Bull. Korean Math. Soc. **58** (2021), No. 6, pp. 1419–1443
https://doi.org/10.4134/BKMS.b200980
pISSN: 1015-8634 / eISSN: 2234-3016

ON THE WEAK LIMIT THEOREMS FOR GEOMETRIC SUMMATIONS OF INDEPENDENT RANDOM VARIABLES TOGETHER WITH CONVERGENCE RATES TO ASYMMETRIC LAPLACE DISTRIBUTIONS

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Abstract. The asymmetric Laplace distribution arises as a limiting distribution of geometric summations of independent and identically distributed random variables with finite second moments. The main purpose of this paper is to study the weak limit theorems for geometric summations of independent (not necessarily identically distributed) random variables together with convergence rates to asymmetric Laplace distributions. Using Trotter-operator method, the orders of approximations of the distributions of geometric summations by the asymmetric Laplace distributions are established in term of the "large–$\mathcal{O}$" and "small–o" approximation estimates. The obtained results are extensions of some known ones.

1. Introduction

The asymmetric Laplace distribution arises as a limiting distribution of geometric summations of independent and identically distributed (i.i.d.) random variables having finite second moments. We introduce the notion of asymmetric Laplace distributed random variables, following Kotz et al. [16] (Chapter 3, Section 3.1, page 136). A random variable $L_{\mu,\sigma}$ is said to have an asymmetric Laplace (AL) distribution, denoted by $L_{\mu,\sigma} \sim \mathcal{AL}(\mu,\sigma)$, if there are parameters $\mu \in \mathbb{R}$ (location parameter) and $\sigma > 0$ (scale parameter) such that the characteristic function of $L_{\mu,\sigma}$ has the form

$$\varphi(t) = \left(1 - i\mu t + \frac{1}{2}\sigma^2 t^2\right)^{-1}. \tag{1.1}$$

It is to be noticed that the asymmetric Laplace distributed random variable $L_{\mu,\sigma}$ with characteristic function in form (1.1) is a geometric infinitely divisible

Received November 22, 2020; Revised April 4, 2021; Accepted May 7, 2021.
2010 *Mathematics Subject Classification.* Primary 60G50, 60F05, 60E07; Secondary 41A36.
Key words and phrases. Geometric summation, asymmetric Laplace distribution, weak limit theorem, geometric Lindeberg condition, rate of convergence, Trotter's operator.

(G.I.D) random variable (see for instance [16], Proposition 3.4.3, page 151). For a deeper discussion of G.I.D. random variables we refer the reader to [1], [7], [13], [15,16,21] and [20]. With the notation $\kappa = \sqrt{2}\sigma(\mu + \sqrt{\mu^2 + 2\sigma^2})^{-1}$, we can express the probability density function $p_{\mu,\sigma}(x)$ and probability distribution functions $F_{\mu,\sigma}(x)$ of an asymmetric Laplace distributed random variable $L_{\mu,\sigma}$, as follows

$$p_{\mu,\sigma}(x) = \begin{cases} \frac{\sqrt{2}}{\sigma}\frac{\kappa}{1+\kappa^2}\exp\left(-\frac{\sqrt{2}\kappa}{\sigma}x\right), & \text{if } x \geq 0, \\ \frac{\sqrt{2}}{\sigma}\frac{\kappa}{1+\kappa^2}\exp\left(\frac{\sqrt{2}}{\sigma\kappa}x\right), & \text{if } x < 0, \end{cases} \tag{1.2}$$

and

$$F_{\mu,\sigma}(x) = \begin{cases} 1 - \frac{1}{1+\kappa^2}\exp\left(-\frac{\sqrt{2}\kappa}{\sigma}x\right), & \text{if } x \geq 0, \\ \frac{\kappa^2}{1+\kappa^2}\exp\left(\frac{\sqrt{2}}{\sigma\kappa}x\right), & \text{if } x < 0. \end{cases} \tag{1.3}$$

Direct computation shows that, for $r \in \mathbb{N}$,

$$\begin{aligned} &E(L_{\mu,\sigma}) = \mu, E(L^2_{\mu,\sigma}) = 2\mu^2 + \sigma^2,\ Var(L_{\mu,\sigma}) = \mu^2 + \sigma^2, \\ &E|L_{\mu,\sigma}|^r = \left(\frac{\sigma}{\sqrt{2}\kappa}\Gamma(r+1)\frac{1+\kappa^{2(r+1)}}{1+\kappa^2}\right)^r. \end{aligned} \tag{1.4}$$

The asymmetric Laplace distribution which was introduced by Hinkley and Revankar [10] and has been studied more recently by many authors. Although not wide known, the asymmetric Laplace distribution plays an important role in probability theory and it appears in a number of applications in sciences, in business, and in branches of engineering. In view of (1.2)–(1.4), the asymmetric Laplace distribution has finite moments of all orders, explicit formulas for density and distribution functions, the asymmetric Laplace distribution is much easier to work with in practice than several geometric stable laws (see [20] and the references given there). Especially, the asymmetric Laplace distribution has been used for modeling data exhibiting asymmetry and heavy tails. Furthermore, if the variable of interest can be viewed as a geometric summations of increment with finite second moment, then its distribution can be approximated by an asymmetric Laplace distribution. A comprehensive review of the asymmetric Laplace distribution can be found in [12], [16–20] and the references given there.

Let $(\Omega, \mathcal{B}, \mathbb{P})$ be a probability space and suppose that all random variables under consideration are defined on this probability space. Let $X, X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with common mean $E(X) = \mu$ and finite variance $0 < \sigma^2 = Var(X) < +\infty$. Further, let ν_p be a geometric random variable, denoted by $\nu_p \sim Geo(p)$, having mean $p^{-1}, p \in (0,1)$ and probability mass function

$$P(\nu_p = k) = p(1-p)^{k-1}, \quad k \geq 1.$$

From now on we suppose that for each $p \in (0,1)$ the geometric random variable ν_p is independent of all X_j, $j \geq 1$. It is worth noticing that $\nu_p \xrightarrow{P} +\infty$ when

$p \to 0$ (see Appendix, Proposition A1), here and from now on the symbol $\xrightarrow{P}$ denotes convergence in probability. As discussed in Kalashnikov [14], the geometric summations arise naturally in diverse fields of applications such as risk theory, modeling financial asset returns, insurance mathematics and others. Consequently the Laplace distribution is applicable for stochastic modeling, stochastic analysis, etc. (see e.g., [17–20]). According to Gnedenko's transfer theorem [6], Sakalauskas has established the order of approximations for distribution of geometric summation by a stable distribution with exponent $\alpha \in (0, 2]$ (see [25], Theorems 2, 3 and 5 with Corollaries). Further, using Stein's method, some results related to Berry–Esseen type theorems for convergence of the distributions of geometric summations to symmetric Laplace distribution have been obtained by Pike and Ren [23], and by Dobler [5]. However, these results have been received only for symmetric Laplace distributions (see [5] and [23] for more details).

It is well known that under desired conditions, the asymmetric Laplace distribution with characteristic function in (1.1) arises as a limiting distribution of (appropriately normalized) geometric summation of i.i.d. random variables with finite second moments

$$p^{1/2} \sum_{j=1}^{\nu_p} \Big(X_j - \mu + p^{1/2} \mu \Big) \xrightarrow{d} L_{\mu,\sigma} \quad \text{as} \quad p \to 0, \tag{1.5}$$

where $L_{\mu,\sigma} \sim \mathcal{AL}(\mu, \sigma)$, $\mu \in \mathbb{R}$, $\sigma > 0$, the symbol $\xrightarrow{d}$ stands for convergence in distribution. The detailed proof of limiting assertion (1.5) was presented in Kotz et al. [16] by method of characteristic functions (see [16], Proposition 3.4.4, page 152). It seems that the way of using the characteristic functions is complicated for the case of independent and non-identically distributed random variables. This problem was discussed by Toda [27], using the idea of Lindeberg [22], with a so–called Lindeberg–type condition (see [27], condition (2.1), page 3) and notion of uniformly integrability for a sequence of independent non-identically distributed random variables X_j^2 for $j \geq 1$ (see [27], Theorem 2.1, Corollaries 2.2 and 2.3, page 3). However, in [27], the rates of convergence in (1.5) were omitted. When studying limit theorems in probability theory it is important to try to assess the rates at which these converge. Therefore, the estimates of convergence rate in weak limit theorems for geometric summation in the type of (1.5) are main results of this paper.

In the paper, with an extension of classical Lindeberg condition (see [26], page 112), a weak limit theorem for geometric summation of independent non–identically distributed random variables, will be proved by method of Trotter operator [28]. It is worth noted that the limiting result in (1.5) may be followed directly from Theorem 3.2. We also investigate the convergence in distribution of normalized geometric summations of a sequence of independent (not necessarily identically distributed) random variables to the asymmetric Laplace distributed random variables when parameter p tends to zero. The obtained results related to sequences of independent but non-identically distributed random variables satisfied the so–called geometric Lindeberg condition. Some rates of convergence in type of the "large–$\mathcal{O}$" and "small–o" approximation estimates for limiting assertion (1.5) are established, in term of Trotter's operator.

In the paper, we follow the notations used in [16] (see Abbreviation and Notation, page 16), where $a_n = \mathcal{O}(b_n)$ means $\big|a_n/b_n\big|$ is bounded for $n \to \infty, a_n = o(b_n)$ expresses that $\lim\limits_{n\to\infty} a_n/b_n = 0$ and $a_n = o(1)$ if $\lim\limits_{n\to\infty} a_n = 0$. The techniques used in this paper for estimating the rates of convergence may be found in [2–4], [8], [24] and [28].

