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Existence, Blow-up and Exponential Decay Estimates for the Nonlinear Kirchhoff-Carrier Wave Equation in an Annular with Robin-Dirichlet Conditions

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Abstract. This paper is devoted to the study of a nonlinear Kirchhoff-Carrier wave equation in an annulus associated with Robin-Dirichlet conditions. At first, by applying the Faedo-Galerkin method, we prove existence and uniqueness results. Then, by constructing a Lyapunov functional, we prove a blow up result for solutions with a negative initial energy and establish a sufficient condition to obtain the exponential decay of weak solutions.

1. Introduction

In this paper, we consider the following nonlinear Kirchhoff-Carrier wave equa-

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tion on an annulus

$$u_{tt} - \mu(t, u(1,t), \|u(t)\|_0^2, \|u_x(t)\|_0^2)(u_{xx} + \frac{1}{x}u_x) = f(x,t,u,u_x,u_t), \quad 0 < x < 1,\ 0 < t < T, \tag{1.1}$$

associated with Robin-Dirichlet conditions

$$u(\rho,t) = u_x(1,t) + \zeta u(1,t) = 0, \tag{1.2}$$

and initial conditions

$$u(x,0) = \tilde{u}_0(x),\ u_t(x,0) = \tilde{u}_1(x), \tag{1.3}$$

where μ, f, $\tilde{u}_0$, $\tilde{u}_1$ are given functions; and $\rho \in (0,1)$ and $\zeta \geq 0$ are given constants. In Eq. (1.1), the nonlinear term $\mu(t, u(1,t), \|u(t)\|_0^2, \|u_x(t)\|_0^2)$ depends on the integrals $\|u(t)\|_0^2 = \int_\rho^1 xu^2(x,t)\,dx$ and $\|u_x(t)\|_0^2 = \int_\rho^1 xu_x^2(x,t)\,dx$.

Eq. (1.1) above is the bidimensional nonlinear wave equation describing nonlinear vibrations of the annular membrane $\Omega_1 = \{(x,y) : \rho^2 < x^2 + y^2 < 1\}$. In the vibration processing, the area of the annular membrane and the tension at various points change in time. The condition $u_x(1,t) + \zeta u(1,t) = 0$ on the boundary $\Gamma_1 = \{(x,y) : x^2 + y^2 = 1\}$ describes elastic constraints where the constant ζ has a mechanical signification. With the boundary condition on $\Gamma_\rho = \{(x,y) : x^2 + y^2 = \rho^2\}$ requiring $u(\rho,t) = 0$, the annular membrane is fixed.

Eq. (1.1) has its origin in the nonlinear vibration of an elastic string (Kirchhoff [6]), for which the associated equation is

$$\rho h u_{tt} = \left(P_0 + \frac{Eh}{2L} \int_0^L \left| \frac{\partial u}{\partial y}(y,t) \right|^2 dy \right) u_{xx}, \tag{1.4}$$

here u is the lateral deflection, L is the length of the string, h is the crosssectional area, E is Young's modulus, ρ is the mass density, and P_0 is the initial tension.

In [3], Carrier established the equation which models vibrations of an elastic string when changes in tension are not small

$$\rho u_{tt} - \left(1 + \frac{EA}{LT_0} \int_0^L u^2(y,t)dy \right) u_{xx} = 0, \tag{1.5}$$

where $u(x,t)$ is the x-derivative of the deformation, T_0 is the tension in the rest position, E is the Young modulus, A is the cross-section of a string, L is the length of the string and ρ is the density of the material. Clearly, if properties of a material vary with x and t, then there is a hyperbolic equation of the type ([7])

$$u_{tt} - \mu\left(x, t, \int_0^1 u^2(y,t)\,dy \right) u_{xx} = 0. \tag{1.6}$$

The Kirchhoff-Carrier equations of the form Eq. (1.1) have received much attention. We refer the reader to, e.g., [1], [2], [4], [5], [7], [8], [11] - [14], [19] - [21], [23], [24] for many interesting results and further references. In these works, the results concerning local existence, global existence, asymptotic expansion, asymptotic behavior, decay and blow-up of solutions have been examined.

