=(\alpha_1,\dots,\alpha_n) : \alpha_1\ge 0,\ \sum_{i=1}^{j}\alpha_i \ge 0,\quad j=2,\dots,n\right\}. \tag{15}$$

By Lemma 2.2, we know that

$$\|z^\alpha\|_2^2 = \frac{\pi^n}{\prod_{j=1}^{n}\left(\sum_{i=1}^{j}\alpha_j + j\right)}.$$

Combining this with (15), we have that

$$B_1^\infty(z,w) = \frac{1}{\pi^n}\sum_{\alpha_1=0}^{\infty}\sum_{\alpha_2=-\alpha_1}^{\infty}\cdots\sum_{\alpha_n=-\alpha_1-\cdots-\alpha_{n-1}}^{\infty}\left[\prod_{j=1}^{n}\left(\sum_{i=1}^{j}\alpha_j + j\right)r_j^{\alpha_j}\right]. \tag{16}$$

Denote $t_j := \sum_{i=1}^{j}\alpha_j$ for $j=1,\dots,n$ and $t_0 := 0$. Then by changing the summation indices from α_j to t_j in (16) and we get

$$\begin{aligned}
(16) &= \frac{1}{\pi^n}\sum_{t_1=0}^{\infty}\cdots\sum_{t_n=0}^{\infty}\left[\prod_{j=1}^{n}\left(t_j+j\right)r_j^{t_j-t_{j-1}}\right]\\
&= \frac{1}{\pi^n}\sum_{t_1=0}^{\infty}\cdots\sum_{t_n=0}^{\infty}\left[\prod_{j=1}^{n}\left(t_j+j\right)\left(\frac{r_j}{r_{j+1}}\right)^{t_j}\right]\\
&= \frac{1}{\pi^n}\prod_{j=1}^{n}\sum_{t_j=0}^{\infty}(t_j+j)\left(\frac{r_j}{r_{j+1}}\right)^{t_j}\\
&= \frac{1}{\pi^n}\prod_{j=1}^{n}\frac{(1-j)r_j+j}{\left(1-\frac{r_j}{r_{j+1}}\right)^2}\\
&= \frac{1}{\pi^n}\frac{\left[(1-n)r_n^3+nr_n^2\right]\left[(2-n)\frac{r_{n-1}^3}{r_n}+(n-1)r_{n-1}^2\right]\cdots\left(-\frac{r_2^3}{r_3}+2r_2^2\right)}{(1-r_n)^2(r_n-r_{n-1})^2\cdots(r_2-r_1)^2}, \qquad (17)
\end{aligned}$$

where $r_{n+1} := 1$. In the above calculations we used the formula

$$\sum_{n=0}^{\infty}(An+B)z^n = \frac{(A-B)z+B}{(1-z)^2},\quad |z|<1,\ A,B\in\mathbb{C}.$$

From (17) it is easily seen that

$$|B_1^\infty(z,w)| \lesssim \frac{\prod_{j=2}^{n}|z_j|^2|w_j|^2}{|1-r_n|^2|r_n-r_{n-1}|^2\cdots|r_2-r_1|^2},\quad (z,w)\in\mathbb{H}_{\mathbf{1}}^n\times\mathbb{H}_{\mathbf{1}}^n. \tag{18}$$

Step 2: Now we come to obtain the boundedness range for the L^∞ sub-Bergman projection $\mathbf{B}_{\mathbf{1}}^\infty$ on the Sobolev space $L_1^p(\mathbb{H}_{\mathbf{1}}^n)$. By an argument similar to the proof of Theorem 1.1, for any $r \in \{2, \ldots, n\}$ and $f \in L_1^p(\mathbb{H}_{\mathbf{1}}^n)$ with $1 < p < \infty$ we have that

$$\begin{aligned}&\frac{\partial}{\partial z_r} \circ \mathbf{B}_{\mathbf{1}}^\infty f(z)\\ &= \int_{\mathbb{H}_{\mathbf{1}}^n} \frac{w_r}{z_r} B_1^\infty(z,w) \frac{\partial f}{\partial w_r}(w) dV(w) - \int_{\mathbb{H}_{\mathbf{1}}^n} \frac{\overline{w}_r}{z_r} B_1^\infty(z,w) \frac{\partial f}{\partial \overline{w}_r}(w) dV(w).\end{aligned}$$

From (18) we know that the integral operator with kernel $\frac{w_r}{z_r} B_1^\infty(z,w)$ or $\frac{\overline{w}_r}{z_r} B_1^\infty(z,w)$ is of general type-$(2, \ldots, 1, \ldots, 2; 2, \ldots, 3, \ldots, 2)$, where 1 and 3 appear in the r-th position. Then Lemma 2.7 implies that $\frac{\partial}{\partial z_r} \circ \mathbf{B}_{\mathbf{1}}^\infty$ is bounded from $L_1^p(\mathbb{H}_{\mathbf{1}}^n)$ to $L^p(\mathbb{H}_{\mathbf{1}}^n)$ for all $1 < p < 2n$.