This paper is organized as follows. Section 2 is a brief introduction to Trotter's operator with main properties. The class of the modulus of continuity and Lipschitz functions is recalled. The Lindeberg type and Liapunov type conditions of order r for sequences of independent random variables are defined in this section, too. Section 3 is devoted to the discussion of the weak limit theorems for normalized geometric summations of independent random variables with convergence rates to the asymmetric Laplace distributed random variables in term of large–$\mathcal{O}$ and small–o approximation estimates. The obtained results are extensions of some known ones.

2. Some notations, definitions and auxiliary results

Before proceeding to the main results we introduce some notations and preliminaries concerning Trotter's operator [28] and related results.

2.1. Trotter's operator

Throughout this paper, the symbol $C_B(\mathbb{R})$ will denote the set of all bounded uniformly continuous functions on $\mathbb{R}$ and $C_B^r(\mathbb{R}) := \{f \in C_B(\mathbb{R}) : f^{(j)} \in C_B(\mathbb{R}), 1 \le j \le r; r \in \mathbb{N}\}$. The norm of function $f \in C_B(\mathbb{R})$ is defined by $||f|| = \sup\limits_{x\in\mathbb{R}} |f(x)|$. We introduce the notion of Trotter's operator, following [28].

Definition ([28])**.** Let X be a random variable. For each $f \in C_B(\mathbb{R})$, Trotter's operator A_X associated with the random variable X is defined by

$$A_X f(t) := E\big[f(X+t)\big] = \int_{\mathbb{R}} f(x+t)dF_X(x), \quad t \in \mathbb{R}, \tag{2.1}$$

where F_X is the distribution function of X.

In the sequel, we shall use the following properties of Trotter's operator A_X defined in (2.1). The proofs may be found in [4], [8], [24] and [28].

(1) The operator A_X is a linear positive "contraction" operator, that is

$$\| A_X f \| \le \| f \|$$

for each $f \in C_B(\mathbb{R})$.

(2) The equation $A_X f(t) = A_Y f(t)$ for each $f \in C_B(\mathbb{R})$, $t \in \mathbb{R}$, provided that X and Y are identically distributed random variables, in short $X \stackrel{d}{=} Y$.

(3) If $X_1, X_2, \ldots, X_n$ are independent random variables, then for $f \in C_B(\mathbb{R})$

$$A_{\sum\limits_{j=1}^{n} X_j}(f) = A_{X_1} \circ \cdots \circ A_{X_n}(f).$$

(4) If $X, X_1, X_2, \ldots, X_n$ are i.i.d. random variables, then for $f \in C_B(\mathbb{R})$

$$A_{\sum\limits_{j=1}^{n} X_j}(f) = A_X^n(f).$$

(5) Suppose that $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$, are independent random variables (in each group) and they are independent. Then, for each $f \in C_B(\mathbb{R})$

$$\left|\left|A_{\sum\limits_{j=1}^{n} X_j}(f) - A_{\sum\limits_{j=1}^{n} Y_j}(f)\right|\right| \leq \sum_{j=1}^{n} \left|\left|A_{X_j}(f) - A_{Y_j}(f)\right|\right|.$$

Furthermore, for two independent random variables X and Y, for each $f \in C_B(\mathbb{R})$ and for $n \geq 1$,

$$\left|\left|A_X^n(f) - A_Y^n(f)\right|\right| \leq n\left|\left|A_X(f) - A_Y(t)\right|\right|.$$

The following lemma is playing a key role in weak limit theorems (see [2], [4], [8] and [28] for more details). The proof of the lemma may be found in [8] and [28].

Lemma 2.1 ([28]). *A sufficient condition for a sequence of random variables $X_1, X_2, \ldots$ to converge in distribution to a random variable X, in short $X_n \xrightarrow{d} X$, is that*

$$\left|\left|A_{X_n} f - A_X f\right|\right| = o(1) \quad as \quad n \to 0.$$

Next lemmas will be needed in proofs in the sequel. The proofs are based on Trotter–operator's properties [28] and random summations in [9] (see Chapter 2, Section 15, pages 85–86).

Lemma 2.2. *Suppose that $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$, are independent random variables (in each group) and they are independent. Let ν_p be a geometric random variable with mean p^{-1}, $p \in (0,1)$, independent of all X_j and Y_j, $1 \leq j \leq n$. Then, for each $f \in C_B(\mathbb{R})$*

$$\left|\left|A_{\sum\limits_{j=1}^{\nu_p} X_j}(f) - A_{\sum\limits_{j=1}^{\nu_p} Y_j}(f)\right|\right| \leq E\bigg(\sum_{j=1}^{\nu_p} \left|\left|A_{X_j}(f) - A_{Y_j}(f)\right|\right|\bigg).$$

Lemma 2.3. *Suppose that $X, X_1, X_2, \ldots, X_n$ and $Y, Y_1, Y_2, \ldots, Y_n$, are i.i.d. random variables (in each group) and they are independent. Let ν_p be a geometric random variable with mean p^{-1}, $p \in (0,1)$, independent of all X_j and Y_j, $1 \leq j \leq n$. Then, for each $f \in C_B(\mathbb{R})$*

$$\left|\left|A_{\sum\limits_{j=1}^{\nu_p} X_j}(f) - A_{\sum\limits_{j=1}^{\nu_p} Y_j}(f)\right|\right| \leq E(\nu_p)\left|\left|A_X(f) - A_Y(f)\right|\right|.$$

2.2. The modulus of continuity and Lipschitz condition

We follow the definitions and notations in [4].

Definition ([4]). For any $f \in C_B(\mathbb{R})$, the modulus of continuity with $\delta > 0$ is defined by

$$\omega(f,\delta) = \sup_{|h|\leq\delta}\left\{|f(t+h) - f(t)|\right\}, \quad t \in \mathbb{R}. \tag{2.2}$$

We shall need in the sequel following properties of the modulus of continuity $\omega(f,\delta)$ as follows.

1. The modulus of continuity $\omega(f,\delta)$ is a monotone decreasing function of δ with $\omega(f,\delta) \to 0$ for $\delta \to 0^+$.
2. For each $\lambda > 0$, we have $\omega(f,\lambda\delta) \leq (1+\lambda)\omega(f,\delta)$.

Definition ([4]). A function $f \in C_B(\mathbb{R})$ is said to satisfy a Lipschitz condition of order $\alpha, (0 < \alpha \leq 1)$, in symbols $f \in Lip(\alpha)$, if

$$\omega(f;\delta) = O(\delta^\alpha). \tag{2.3}$$

It is obvious that $f' \in C_B(\mathbb{R})$ implies $f \in Lip(1)$.

2.3. Moment inequality

Next proposition will be needed in proofs of Theorems 3.6 and 3.7.

Proposition 2.4 ([4]). *Let X be a random variable with $E(|X|^r) < +\infty$, $r \in \mathbb{N}$. Then $E(|X|^j) < +\infty$, for any $1 \leq j \leq r$, and*

$$E\left(|X|^j\right) \leq 1 + E\left(|X|^r\right). \tag{2.4}$$

Proof. The proof may be found in [11] (Appendix, page 230). □

2.4. Asymmetric Laplace distributed random variable

Lemma 2.5. *Let $L_{\mu,\sigma}$ be an asymmetric Laplace distributed random variable with location parameter $\mu \in \mathbb{R}$ and scale parameter $\sigma > 0$, (in short, $L_{\mu,\sigma} \sim \mathcal{AL}(\mu,\sigma)$). Then, for each $p \in (0,1)$, there exist independent, identically asymmetric Laplace distributed random variables $L_p^{(1)}, L_p^{(2)}, \ldots, L_p^{(\nu_p)}$, such that*

$$L_{\mu,\sigma} \stackrel{d}{=} p^{1/2}\sum_{j=1}^{\nu_p} L_p^{(j)}, \tag{2.5}$$

where ν_p is a geometric random variable having mean p^{-1}, independent of all $L_p^{(j)}$, $j \geq 1$. The random variables $L_p^{(j)}$ are independent, asymmetric Laplace distributed with location parameter $p^{1/2}\mu \in \mathbb{R}$, and scale parameter $\sigma > 0$, i.e., $L_p^{(j)} \sim \mathcal{AL}(p^{1/2}\mu, \sigma)$, $1 \leq j \leq \nu_p$. From now on, the notation $\stackrel{d}{=}$ expresses the equality in the sense of distributions.

Proof. Since $L_p^{(j)} \sim \mathcal{AL}(p^{\frac{1}{2}}\mu, \sigma)$, it follows that the characteristic function of an asymmetric Laplace distributed random variable $L_p^{(j)}$ at point $p^{1/2}t$ is given by

$$\varphi_{L_p^{(j)}}(p^{1/2}t) = \frac{1}{1 - ipt\mu + \frac{1}{2}p\sigma^2 t^2}.$$

Using the formula of the characteristic function of a random sum ([9], Theorem 9.1, page 193) and Equation (1.1), for $t \in (-\infty, +\infty)$, we have

$$\begin{aligned}\varphi_{p^{1/2}\sum\limits_{j=1}^{\nu_p} L_p^{(j)}}(t) = \psi_{\nu_p}\big(\varphi_{L_p^{(j)}}(p^{1/2t})\big) &= \frac{p\varphi_{L_p^{(j)}}(p^{1/2t})}{1 - (1-p)\varphi_{L_p^{(j)}}(p^{1/2t})} \\ &= \frac{1}{1 - it\mu + \frac{1}{2}t^2\sigma^2} = \varphi_{L_{\mu,\sigma}}(t),\end{aligned}$$

where $\psi_{\nu_p}(t)$ is moment generating function of ν_p. According to the Continuity Theorem for characteristic function ([9], Theorem 9.1, page 238), it finishes the proof. □

2.5. Geometric Lindeberg condition

Definition (Geometric Lindeberg condition)**.** Let $X_1, X_2, \ldots$ be a sequence of independent random variables with means $E(X_j) = \mu_j$ and $E(|X_j - \mu_j|^r) < +\infty$, $r \geq 2$, $j = 1, 2, \ldots$. Let ν_p be a geometric variable having mean p^{-1}, $p \in (0,1)$, independent of all X_j, $j \geq 1$. The sequence $X_1, X_2, \ldots$ is said to satisfy geometric Lindeberg condition of order r, if for every $\tau > 0$,

$$\begin{aligned}(2.6)\quad L_{\nu_p}^r(\tau) := E&\left(\sum_{j=1}^{\nu_p} E\left[|X_j - \mu_j|^r \mathbf{1}_{\{|X_j - \mu_j| \geq \tau p^{-1/2}\}}\right] \Big/ \sum_{j=1}^{\nu_p} E|X_j - \mu_j|^r\right) \\ &= o(1)\end{aligned}$$

as $p \to 0$, where $\mathbf{1}_{\{A\}}$ stands for indicator function of a set A.