In [14], Messaoudi established a blow-up result for solutions with negative initial energy and a global existence result for arbitrary initial data of a nonlinear viscoelastic wave equation associated with initial and Dirichlet boundary conditions.

In [11], [23], [24], the existence, regularity, blow-up, and exponential decay estimates of solutions for nonlinear wave equations associated with two-point boundary conditions have been established. The proofs are based on the Galerkin method associated to a priori estimates, weak convergence, and compactness techniques and also by the construction of a suitable Lyapunov functional. The authors in [23], [24] proved that any weak solution with negative initial energy will blow up in finite time.

The paper consists of four sections. Preliminaries are done in Section 2, with the notations, definitions, list of appropriate spaces and required lemmas. The main results are presented in Sections 3 and 4.

In Sections 3, by combining the linearization method for nonlinear terms, the Faedo-Galerkin method and the weak compact method, we prove that Prob. (1.1) - (1.3) has a unique weak solution.

In Sections 4 and 5, Prob. (1.1) - (1.3) is considered in the case

$$\zeta = 0,\ f = -\lambda u_t + f(u) + F(x,t), \tag{1.7}$$

with constant $\lambda > 0$. In Section 4, with $F(x,t) \equiv 0$ and a negative initial energy, we prove that the solution of (1.1)-(1.3) and (1.7) blows up in finite time. In Section 5, we give a sufficient condition, in which the initial energy is positive and small, any the global weak solution is exponential decaying. In the proof, a suitable Lyapunov functional is constructed. Our results can be regarded as an extension and improvement of the corresponding results of [10], [11], [15] - [18], [23], [24].

2. Preliminaries

First, put $\Omega = (\rho, 1)$, $Q_T = \Omega \times (0,T)$, $T > 0$. We omit the definitions of the usual function spaces and denote them by the notations $L^p = L^p(\Omega)$, $H^m = H^m(\Omega)$. Let $(\cdot,\cdot)$ be a scalar product in L^2. The notation $\|\cdot\|$ stands for the norm in L^2 and we let $\|\cdot\|_X$ denote the norm in the Banach space X. We call X' the dual space of X. We let $L^p(0,T;X)$, $1 \leq p \leq \infty$ denote the Banach space of real measurable functions $u : (0,T) \to X$ such that $\|u\|_{L^p(0,T;X)} < +\infty$, with

$$\|u\|_{L^p(0,T;X)} = \begin{cases} \left(\int_0^T \|u(t)\|_X^p \, dt\right)^{1/p}, & \text{if} \quad 1 \leq p < \infty, \\ \underset{0<t<T}{ess\sup} \|u(t)\|_X, & \text{if} \quad p = \infty. \end{cases}$$

With $f \in C^k([\rho,1] \times \mathbb{R}_+ \times \mathbb{R}^3)$, $f = f(x,t,y_1,y_2,y_3)$, we put $D_1 f = \frac{\partial f}{\partial x}$, $D_2 f = \frac{\partial f}{\partial t}$, $D_{i+2} f = \frac{\partial f}{\partial y_i}$ with $i = 1, \cdots, 3$, and $D^\alpha f = D_1^{\alpha_1} \cdots D_5^{\alpha_5} f$, $\alpha = (\alpha_1, \cdots, \alpha_5) \in \mathbb{Z}_+^5$, $|\alpha| = \alpha_1 + \cdots + \alpha_5 = k$, $D^{(0,\cdots,0)} f = f$;

With $\mu \in C^k([0,T^*] \times \mathbb{R} \times \mathbb{R}_+^2)$, $\mu = \mu(t,y_1,y_2,y_3)$, we put $D_1 \mu = \frac{\partial \mu}{\partial t}$, $D_{i+1} \mu = \frac{\partial \mu}{\partial y_i}$ with $i = 1, \cdots, 3$, and $D^\beta \mu = D_1^{\beta_1} \cdots D_4^{\beta_4} \mu$, $\beta = (\beta_1, \cdots, \beta_4) \in \mathbb{Z}_+^4$, $|\beta| = \beta_1 + \cdots + \beta_4 = k$, $D^{(0,\cdots,0)} \mu = \mu$.