For the z_1 differentiated operator $\frac{\partial}{\partial z_1} \circ \mathbf{B}_{\mathbf{1}}^\infty$, again by an argument similar to that in the proof of Theorem 1.1, we have that

$$\begin{aligned}&\frac{\partial}{\partial z_1} \circ \mathbf{B}_{\mathbf{1}}^\infty f(z)\\ &= \int_{\mathbb{H}_{\mathbf{1}}^n} \frac{w_1}{z_1} \widetilde{B_1^\infty}(z,w) \frac{\partial f}{\partial w_1}(w) dV(w) - \int_{\mathbb{H}_{\mathbf{1}}^n} \frac{\overline{w}_l}{\overline{z}_l} \widetilde{B_1^\infty}(z,w) \frac{\partial f}{\partial \overline{w}_1}(w) dV(w),\end{aligned}$$

holds for any $f \in L_1^p(\mathbb{H}_{\mathbf{1}}^n)$, where $\widetilde{B_1^\infty}(z,w)$ is the "modified" kernel defined by

$$\widetilde{B_1^\infty}(z,w) = B_1^\infty(z,w) - \sum_{\alpha \in \mathcal{A}^\infty(\mathbb{H}_{\mathbf{1}}^n), \alpha_1 = 0} \frac{z^\alpha \overline{w}^\alpha}{\|z^\alpha\|_2^2}, \qquad (z,w) \in \mathbb{H}_{\mathbf{1}}^n \times \mathbb{H}_{\mathbf{1}}^n. \tag{19}$$

From (17) we can obtain that the integral operator with kernel $\frac{w_1}{z_1} \widetilde{B_1^\infty}(z,w)$ or $\frac{\overline{w}_l}{z_1} \widetilde{B_1^\infty}(z,w)$ is of general type-$(1, 2, \ldots, 2; 3, 2, \ldots, 2)$. By Lemma 2.7, we can conclude that $\frac{\partial}{\partial z_1} \circ \mathbf{B}_{\mathbf{1}}^\infty$ is bounded from $L_1^p(\mathbb{H}_{\mathbf{1}}^n)$ to $L^p(\mathbb{H}_{\mathbf{1}}^n)$ for $1 < p < 2n$. Note that the operator B_1^∞ is bounded on the ordinary L^p space $L^p(\mathbb{H}_{\mathbf{1}}^n)$ for all $1 < p < \infty$, it follows from the above analysis that the proof of the case $k = 1$ is complete.

For the cases of $k = 2$ and $k = 3$, we can easily obtain the desired results by using the same method as in the proof of the case $k = 1$, except that we need to change the "modified" kernel in (19) to

$$\widetilde{B_1'^\infty}(z,w) := \sum_{\alpha \in \mathcal{A}^\infty(\mathbb{H}_{\mathbf{1}}^n), \alpha_1 \geq 2} \frac{z^\alpha \overline{w}^\alpha}{\|z^\alpha\|_2^2}, \qquad (z,w) \in \mathbb{H}_{\mathbf{1}}^n \times \mathbb{H}_{\mathbf{1}}^n$$

and

$$\widetilde{B_1''^\infty}(z,w) := \sum_{\alpha \in \mathcal{A}^\infty(\mathbb{H}_{\mathbf{1}}^n), \alpha_1 \geq 3} \frac{z^\alpha \overline{w}^\alpha}{\|z^\alpha\|_2^2}, \qquad (z,w) \in \mathbb{H}_{\mathbf{1}}^n \times \mathbb{H}_{\mathbf{1}}^n$$

in studying the boundedness range for the differentiated operators $\frac{\partial^2}{\partial z_1^2} \circ \mathbf{B}_{\mathbf{1}}^\infty$ (in the case $k = 2$) and $\frac{\partial^3}{\partial z_1^3} \circ \mathbf{B}_{\mathbf{1}}^\infty$ (in the case $k = 3$). The proof is complete. $\square$

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