Lemma 2.6. *Let $X, X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with common mean $E(X) = \mu$ and $E|X - \mu|^r < +\infty$, $r \geq 2$. Let ν_p be a geometric variable having mean p^{-1}, $p \in (0,1)$, independent of all X_j, $j \geq 1$. Then, the geometric Lindeberg condition* (2.6) *holds for the sequence $X, X_1, X_2, \ldots$.*

Proof. According to [9] (formula (15.4), page 84), an trivial verification shows that, for the i.i.d. random variables $X, X_1, X_2, \ldots$

$$\begin{aligned} L^r_{\nu_p}(\tau) &:= E\left(\sum_{j=1}^{\nu_p} E\left[|X-\mu|^r \mathbf{1}_{\{|X-\mu|\geq \tau p^{-1/2}\}}\right] \Big/ \sum_{j=1}^{\nu_p} E|X-\mu|^r\right) \\ &= \sum_{n=1}^{\infty} P(\nu_p = n) E\left(\sum_{j=1}^{n} E\left[|X-\mu|^r \mathbf{1}_{\{|X-\mu|\geq \tau p^{-1/2}\}}\right] \Big/ \sum_{j=1}^{n} E|X-\mu|^r\right) \\ &= E\left[|X-\mu|^r \mathbf{1}_{\{|X-\mu|\geq \tau p^{-1/2}\}}\right] \Big/ E|X-\mu|^r \\ &= o(1) \quad \text{as} \quad p \to 0, \end{aligned}$$

since $E|X-\mu|^r < +\infty, r \geq 2$, and therefore

$$E\left[|X-\mu|^r \mathbf{1}_{\{|X-\mu|\geq \tau p^{-1/2}\}}\right] = o(1) \quad \text{as} \quad p \to 0.$$

□

Remark 2.7.
(1) For the case $P(\nu_p = n) = 1$, $p = n^{-1}$, $r = 2$, the geometric Lindeberg condition (2.6) reduces to classical Lindeberg condition (see, e.g. [4], condition (1.16), page 330 or [9], condition (2.2), p. 330).
(2) Results related to random version of Lindeberg's condition may be found in [24] (condition (5), page 148).
(3) The condition (2.1) in [27] is followed from geometric Lindeberg condition (2.6) of order 2 (see Appendix, Proposition A2 for more details).

A slightly stronger condition is the following geometric Lyapunov condition.

Definition. (Geometric Lyapunov condition) Let $X_1, X_2, \ldots$ be a sequence of random variables with $E(X_j) = \mu_j$ and $E(|X_j - \mu_j|^r) < +\infty$, $r \geq 2$, $j = 1, 2, \ldots$. Let ν_p be a geometric variable having mean p^{-1}, $p \in (0, 1)$, independent of all $X_j, j \geq 1$. The sequence $X_1, X_2, \ldots$ is said to satisfy geometric Lyapunov condition of order r, if for every $\epsilon > 0$,

$$E\left(\sum_{j=1}^{\nu_p} E\left(|X_j-\mu_j|^{r+\epsilon}\right) \Big/ p^{\epsilon/2} \sum_{j=1}^{\nu_p} E|X_j-\mu_j|^r\right) = o(1) \quad \text{as} \quad p \to 0. \tag{2.7}$$

Remark 2.8. For the case $P(\nu_p = n) = 1$, $p = n^{-1}$, $r = 2$, the geometric Lyapunov condition (2.7) reduces to classical Lyapunov condition (see, e.g. [9], condition (2.20), p. 339).

Proposition 2.9. *If a sequence $X_1, X_2, \ldots$ satisfies the geometric Lyapunov condition of order r in* (2.7)*, then it also satisfies the geometric Lindeberg condition of order r in* (2.6)*.*

Proof. It is clear that from $|X_j-\mu_j|\geq \tau p^{-1/2}$ implies $|X_j-\mu_j|^\epsilon \geq \tau^\epsilon p^{-\epsilon/2}$ for arbitrary $\epsilon>0$. Then

$$0\leq L^r_{\nu_p}(\tau)=E\left(\sum_{j=1}^{\nu_p}E\left[|X_j-\mu_j|^r\mathbf{1}_{\{|X_j-\mu_j|\geq\tau p^{-1/2}\}}\right]\Big/\sum_{j=1}^{\nu_p}E|X_j-\mu_j|^r\right)$$
$$\leq \tau^{-\epsilon}E\left(\sum_{j=1}^{\nu_p}E\left(|X_j-\mu_j|^{r+\epsilon}\right)\Big/p^{\epsilon/2}\sum_{j=1}^{\nu_p}E|X_j-\mu_j|^r\right).$$

Since τ is arbitrary, the assertion is followed. □

3. Main results

3.1. Weak limit theorems for geometric summations

Throughout the forthcoming, unless otherwise specified, we shall denote that for $n\geq 1$, the $X_{n,1},X_{n,2},\ldots$ be a sequence of independent (not necessarily identically distributed) random variables with mean $E(X_{n,j})=\mu_j$ and finite variance $0<Var(X_{n,j})=\sigma_j^2<+\infty$ for $j=1,2,\ldots,n$. Write $\mu=n^{-1}\sum_{j=1}^n\mu_j$ and $\sigma^2=n^{-1}\sum_{j=1}^n\sigma_j^2$. Let $\nu_p\sim Geo(p)$ be a geometric distributed random variable with mean p^{-1}, $p\in(0,1)$, independent of $X_{n,j}$ for $j=1,2,\ldots,n;n\geq 1$. Let $L_{\mu,\sigma}\sim\mathcal{AL}(\mu,\sigma)$ be an asymmetric Laplace distributed random variable with location parameter μ, scale parameter $\sigma>0$ and its characteristic function is given in (1.1).

The following weak limit theorem for geometric summation of $X_{n,1},X_{n,2},\ldots$ states as follows.

Theorem 3.1 (Weak limit theorem for geometric summations of independent non–identically distributed random variables)**.** *Suppose that for the sequence $X_{n,1},X_{n,2},\ldots$ the geometric Lindeberg condition of order* 2

$$E\left(\frac{\sum_{j=1}^{\nu_p}E\left[\left|X_{n,j}-\mu+p^{1/2}\mu\right|^2\mathbf{1}_{\{|X_{n,j}-\mu+p^{1/2}\mu|\geq\tau p^{-1/2}\}}\right]}{\sum_{j=1}^{\nu_p}E\left|X_{n,j}-\mu+p^{1/2}\mu\right|^2}\right)=o(1)$$

holds when $p\to 0$ for $\tau>0$. Then, the weak limit theorem states that

$$p^{1/2}\sum_{j=1}^{\nu_p}(X_{n,j}-\mu+p^{1/2}\mu)\xrightarrow{d}L_{\mu,\sigma}\quad as\quad p\to 0. \tag{3.1}$$

Proof. From now on, let us write $W_{n,j}=X_{n,j}-\mu+p^{1/2}\mu=X_{n,j}+\mu(p^{1/2}-1)$ for $j=1,2,\ldots,n;n\geq 1$. It is clear that the random variables $W_{n,j}$ are independent for $n\geq 1$ and for all $j=1,2,\ldots,n$. A direct computation shows that

$$E(W_{n,j})=p^{\frac{1}{2}}\mu\quad\text{and}\quad E(W^2_{n,j})=\sigma^2+p\mu^2.$$

According to properties of Trotter's operator in (2.1) and Lemma 2.2, for each $f \in C_B^2(\mathbb{R})$, we have

$$\left\|A_{p^{1/2}\sum_{j=1}^{\nu_p} W_{n,j}} f - A_{L_{\mu,\sigma}} f\right\| = \left\|A_{p^{1/2}\sum_{j=1}^{\nu_p} W_{n,j}} f - A_{p^{1/2}\sum_{j=1}^{\nu_p} L_p(j)} f\right\|.$$

Therefore, a sufficient condition for the validity of limiting assertion (3.1) is

$$\left\|A_{p^{1/2}\sum_{j=1}^{\nu_p} W_{n,j}} f - A_{p^{1/2}\sum_{j=1}^{\nu_p} L_p(j)} f\right\| = o(1) \quad \text{as} \quad p \to 0 \tag{3.2}$$

for each $f \in C_B^2(\mathbb{R})$.