On H^1, H^2, we shall use the following norms

$$\|v\|_{H^1} = \left(\|v\|^2 + \|v_x\|^2 \right)^{\frac{1}{2}}, \tag{2.1}$$

and

$$\|v\|_{H^2} = \left(\|v\|^2 + \|v_x\|^2 + \|v_{xx}\|^2 \right)^{\frac{1}{2}}, \tag{2.2}$$

respectively.

We remark that L^2, H^1, H^2 are the Hilbert spaces with respect to the corresponding scalar products

$$\langle u, v \rangle = \int_\rho^1 x u(x) v(x) \, dx, \ \langle u, v \rangle + \langle u_x, v_x \rangle, \ \langle u, v \rangle + \langle u_x, v_x \rangle + \langle u_{xx}, v_{xx} \rangle. \tag{2.3}$$

The norms in L^2, H^1 and H^2 induced by the corresponding scalar products (2.3) are denoted by $\|\cdot\|_0$, $\|\cdot\|_1$ and $\|\cdot\|_2$.

Consider the set

$$V = \left\{ v \in H^1 : v(\rho) = 0 \right\}. \tag{2.4}$$

It is obvious that V is a closed subspace of H^1 and on V two norms $\|v\|_{H^1}$ and $\|v_x\|$ are equivalent norms. On the other hand, V is continuously and densely embedded in L^2. Identifying L^2 with $(L^2)'$ (the dual of L^2), we have $V \hookrightarrow L^2 \hookrightarrow V'$. We note more that the notation $\langle \cdot, \cdot \rangle$ is also used for the pairing between V and V'.

We then have the following lemmas, the proofs of which can be found in the paper [17].

Lemma 2.1. *The following inequalities are fulfilled*

(i) $\sqrt{\rho} \|v\| \leq \|v\|_0 \leq \|v\|$ *for all* $v \in L^2$,

(ii) $\sqrt{\rho} \|v\|_{H^1} \leq \|v\|_1 \leq \|v\|_{H^1}$ *for all* $v \in H^1$.

Lemma 2.2. *The embedding* $V \hookrightarrow C^0(\overline{\Omega})$ *is compact and for all* $v \in V$, *we have*

(i) $\|v\|_{C^0(\overline{\Omega})} \leq \sqrt{1-\rho} \|v_x\|$,

(ii) $\|v\| \leq \frac{1-\rho}{\sqrt{2}} \|v_x\|$,

(iii) $\|v\|_0 \leq \frac{1-\rho}{\sqrt{2\rho}} \|v_x\|_0$,

(iv) $\|v_x\|_0^2 + v^2(1) \geq \|v\|_0^2$,

(v) $|v(1)| \leq \sqrt{3} \|v\|_1$.

Remark 2.3. On L^2, two norms $v \mapsto \|v\|$ and $v \mapsto \|v\|_0$ are equivalent. So are two norms $v \mapsto \|v\|_{H^1}$ and $v \mapsto \|v\|_1$ on H^1, and five norms $v \mapsto \|v\|_{H^1}$, $v \mapsto \|v\|_1$, $v \mapsto \|v_x\|$, $v \mapsto \|v_x\|_0$ and $v \mapsto \sqrt{\|v_x\|_0^2 + v^2(1)}$ on V.

Lemma 2.4. *We have*

$$\|v\|_{C^0(\overline{\Omega})} \leq \alpha_0 \|v\|_{H^1} \text{ for all } v \in H^1, \tag{2.5}$$

where $\alpha_0 = \sqrt{\frac{1+\sqrt{1+16(1-\rho)^2}}{2(1-\rho)}}$.