Applying Trotter's operator [28] to a function $f \in C_B^2(\mathbb{R})$, with Taylor series expansion (see [26], formula (1.4.6), page 17) for $r = 2$, this yields

$$\begin{aligned}
(3.3)\qquad & E\left[f(p^{1/2}W_{n,j} + y)\right] \\
&= E\left[f(y) + f'(y)p^{1/2}W_{n,j} + f''(y)\frac{pW_{n,j}^2}{2} + (f''(\eta) - f''(y))\frac{pW_{n,j}^2}{2}\right] \\
&= f(y) + p^{1/2}f'(y)E(W_{n,j}) + \frac{pf''(y)}{2}E(W_{n,j}^2) \\
&\quad + E\left[(f''(\eta) - f''(y))\frac{pW_{n,j}^2}{2}\right] \\
&= f(y) + f'(y)p\mu + \frac{f''(y)}{2}p(\sigma^2 + p\mu^2) + \frac{p}{2}E\left[(f''(\eta) - f''(y))W_{n,j}^2\right],
\end{aligned}$$

where $|\eta - y| < p^{1/2}|x|$, $y \in \mathbb{R}$. By an argument analogous, for $f \in C_B^2(\mathbb{R})$, $y \in \mathbb{R}$,

$$\begin{aligned}
(3.4)\qquad & E\left[f(p^{1/2}L_p(j) + y)\right] \\
&= E\left[f(y) + f'(y)p^{1/2}L_p(j) + f''(y)\frac{pL_p^2(j)}{2} + (f''(\eta) - f''(y))\frac{pL_p^2(j)}{2}\right] \\
&= f(y) + p^{1/2}f'(y)E(L_p(1)) + \frac{pf''(y)}{2}E\left(L_p^2(j)\right) \\
&\quad + E\left[(f''(\eta) - f''(y))\frac{pL_p^2(j)}{2}\right] \\
&= f(y) + f'(y)p\mu + \frac{f''(y)}{2}p(\sigma^2 + p\mu^2) + \frac{p}{2}E\left[(f''(\eta) - f''(y))L_p^2(j)\right],
\end{aligned}$$

where $|\eta - y| < p^{1/2}|x|$. Combining (3.3) and (3.4), for $\delta > 0$, it follows that

$$\begin{aligned}
(3.5)\qquad & \left|E\left[f(p^{1/2}W_{n,j} + y)\right] - E\left[f(p^{1/2}L_p(j) + y)\right]\right| \\
&\le \frac{p}{2}E\left[|f''(\eta) - f''(y)|W_{n,j}^2\right] + \frac{p}{2}E\left[|f''(\eta) - f''(y)|L_p^2(j)\right]
\end{aligned}$$

$$\leq \frac{p}{2}E\left[|f''(\eta)-f''(y)|W_{n,j}^2\mathbf{1}_{\{|W_{n,j}|<p^{-1/2}\delta\}}\right]$$
$$+\frac{p}{2}E\left[|f''(\eta)-f''(y)|W_{n,j}^2\mathbf{1}_{\{|W_{n,j}|\geq p^{-1/2}\delta\}}\right]$$
$$+\frac{p}{2}E\left[|f''(\eta)-f''(y)|L_p^2(j)\mathbf{1}_{\{|L_p^2(j)|<p^{-1/2}\delta\}}\right]$$
$$+\frac{p}{2}E\left[|f''(\eta)-f''(y)|L_p^2(j)\mathbf{1}_{\{|L_p(j)|\geq p^{-1/2}\delta\}}\right].$$

Since $f\in C_B^2(\mathbb{R})$, for every $\epsilon>0$, there exists $\delta>0$, such that for $|\eta-y|<\delta$, we have $|f''(\eta)-f''(y)|<\epsilon$. Therefore, for $|x|<p^{-1/2}\delta$ we have $|f''(\eta)-f''(y)|<\epsilon$ and $|f''(\eta)-f''(y)|\leq 2||f''||$ for $|x|\geq p^{-1/2}\delta$. Then, from (3.5) it follows that

$$\sup_{y\in\mathbb{R}}\left|E\left[f(p^{1/2}W_{n,j}+y)\right]-E\left[f(p^{1/2}L_p(j)+y)\right]\right| \tag{3.6}$$
$$=\left|\left|A_{p^{-1/2}W_{n,j}}f-A_{p^{-1/2}L_p(j)}f\right|\right|$$
$$\leq\frac{p}{2}\epsilon E(W_{n,j}^2)+p||f''||\left[EW_{n,j}^2\mathbf{1}_{\{|W_{n,j}|\geq p^{-1/2}\delta\}}\right]$$
$$+\frac{p}{2}\epsilon EL_p^2(j)+p||f''||\left[EL_p^2(j)\mathbf{1}_{\{|L_p(j)|\geq p^{-1/2}\delta\}}\right].$$

Multiplying by $(2p^{-1})$, summing over the j's and dividing by

$$\sum_{j=1}^{n}\left(EW_{n,j}^2+EL_p^2(j)\right),$$

from (3.6) one has

$$\sum_{j=1}^{n}\left|\left|A_{p^{-1/2}W_{n,j}}f-A_{p^{-1/2}L_p(j)}f\right|\right|\left(\frac{p}{2}\left[\sum_{j=1}^{n}EW_{n,j}^2+\sum_{j=1}^{n}EL_p^2(j)\right]\right)^{-1} \tag{3.7}$$
$$\leq\epsilon+2||f''||\left(\sum_{j=1}^{n}EW_{n,j}^2\mathbf{1}_{\{|W_{n,j}|\geq p^{-1/2}\delta\}}\Big/\sum_{j=1}^{n}EW_{n,j}^2\right)$$
$$+2||f''||\left(\sum_{j=1}^{n}EL_p^2(j)\mathbf{1}_{\{|L_p(j)|\geq p^{-1/2}\delta\}}\Big/\sum_{j=1}^{n}EL_p^2(j)\right).$$

According to [9] (see formula (15.4), page 84), from (3.7), it may be concluded that

$$\sum_{n=1}^{\infty}P(\nu_p=n)E\left\{\frac{\sum_{j=1}^{n}\left|\left|A_{p^{-1/2}W_{n,j}}f-A_{p^{-1/2}L_p(j)}f\right|\right|}{\frac{p}{2}\left[\sum_{j=1}^{n}EW_{n,j}^2+\sum_{j=1}^{n}EL_p^2(j)\right]}\right\} \tag{3.8}$$
$$=E\left\{\sum_{j=1}^{\nu_p}\left|\left|A_{p^{-1/2}W_{n,j}}f-A_{p^{-1/2}L_p(j)}f\right|\right|\right\}E\left(\frac{p}{2}\left[\sum_{j=1}^{\nu_p}EW_{n,j}^2+\sum_{j=1}^{\nu_p}EL_p^2(j)\right]\right)^{-1}$$

$$\leq \epsilon + 2||f''||E\Big(\sum_{j=1}^{\nu_p} EW_{n,j}^2 \mathbf{1}_{\{|W_{n,j}|\geq p^{-1/2}\delta\}} \Big/ \sum_{j=1}^{\nu_p} EW_{n,j}^2\Big)$$

$$+ 2||f''||E\Big(\sum_{j=1}^{\nu_p} EL_p^2(j) \mathbf{1}_{\{|L_p(j)|\geq p^{-1/2}\delta\}} \Big/ \sum_{j=1}^{\nu_p} EL_p^2(j)\Big).$$

Thus, in view of Lemma 2.2, from (3.8) it may be concluded that

(3.9)

$$\Big|\Big|A_{p^{-1/2}\sum_{j=1}^{\nu_p} W_{n,j}} f - A_{p^{-1/2}\sum_{j=1}^{\nu_p} L_p} f\Big|\Big| E\Big(\frac{p}{2}\Big[\sum_{j=1}^{\nu_p} EW_{n,j}^2 + \sum_{j=1}^{\nu_p} EL_p^2(j)\Big]\Big)^{-1}$$

$$\leq E\Big\{\sum_{j=1}^{\nu_p}\Big|\Big|T_{p^{-1/2}W_{n,j}} f - T_{p^{-1/2}L_p(j)} f\Big|\Big|\Big\} E\Big(\frac{p}{2}\Big[\sum_{j=1}^{\nu_p} EW_{n,j}^2 + \sum_{j=1}^{\nu_p} EL_p^2(j)\Big]\Big)^{-1}$$

$$\leq \epsilon + 2||f''||E\Big(\sum_{j=1}^{\nu_p} EW_{n,j}^2 \mathbf{1}_{\{|W_{n,j}|\geq p^{-1/2}\delta\}} \Big/ \sum_{j=1}^{\nu_p} EW_{n,j}^2\Big)$$

$$+ 2||f''||E\Big(\sum_{j=1}^{\nu_p} EL_p^2(j) \mathbf{1}_{\{|L_p(j)|\geq p^{-1/2}\delta\}} \Big/ \sum_{j=1}^{\nu_p} EL_p^2(j)\Big)$$

for $f \in C_B^2(\mathbb{R})$.

On the left-hand side of (3.9) the term $E\Big(\frac{p}{2}\Big[\sum_{j=1}^{\nu_p} EW_{n,j}^2 + \sum_{j=1}^{\nu_p} EL_p^2(j)\Big]\Big)^{-1} \to \sigma^{-2} < +\infty$ as $p \to 0$. Since $\epsilon(> 0)$ on the right-hand side of (3.9) is arbitrary, in view of geometric Lindeberg condition for sequence of independent random variables $\{W_{n,j}, j \geq 1\}$ and sequence of i.i.d. random variables $\{L_p(j), j \geq 1\}$, it follows that

$$\Big|\Big|A_{p^{-1/2}\sum_{j=1}^{\nu_p} W_{n,j}} f - A_{p^{-1/2}\sum_{j=1}^{\nu_p} L_p} f\Big|\Big| = o(1)$$

as $p \to 0$. The proof is complete. □

The following theorem is a weak limit theorem for geometric summation of i.i.d. random variables. The proof will be given by the same way as Theorem 3.1.

Theorem 3.2 (Weak limit theorem for geometric summation of i.i.d. random variables)**.** *For $n \geq 1$, let $X_{n,1}, X_{n,2}, \ldots$ be a sequence of i.i.d. random variable with common mean μ and variance $0 < \sigma^2 < +\infty$ for $j \geq 1$. Let $\nu_p \sim Geo(p)$ be a geometric distributed random variable with mean p^{-1}, $p \in (0,1)$, independent of all $X_{n,j}$ for $j \geq 1$ and $n \geq 1$. Then (1.5) holds.*

Proof. A sufficient condition for the validity of limiting assertion (1.5) is

(3.10) $$E(\nu_p) \parallel A_{p^{-1/2}W_{n,1}} f - A_{p^{-1/2}L_p(1)} f \parallel = o(1) \quad \text{as} \quad p \to 0$$

for each $f \in C_B^2(\mathbb{R})$. Using Taylor series expansion (see [26], formula (1.4.6), page 17) for $f \in C_B^2(\mathbb{R})$ and $y \in \mathbb{R}$, we have

$$\begin{aligned}
(3.11)\quad & E\left[f(p^{1/2}W_{n,1}+y)\right]\\
&= E\left[f(y)+f'(y)p^{1/2}W_{n,1}+f''(y)\frac{pW_{n,1}^2}{2}+\Big(f''(\eta)-f''(y)\Big)\frac{pW_{n,1}^2}{2}\right]\\
&= f(y)+p^{1/2}f'E(W_{n,1})+\frac{pf''(y)}{2}E(W_{n,1}^2)\\
&\quad + E\left[\Big(f''(\eta)-f''(y)\Big)\frac{pW_{n,1}^2}{2}\right]\\
&= f(y)+f'(y)p\mu+\frac{f''(y)}{2}p(\sigma^2+p\mu^2)+\frac{p}{2}E\left[\Big(f''(\eta)-f''(y)\Big)W_{n,1}^2\right],
\end{aligned}$$

where $\mid \eta - y \mid < p^{1/2} \mid x \mid$.

By an argument analogous, for $f \in C_B^2(\mathbb{R})$, $y \in \mathbb{R}$, it follows that

$$\begin{aligned}
(3.12)\quad & E\left[f(p^{1/2}L_p(1)+y)\right]\\
&= E\left[f(y)+f'(y)p^{1/2}L_p(1)+f''(y)\frac{pL_p^2(1)}{2}+\Big(f''(\eta)-f''(y)\Big)\frac{pL_p^2(1)}{2}\right]\\
&= f(y)+p^{1/2}f'E(L_p(1))+\frac{pf''(y)}{2}E(L_p^2(1))\\
&\quad + E\left[\Big(f''(\eta)-f''(y)\Big)\frac{pL_p^2(1)}{2}\right]\\
&= f(y)+f'(y)p\mu+\frac{f''(y)}{2}p(\sigma^2+p\mu^2)+\frac{p}{2}E\left[\Big(f''(\eta)-f''(y)\Big)L_p^2(1)\right],
\end{aligned}$$

where $\mid \eta - y \mid < p^{1/2}|x|$.

Combining the (3.11) and (3.12), for $\delta > 0$, it follows that

$$\begin{aligned}
(3.13)\quad & \left|E\left[f(p^{1/2}W_{n,1}+y)\right]-E\left[f(p^{1/2}L_p(1)+y)\right]\right|\\
&\le \frac{p}{2}E\left[|f''(\eta)-f''(y)|W_{n,1}^2\right]+\frac{p}{2}E\left[|f''(\eta)-f''(y)|L_p^2(1)\right]\\
&\le \frac{p}{2}E\left[|f''(\eta)-f''(y)|W_{n,1}^2\mathbf{1}_{\{|W_{n,1}|<p^{-1/2}\delta\}}\right]\\
&\quad +\frac{p}{2}E\left[|f''(\eta)-f''(y)|W_{n,1}^2\mathbf{1}_{\{|W_{n,1}|\ge p^{-1/2}\delta\}}\right]\\
&\quad +\frac{p}{2}E\left[|f''(\eta)-f''(y)|L_p^2(1)\mathbf{1}_{\{|L_p(1)|<p^{-1/2}\delta\}}\right]\\
&\quad +\frac{p}{2}E\left[|f''(\eta)-f''(y)|L_p^2(1)\mathbf{1}_{\{|L_p(1)|\ge p^{-1/2}\delta\}}\right].
\end{aligned}$$

Since $f \in C_B^2(\mathbb{R})$, for every $\epsilon > 0$, there exists $\delta > 0$, such that for $|\eta - y| < \delta$, we have $|f''(\eta) - f''(y)| < \epsilon$. Therefore, we have $|f''(\eta) - f''(y)| < \epsilon$ for $|x| < p^{-1/2}\delta$ and $|f''(\eta) - f''(y)| \le 2||f''||$ for $|x| \ge p^{-1/2}\delta$. Then, from (3.13) it follows that

$$\begin{aligned}
(3.14) \qquad & \sup_{y\in\mathbb{R}} \Big| E\Big[f(p^{1/2}W_{n,1} + y)\Big] - E\Big[f(p^{1/2}L_p(1) + y)\Big]\Big| \\
&= \Big|\Big| A_{p^{-1/2}W_{n,1}} f - A_{p^{-1/2}L_p(1)} f \Big|\Big| \\
&\le \frac{p}{2}\epsilon E(W_{n,1}^2) + p||f''|| \Big[EW_{n,j}^2 \mathbf{1}_{\{|W_{n,1}| \ge p^{-1/2}\delta\}}\Big] \\
&\quad + \frac{p}{2}\epsilon E(L_p^2(1)) + p||f''|| \Big[EL_p^2(1)\mathbf{1}_{\{|L_p(1)| \ge p^{-1/2}\delta\}}\Big].
\end{aligned}$$

Multiplying by $2p^{-1}$, and dividing by $\Big[EW_{n,1}^2 + EL_p^2(1)\Big]$, one has

$$\begin{aligned}
(3.15) \qquad & n\Big|\Big| A_{p^{-1/2}W_{n,1}} f - A_{p^{-1/2}L_p(1)} f \Big|\Big| \times \left(n\frac{p}{2}\Big[EW_{n,1}^2 + EL_p^2(1)\Big]\right)^{-1} \\
&\le \epsilon + 2||f''||\left(EW_{n,1}^2 \mathbf{1}_{\{|W_{n,1}| \ge p^{-1/2}\delta\}} \Big/ EW_{n,1}^2\right) \\
&\quad + \epsilon + 2|f''||\left(EL_p^2(1)\mathbf{1}_{\{|L_p(1)| \ge p^{-1/2}\delta\}} \Big/ EL_p^2(1)\right).
\end{aligned}$$

According to Lemma 2.3, from (3.15), it may be concluded that

$$\begin{aligned}
(3.16) \quad & E(\nu_p)\Big|\Big| A_{p^{-1/2}W_{n,1}} f - A_{p^{-1/2}L_p(1)} f \Big|\Big| \times \left(\frac{p}{2}E(\nu_p)\big[EW_{n,1}^2 + EL_p^2(1)\big]\right)^{-1} \\
&\le \epsilon + 2||f''||\left(EW_{n,1}^2 \mathbf{1}_{\{|W_{n,1}| \ge p^{-1/2}\delta\}} \Big/ EW_{n,1}^2\right) \\
&\quad + \epsilon + 2||f''||\left(EL_p^2(1)\mathbf{1}_{\{|L_p(1)| \ge p^{-1/2}\delta\}} \Big/ EL_p^2(1)\right).
\end{aligned}$$

Since $EW_{n,1}^2$ and EL_p^2 are finite, both terms $EW_{n,1}^2 \mathbf{1}_{\{|W_{n,1}| \ge p^{-1/2}\delta\}} \to 0$ and $EL_p^2 \mathbf{1}_{\{|W_p| \ge p^{-1/2}\delta\}} \to 0$ for $p \to 0$. On the other hand, the term

$$\left(\frac{p}{2}E(\nu_p)\Big[EW_{n,1}^2 + EL_p^2(1)\Big]\right)^{-1} = \sigma^{-2}$$

is finite. Therefore, as $p \to 0$, we obtain

$$E(\nu_p)\Big|\Big| A_{p^{-1/2}W_{n,j}} f - A_{p^{-1/2}L_p(j)} f \Big|\Big| = o(1).$$

The proof is complete. $\square$

3.2. Small–o approximation estimates

The order of approximations of the distribution of geometric summation will be established by the following theorems.

Theorem 3.3 ("Small–o" approximation estimate for geometric summation of independent (not necessarily identically distributed) random variables)**.** *Under the hypotheses of Theorem 3.1 with assumptions* $E(|X_{n,j}|^r) < +\infty$ *and* $E|L_p(j)|^r < +\infty$ *for* $r \geq 2$, $j \geq 1$, $n \geq 1$, *suppose that the following condition*

$$E\Big(W_{n,j}^k\Big) = E\Big(L_p^k(j)\Big) \tag{3.17}$$

holds, for $k \geq 2$, $j \geq 1$, $n \geq 1$. *Furthermore, suppose that for* $r \geq 2$, $j \geq 1$, $n \geq 1$, *the sequence* $X_{n,1}, X_{n,1}, \ldots$ *satisfies geometric Lindeberg condition* (2.7). *Then, for any function* $f \in C_B^r(\mathbb{R})$, *as* $p \to 0$,

$$\left|\left|A_{p^{1/2}\sum\limits_{j=1}^{\nu_p} W_{n,j}} f - A_{L_{\mu,\sigma}} f\right|\right| = o\left(\frac{p^{r/2}}{r!} E\left[\sum_{j=1}^{\nu_p}\Big(E|W_{n,j}|^r + E|L_p(j)|^r\Big)\right]\right), \tag{3.18}$$

if $E\left(\sum\limits_{j=1}^{\nu_p} E|W_{n,j}|^r\right) < +\infty$ *and* $E\left(\sum\limits_{j=1}^{\nu_p} E|L_p^{(j)}|^r\right) < +\infty$ *for* $r \geq 2$.

Proof. Using Taylor series expansion (see [26], formula (1.4.6), page 17), for $f \in C_B^r(\mathbb{R})$, we have

$$\begin{aligned}
&\left|E\Big(f\big(p^{1/2}W_{n,j} + y\big)\Big) - f(y) - \sum_{k=1}^{r}\frac{f^{(k)}(y)}{k!}p^{k/2}E(W_{n,j}^k)\right| \qquad (3.19)\\
&\leq \frac{p^{r/2}}{r!}E\Big|f^{(r)}(\eta) - f^r(y)\Big|\Big|W_{n,j}^r\Big|\\
&\leq \frac{p^{r/2}}{r!}\epsilon E|W_{n,j}^r| + \frac{p^{r/2}}{r!}2||f^{(r)}||E|W_{n,j}^r|\mathbf{1}_{\{|W_{n,j}| \geq p^{-1/2}\delta\}},
\end{aligned}$$

where η is some number between y and $y + p^{1/2}x$, and $\delta > 0$ is chosen as in the proof of Theorem 3.1. By an argument analogous, for $f \in C_B^r(\mathbb{R})$, $y \in \mathbb{R}$, we have

$$\begin{aligned}
&\left|E\Big(f\big(p^{1/2}L_p(j) + y\big)\Big) - f(y) - \sum_{k=1}^{r}\frac{f^{(k)}(y)}{k!}p^{k/2}E(L_p^k(j))\right| \qquad (3.20)\\
&\leq \frac{p^{r/2}}{r!}E\Big|f^{(r)}(\eta) - f^r(y)\Big|\Big|L_p^r(j)\Big|\\
&\leq \frac{p^{r/2}}{r!}\epsilon E|L_p^r(j)| + \frac{p^{r/2}}{r!}2||f^{(r)}||E|L_p^r(j)|\mathbf{1}_{\{|L_p(j)| \geq p^{-1/2}\delta\}},
\end{aligned}$$

where η is some number between y and $y + p^{1/2}x$, and $\delta > 0$.

On account of condition (3.17), for every $\epsilon > 0$, $\delta > 0$, $y \in \mathbb{R}$ we obtain

$$
\begin{aligned}
&\sup_{y\in\mathbb{R}} \Big| Ef(p^{1/2}W_{n,j}+y) - Ef(p^{1/2}L_p(j)+y)\Big| \\
&= \Big|\Big| A_{p^{1/2}W_{n,j}}f - A_{p^{1/2}L_p(j)}f \Big|\Big| \\
&\le \epsilon \frac{p^{r/2}}{r!}\left(E|W^r_{n,j}| + E|L^r_p(j)|\right) \\
&\quad + 2||f^{(r)}|| \frac{p^{r/2}}{r!}\left[E\left(|W^r_{n,j}|\mathbf{1}_{\{|W_{n,j}|\ge \delta p^{-1/2}\}}\right) + E\left(|L^r_p(j)|\mathbf{1}_{\{|L_p(j)|\ge \delta p^{-1/2}\}}\right)\right].
\end{aligned} \tag{3.21}
$$

Multiplying by $r!p^{-r/2}$, summing over the j's and dividing by

$$\sum_{j=1}^{n}\left[E|W^r_{n,j}| + E|L^r_p(j)|\right],$$

from inequality (3.21), one has

$$
\begin{aligned}
&\sum_{j=1}^{n} \Big|\Big| A_{p^{-1/2}W_{n,j}}f - A_{p^{-1/2}L_p(j)}f \Big|\Big| \\
&\quad \times \left(\frac{p^{r/2}}{r!}\left[\sum_{j=1}^{n} E|W^r_{n,j}| + E|L^r_p(j)| \right]\right)^{-1} \\
&\le \epsilon + 2||f^{(r)}|| \left(\sum_{j=1}^{n} E\left(|W^r_{n,j}|\mathbf{1}_{\{|W_{n,j}|\ge\delta p^{-1/2}\}}\right)\right) \Big/ \left(\sum_{j}^{n} E|W^r_{n,j}|\right) \\
&\quad + 2||f^{(r)}|| \left(\sum_{j=1}^{n} E\left(|L^r_p(j)|\mathbf{1}_{\{|L_p(j)|\ge\delta p^{-1/2}\}}\right)\right) \Big/ \left(\sum_{j}^{n} E|L^r_p(j)|\right).
\end{aligned} \tag{3.22}
$$

According to Lemma 2.2, from (3.22), we have

$$
\begin{aligned}
&\Big|\Big| A_{p^{-1/2}\sum_{j=1}^{\nu_p} W_{n,j}} f - A_{p^{-1/2}\sum_{j=1}^{\nu_p} L_p(j)} f \Big|\Big| \\
&\quad \times \left(\frac{p^{r/2}}{r!} E\left[\sum_{j=1}^{\nu_p} \left(E|W^r_{n,j}| + E|L^r_p(j)|\right)\right]\right)^{-1} \\
&\le \epsilon + 2||f^r|| E\left(\sum_{j=1}^{\nu_p} E\left(|W^r_{n,j}|\mathbf{1}_{\{|W_{n,j}|\ge\delta p^{-1/2}\}}\right) \Big/ \sum_{j}^{\nu_p} E|W^r_{n,j}|\right)
\end{aligned} \tag{3.23}
$$

$$+ 2||f^r||E\left(\sum_{j=1}^{\nu_p} E\left(|L_p^r(j)|\mathbf{1}_{\{|L_p(j)|\geq \delta p^{-1/2}\}}\right) \Big/ \sum_{j}^{\nu_p} E|L_p^r(j)|\right).$$

Finally, the validity of geometric Lindeberg's condition (2.7) for the sequence $W_{n,1}, W_{n,2}, \ldots$ and the sequences of i.i.d. random variables $L_p^{(1)}, L_p^{(2)}, \ldots$, shows that

$$\left|\left|A_{p^{1/2}\sum\limits_{j=1}^{\nu_p} W_{n,j}} f - A_{L_{\mu,\sigma}} f\right|\right| = o\left(\frac{p^{r/2}}{r!}E\left[\sum_{j=1}^{\nu_p}\left(E|W_{n,j}^r| + E|L_p^r(j)|\right)\right]\right).$$

The proof is complete. □

Theorem 3.4 ("Small–o" approximation estimate for geometric summation of i.i.d. random variables)**.** *Under the hypotheses of Theorem 3.2 with assumption that $E(|X_{n,1}^r|) < +\infty$ for $r \geq 2$, $n \geq 1$, suppose that the following condition*

$$E\left(W_{n,1}^k\right) = E\left(L_p^k(1)\right), \tag{3.24}$$

holds for $1 \leq k \leq r, r \geq 2, j \geq 1, n \geq 1$, where $W_{n,1} = X_{n,1} - \mu + p^{1/2}\mu$ and the sequence of i.i.d. random variables $L_p(1), L_p(2), \ldots$, is defined by Lemma 2.1.

Then, for any function $f \in C_B^r(\mathbb{R})$,

$$\left|\left|A_{p^{1/2}\sum\limits_{i=1}^{\nu_p} W_{n,1}} f - A_{L_{\mu,\sigma}} f\right|\right| = o\left(p^{\frac{r-2}{2}}\left[E|W_{n,1}^r| + E|L_p^r(1)|\right]\right) \tag{3.25}$$

as $p \to 0$.

Proof. Applying Trotter operator to Taylor series expansion (see [26], formula (1.4.6), page 17) for $f \in C_B^r(\mathbb{R})$, we see that

$$\begin{aligned}
&\left|Ef(p^{1/2}W_{n,1} + y) - f(y) - \sum_{i=1}^{r}\frac{f^{(k)}(y)}{k!}p^{k/2}E(W_{n,1}^k)\right| \\
&\leq E\left\{\left|f^{(r)}(\eta) - f^{(r)}(y)\right|\frac{p^{r/2}}{r!}|W_{n,1}^r|\mathbf{1}_{\{|W_{n,1}|<\delta p^{-1/2}\}}\right\} \\
&\quad + E\left\{\left|f^{(r)}(\eta) - f^{(r)}(y)\right|\frac{p^{r/2}}{r!}|W_{n,1}^r|\mathbf{1}_{\{|W_{n,1}|\geq\delta p^{-1/2}\}}\right\} \\
&\leq \epsilon\frac{p^{r/2}}{r!}E|W_{n,1}^r| + 2\frac{||f^{(r)}||}{r!}p^{r/2}E\left[|W_{n,1}^r|\mathbf{1}_{\{|W_{n,1}|\geq\delta p^{-1/2}\}}\right].
\end{aligned} \tag{3.26}$$

Note that for the inequalities in (3.26) the estimations are used as follows: Since $f \in C_B^r(\mathbb{R})$, for each $\epsilon > 0$ and each $p \in (0,1)$ there exists $\delta > 0$ such that $|f^{(r)}(\eta) - f^{(r)}(y)| < \epsilon$ for $|x| < \delta p^{-1/2}$ and $|f^{(r)}(\eta) - f^{(r)}(y)| \leq 2||f^r||$ for $|x| \geq \delta p^{-1/2}$.

Analogously, since $f \in C_B^r(\mathbb{R})$, we get

$$\left|Ef(p^{1/2}L_p(1) + y) - f(y) - \sum_{k=1}^{r}\frac{f^{(k)}(y)}{k!}p^{k/2}EL_p^k(1)\right| \tag{3.27}$$

$$\leq E\left\{\left|f^{(r)}(\eta)-f^{(r)}(y)\right|\frac{p^{r/2}}{r!}|L_p^r(1)|\mathbf{1}_{\{|L_p(1)|<\delta p^{-1/2}\}}\right\}$$
$$+E\left\{\left|f^{(r)}(\eta)-f^{(r)}(y)\right|\frac{p^{r/2}}{r!}|L_p^r(1)|\mathbf{1}_{\{|L_p(1)|\geq\delta p^{-1/2}\}}\right\}$$
$$\leq \epsilon\frac{p^{r/2}}{r!}E|L_p^r(1)|+2\frac{||f^{(r)}||}{r!}p^{r/2}E\left[|L_p^r(1)|\mathbf{1}_{\{|L_p(1)|\geq\delta p^{-1/2}\}}\right].$$

Combining (3.26) and (3.27), in view of (3.24), it follows that

(3.28)
$$\sup_{y\in\mathbb{R}}\left|Ef(p^{1/2}W_{n,1}+y)-Ef(p^{1/2}L_p(1)+y)\right|$$
$$=\left|\left|A_{p^{1/2}W_{n,1}}f-A_{L_p(1)}f\right|\right|$$
$$\leq \epsilon\frac{p^{r/2}}{r!}\left[E|W_{n,1}^r|+E|L_p^r(1)|\right]$$
$$+2\frac{||f^{(r)}||}{r!}p^{r/2}\left(E|W_{n,1}^r|\mathbf{1}_{\{|W_{n,1}|\geq\delta p^{-1/2}\}}+E|L_p^r(1)|\mathbf{1}_{\{|L_p(1)|\geq\delta p^{-1/2}\}}\right).$$

Multiplying by $r!p^{-r/2}$ and dividing by $\left[E|W_{n,1}^r|+E|L_p^r(1)|\right]>0$, one has

$$n\left|\left|A_{p^{1/2}W_{n,1}}f-A_{L_p(1)}f\right|\right|\times\left(r!p^{-r/2}n\left[E|W_{n,1}^r|+E|L_p^r(1)|\right]\right)^{-1} \tag{3.29}$$
$$\leq \epsilon+2||f^{(r)}||\left[E|W_{n,1}^r|\mathbf{1}_{\{|W_{n,1}|\geq\delta p^{-1/2}\}}\Big/E|W_{n,1}^r|\right]$$
$$+2||f^{(r)}||\left[E|L_p^r(1)|\mathbf{1}_{\{|L_p(1)|\geq\delta p^{-1/2}\}}\Big/|L_p^r(1)|\right].$$

In view of Lemma 2.3, from (3.29), we obtain

(3.30)
$$E(\nu_p)\left|\left|A_{p^{1/2}W_{n,1}}f-A_{L_p(1)}f\right|\right|\times\left(\frac{p^{r/2}}{r!}E(\nu_p)\left[E|W_{n,1}^r|+E|L_p^r(1)|\right]\right)^{-1}$$
$$\leq \epsilon+2||f^{(r)}||\left[E|W_{n,1}^r|\mathbf{1}_{\{|W_{n,1}|\geq\delta p^{-1/2}\}}\Big/E|W_{n,1}^r|\right]$$
$$+2||f^{(r)}||\left[E|L_p^r(1)|\mathbf{1}_{\{|L_p(1)|\geq\delta p^{-1/2}\}}\Big/|L_p^r(1)|\right].$$

Since $E|W_{n,1}^r|$ and $E|L_p^r(1)|$ are finite, therefore both terms

$$E|W_{n,1}^r|\mathbf{1}_{\{|W_{n,1}|\geq\delta p^{-1/2}\}}\to 0$$

and

$$E|L_p^r(1)|\mathbf{1}_{\{|L_p(1)|\geq\delta p^{-1/2}\}}\to 0$$

when $p \to 0$. It follows that

$$\left\|A_{p^{1/2}\sum_{i=1}^{\nu_p}(X_{n,i}-\mu+p^{1/2}\mu)}f - A_{L_{\mu,\sigma}}f\right\| = o\left(p^{\frac{r-2}{2}}\left[E|W_{n,1}^r| + E|L_p^r(1)|\right]\right)$$

as $p \to 0$. The proof is complete. □

Remark 3.5. In case of $r = 2$, Theorem 3.4 deduces Theorem 3.2.

3.3. Large–$\mathcal{O}$ approximation estimates

The order of approximations of the distributions of geometric summation for sequence of independent (not necessarily identically distributed) random variables $X_{n,1}, X_{n,2}, \ldots, n \geq 1$, may be defined by following theorem.

Theorem 3.6 ("Large–$\mathcal{O}$" approximation estimate for geometric summation of independent (not necessarily identically distributed) random variables)**.** *Let $X_{n,1}, X_{n,2}, \ldots$ be a sequence of independent (not necessarily identically distributed) random variables with $E|X_{n,j}|^r < +\infty$, for $r \geq 3$, $j \geq 1$, $n \geq 1$. Suppose that for $1 \leq j \leq r-1$, $r \geq 3$ the condition* (3.17) *holds. Furthermore, assume that $E\left(\sum_{j=1}^{\nu_p} E|W_{n,j}^r|\right) < +\infty$ and $E\left(\sum_{j=1}^{\nu_p} E|L_p^r(j)|\right) < +\infty$ for $r \geq 3$. Then, for any function $f \in C_B^{r-1}(\mathbb{R})$, as $p \to 0$,*

$$\left\|A_{p^{1/2}\sum_{j=1}^{\nu_p}W_{n,j}}f - A_{L_{\mu,\sigma}}f\right\| \leq \frac{2p^{\frac{r-1}{2}}}{(r-1)!}\omega\left(f^{(r-1);p^{1/2}}\right)E\left(\sum_{j=1}^{\nu_p}\left[E|W_{n,j}^r| + E|L_p^r(j)| + 1\right]\right), \tag{3.31}$$

where $W_{n,j} = X_{n,j} - \mu_j + p^{1/2}\mu_j$ for $n \geq 1$, $j \geq 1$, and $L_p^{(1)}, L_p^{(2)}, \ldots$ are i.i.d. random variables, defined by Lemma 2.1.

Let in addition $f^{(r-1)} \in Lip(\alpha), 0 < \alpha \leq 1$. Then

$$\left\|A_{p^{1/2}\sum_{j=1}^{\nu_p}W_{n,j}}f - A_{L_{\mu,\sigma}}f\right\| = \mathcal{O}\left(p^{(r-1+\alpha)/2}E\left[\sum_{j=1}^{\nu_p}\left(E|W_{n,j}|^r + E|L_p^{(j)}|^r + 1\right)\right]\right) \tag{3.32}$$

as $p \to 0$.

Proof. Analogously as in the beginning of the proof of Theorem 3.3, using properties of the modulus of continuity $\omega(f^{(r-1)}; p^{1/2})$ and inequality (2.4), we have

$$\left|Ef(p^{1/2}W_{n,j} + y) - \sum_{k=1}^{r-1}\frac{p^{k/2}}{i!}f^{(k)}E(W_{n,j})^k\right| \tag{3.33}$$

$$\le \frac{p^{(r-1)/2}}{(r-1)!} E\left[|f^{(r-1)}(\eta) - f^{(r-1)}(y)||W_{n,j}|^{r-1}\right]$$
$$\le \frac{p^{(r-1)/2}}{(r-1)!} \omega(f^{(r-1)}; p^{1/2}) \Big(1 + 2E|W_{n,j}^r|\Big),$$

where $|\eta - y| < p^{1/2}|x|$.

Analogously,

$$\left|Ef(p^{1/2}L_p(j) + y) - \sum_{k=1}^{r-1} \frac{p^{k/2}}{k!} f^{(k)} E(L_p^k(j))\right| \tag{3.34}$$
$$\le \frac{p^{(r-1)/2}}{(r-1)!} E\left[|f^{(r-1)}(\eta) - f^{(r-1)}(y)||L_p^{r-1}(j)|\right]$$
$$\le \frac{p^{(r-1)/2}}{(r-1)!} \omega(f^{(r-1)}; p^{1/2}) \Big(1 + 2E|L_p^r(j)|\Big),$$

where $|\eta - y| < p^{1/2}|x|$. Combining (3.33) and (3.34) with assumption (3.17) gives, for $f \in C_B^{(r-1)}(\mathbb{R})$,

$$\Big|Ef(p^{1/2}W_{n,j} + y) - Ef(p^{1/2}L_p(j))\Big| \tag{3.35}$$
$$\le \frac{2p^{(r-1)/2}}{(r-1)!} \omega(f^{(r-1)}; p^{1/2}) \Big[E|W_{n,j}^r| + E|L_p^r(j)| + 1\Big].$$

Then, in view of (3.35), for $f \in C_B^{r-1}(\mathbb{R})$, it follows that

$$\sum_{j=1}^{n} \Big|\Big|A_{p^{1/2}W_{n,j}} f - A_{p^{1/2}L_p^{(j)}} f\Big|\Big| \tag{3.36}$$
$$\le \frac{2p^{(r-1)/2}}{(r-1)!} \omega(f^{(r-1)}; p^{1/2}) \sum_{j=1}^{n} \Big[E|W_{n,j}^r| + E|L_p^r(j)| + 1\Big].$$

According to Lemma 2.2, from (3.36) and based on assumption that $\nu_p, X_{n,j}$, $L_p(j)$ are independent for all $n \ge 1, j \ge 1$ and for each $p \in (0,1)$, we have for $f \in C_B^{r-1}(\mathbb{R})$,

$$E\Big(\sum_{j=1}^{\nu_p} \Big|\Big|A_{p^{1/2}W_{n,j}} f - A_{p^{1/2}L_p^{(j)}} f\Big|\Big|\Big) \tag{3.37}$$
$$\le \frac{2p^{(r-1)/2}}{(r-1)!} \omega(f^{(r-1)}; p^{1/2}) E\left(\sum_{j=1}^{\nu_p} \Big[E|W_{n,j}^r| + E|L_p^r(j)| + 1\Big]\right).$$

Finally, using properties of Trotter operator, from inequality (3.37), for $f \in C_B^{r-1}(\mathbb{R})$, we obtain the final estimate as $p \to 0$,

$$\Big|\Big|A_{p^{1/2}\sum_{j=1}^{\nu_p} W_{n,j}} f - A_{L_{\mu,\sigma}} f\Big|\Big|$$

$$\leq \frac{2p^{(r-1)/2}}{(r-1)!}\omega(f^{(r-1)};p^{1/2})E\left(\sum_{j=1}^{\nu_p}\left[E|W_{n,j}^r|+E|L_p^r(j)|+1\right]\right).$$

It is clear that, if $f^{(r-1)} \in Lip(\alpha), (0 < \alpha \leq 1)$, then

$$\left\|A_{p^{1/2}\sum_{j=1}^{\nu_p} W_{n,j}} f - A_{L_{\mu,\sigma}} f\right\|$$
$$= \mathcal{O}\left(p^{(r-1+\alpha)/2}E\left[\sum_{j=1}^{\nu_p}\left(E|W_{n,j}^r|+E|L_p^r(j)|+1\right)\right]\right)$$

as $p \to 0$. The proof is complete. □

Theorem 3.7 ("Large–$\mathcal{O}$" approximation estimate for geometric summation of i.i.d. random variables)**.** *Under the hypotheses of Theorem 3.4 with assumption that* $E(|X_{n,1}|^r) < +\infty$ *for some fixed* $r \geq 3$, $r \in \mathbb{N}$, *assume that for* $1 \leq j \leq r$, $r \geq 3$, *the condition* (3.26) *holds. Then, for any function* $f \in C_B^{r-1}(\mathbb{R})$,

$$\left\|A_{p^{1/2}\sum_{j=1}^{\nu_p} W_{n,j}} f - A_{L_{\mu,\sigma}} f\right\| = \mathcal{O}\left(\frac{p^{(r-3)/2}}{(r-1)!}\omega(f^{(r-1)},p^{1/2})\right) \tag{3.38}$$

as $p \to 0$.

If, in addition $f^{(r-1)} \in Lip(\alpha), 0 < \alpha \leq 1$, *then*

$$\left\|A_{p^{1/2}\sum_{j=1}^{\nu_p} W_{n,j}} f - A_{L_{\mu,\sigma}} f\right\| = \mathcal{O}\left(\frac{p^{(r-3+\alpha)/2}}{(r-1)!}\right) \tag{3.39}$$

as $p \to 0$.

Proof. Analogously as in the beginning of the proof of Theorem 3.4, using properties of the modulus of continuity $\omega(f^{(r-1)};p^{1/2})$, we have

$$\left|Ef(p^{1/2}W_{n,1}+y) - \sum_{k=1}^{r-1}\frac{p^{k/2}}{k!}f^{(k)}E(W_{n,1}^k)\right| \tag{3.40}$$
$$\leq \frac{p^{(r-1)/2}}{(r-1)!}E\left[|f^{(r-1)}(\eta)-f^{(r-1)}(y)||W_{n,1}^{r-1}|\right]$$
$$\leq \frac{p^{(r-1)/2}}{(r-1)!}\omega(f^{(r-1)};p^{1/2})(1+2E|W_{n,1}^r|).$$

Analogously,

$$\left|Ef(p^{1/2}L_p^{(1)}+y) - \sum_{k=1}^{r-1}\frac{p^{k/2}}{k!}f^{(k)}EL_p^k(1)\right| \tag{3.41}$$
$$\leq \frac{p^{(r-1)/2}}{(r-1)!}E\left[|f^{(r-1)}(\eta)-f^{(r-1)}(y)||L_p^{r-1}(1)|\right]$$

$$\leq \frac{p^{(r-1)/2}}{(r-1)!}\omega(f^{(r-1)}; p^{1/2})(1 + 2E|L_p^r(1)|).$$

Combining (3.40) and (3.41) with assumption (3.24) and in view of Lemma 2.3, for $f \in C_B^{(r-1)}(\mathbb{R})$, it may be concluded that

$$\begin{aligned}(3.42)\qquad & \left\|A_{p^{1/2}\sum_{j=1}^{\nu_p} W_{n,j}} f - A_{L_{\mu,\sigma}} f\right\| \\ &\leq E(\nu_p)\left\|T_{p^{1/2}W_{n,1}} f - T_{p^{1/2}L_p^1} f\right\| \\ &\leq 2\frac{p^{(r-3)/2}}{(r-1)!}\omega(f^{(r-1)}, p^{1/2})\left[E|W_{n,1}^r| + E|L_p^r(1)| + 1\right].\end{aligned}$$

According to boundness of $E|W_{n,1}^r|$ and $E|L_p^r(1)|$, when $p \to 0$, it follows that

$$\left\|A_{p^{1/2}\sum_{j=1}^{\nu_p} W_{n,j}} f - A_{L_{\mu,\sigma}} f\right\| = \mathcal{O}\left(\frac{p^{(r-3)/2}}{(r-1)!}\omega(f^{(r-1)}, p^{1/2})\right).$$

In view of $f^{(r-1)} \in Lip(\alpha), 0 < \alpha \leq 1$, from Definition 2.2, when $p \to 0$, it may be concluded that,

$$\left\|A_{p^{1/2}\sum_{j=1}^{\nu_p} W_{n,j}} f - A_{L_{\mu,\sigma}} f\right\| = \mathcal{O}\left(\frac{p^{(r-3+\alpha)/2}}{(r-1)!}\right).$$

This finishes the proof. □

Acknowledgments. The author is grateful to the anonymous referees for carefully reading the manuscript and for offering comments with useful suggestions which allowed him to substantially improve the paper.

Appendix

For $p \in (0,1)$, A random variable ν_p is said to have geometric distribution with mean p^{-1}, denoted by $\nu_p \sim \text{Geo}(p)$, if its probability distribution is given by

$$\mathbb{P}(\nu_p = k) = p(1-p)^{k-1}, \quad k = 1, 2, \ldots$$

Proposition A1. *Let* $\nu_p \sim \text{Geo}(p)$. *Then*

$$\nu_p \xrightarrow{P} +\infty \quad as \quad p \to 0^+.$$

Proof. Let M be an arbitrary integer number. Then

$$\begin{aligned}\mathbb{P}(\nu_p > M) &= 1 - \mathbb{P}(\nu_p \leq M) = 1 - \sum_{k=1}^{M}\mathbb{P}(\nu_p = k) \\ &= 1 - \sum_{k=1}^{M}\left\{p(1-p)^{k-1}\right\} = 1 - \frac{p}{1-p}\sum_{k=1}^{M}(1-p)^k.\end{aligned}$$

It is clear that

$$\sum_{k=1}^{M}(1-p)^k = \frac{(1-p)\Big[(1-p)^M - 1\Big]}{1-(1-p)} = \frac{1-p}{p}\Big[(1-p)^M - 1\Big].$$

Therefore

$$\begin{aligned}\mathbb{P}(\nu_p > M) &= 1 - \frac{p}{1-p}\sum_{k=1}^{M}(1-p)^k \\ &= 1 - \frac{p}{1-p} \times \frac{1-p}{p}\Big[(1-p)^M - 1\Big] \\ &= 2 - (1-p)^M \longrightarrow 1 \quad \text{khi} \quad p \to 0^+.\end{aligned}$$

Thus

$$\nu_p \xrightarrow{P} +\infty \quad \text{khi} \quad p \to 0^+. \qquad \square$$

Proposition A2. *Let $X_{n,1}, X_{n,2}, \ldots$ be a sequence of independent random variables with $E|X_{n,j}^2| < +\infty$. Let $\nu_p \sim Geo(p)$, $p \in (0,1)$, be a geometric random variable with mean p^{-1}, independent of $X_{n,j}$ for $n \geq 1$, $j \geq 1$. Suppose that, for $\delta > 0$, the geometric Lindeberg condition*

$$E\bigg(\sum_{j=1}^{\nu_p} EX_{n,j}^2 \mathbf{1}_{\{|X_{n,j}| \geq \delta p^{-1/2}\}} \Big/ \sum_{j=1}^{\nu_p} EX_{n,j}^2\bigg) = o(1) \quad \textit{as} \quad p \to 0$$

holds. Then

$$\sum_{j=1}^{\infty} P(\nu_p = j) EX_{n,j}^2 \mathbf{1}_{\{|X_{n,j}| \geq \delta p^{-1/2}\}} = o(1) \quad \textit{as} \quad p \to 0.$$

Proof. It is clear that

$$\begin{aligned}0 &\leq \sum_{j=1}^{\infty} P(\nu_p = j)\bigg(\sum_{j=1}^{n} EX_{n,j}^2 \mathbf{1}_{\{|X_{n,j}| \geq \delta p^{-1/2}\}} \Big/ \sum_{j=1}^{n} EX_{n,j}^2\bigg) \\ &\leq E\bigg(\sum_{j=1}^{\nu_p} EX_{n,j}^2 \mathbf{1}_{\{|X_{n,j}| \geq \delta p^{-1/2}\}} \Big/ \sum_{j=1}^{\nu_p} EX_{n,j}^2\bigg).\end{aligned}$$

In view of finiteness of $EX_{n,j}^2$, the proof is straightforward. $\square$

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