Proof. Since $C^1(\overline{\Omega})$ is dense in H^1, we only show that (2.5) holds for all $v \in C^1(\overline{\Omega})$.

For all $v \in C^1(\overline{\Omega})$, and $x,\, y \in \overline{\Omega}$, we have

$$v^2(x) = v^2(y) + 2\int_y^x v(z)\, v_x(z)\, dz.$$

Integrating over y from ρ to 1 to get

$$\begin{aligned}
(1-\rho)\, v^2(x) &= \|v\|^2 + 2\int_\rho^1 dy \int_y^x v(z)\, v_x(z)\, dz \\
&= \|v\|^2 + 2\int_\rho^1 dy \int_\rho^x v(z)\, v_x(z)\, dz - 2\int_\rho^1 dy \int_\rho^y v(z)\, v_x(z)\, dz \\
&\leq \|v\|^2 + 2(1-\rho)\int_\rho^x |v(z)\, v_x(z)|\, dz \\
&+ 2\int_\rho^1 (1-z)\, |v(z)\, v_x(z)|\, dz \\
&\leq \|v\|^2 + 4(1-\rho)\int_\rho^1 |v(z)\, v_x(z)|\, dz. \qquad (2.6)
\end{aligned}$$

Note that $\alpha_0^2 = \frac{1+\sqrt{1+16(1-\rho)^2}}{2(1-\rho)}$ satisfies $1 + 4(1-\rho)\frac{1}{\alpha_0^2} = (1-\rho)\,\alpha_0^2$, applying the inequality $2ab \leq \dfrac{2}{\alpha_0^2} a^2 + \frac{\alpha_0^2}{2} b^2$, for all $a,\, b \in \mathbb{R}$, we deduce from (2.6), that

$$\begin{aligned}
(1-\rho)\, v^2(x) &\leq \|v\|^2 + 2(1-\rho)\left(\frac{2}{\alpha_0^2}\|v\|^2 + \frac{\alpha_0^2}{2}\|v_x\|^2\right) \\
&= \left(1 + \frac{4(1-\rho)}{\alpha_0^2}\right)\|v\|^2 + (1-\rho)\,\alpha_0^2\,\|v_x\|^2 \\
&= (1-\rho)\,\alpha_0^2\,\|v\|_{H^1}^2 .
\end{aligned}$$

Hence (2.5) holds. Lemma 2.4 is complete. □

Now, we define the following bilinear form

$$a\,(u,v)=\zeta u\,(1)\,v\,(1)+\int_{\rho}^{1}xu_x\,(x)\,v_x\,(x)\,dx,\ \text{for all } u,\,v\in V, \tag{2.7}$$

where $\zeta\geq 0$ is a constant.

Lemma 2.5. *The symmetric bilinear form* $a\,(\cdot,\cdot)$ *defined by* (2.7) *is continuous on* $V\times V$ *and coercive on* V, i.e.,

(i) $|a\,(u,v)|\leq C_1\,\|u\|_1\,\|v\|_1\,,$

(ii) $a\,(v,v)\geq C_0\,\|v\|_1^2\,,$

for all $u,\,v\in V$, *where* $C_0=\frac{1}{2}\min\{1,\frac{2\rho}{(1-\rho)^2}\}$ *and* $C_1=1+3\zeta$.

Lemma 2.6. *There exists the Hilbert orthonormal base* $\{w_j\}$ *of the space* L^2 *consisting of eigenfunctions* w_j *corresponding to eigenvalues* λ_j *such that*

(i) $0<\lambda_1\leq\lambda_2\leq\cdots\leq\lambda_j\leq\lambda_{j+1}\leq\cdots,\ \lim_{j\to+\infty}\lambda_j=+\infty,$

(ii) $a\,(w_j,v)=\lambda_j\,\langle w_j,v\rangle$ *for all* $v\in V,\quad j=1,2,\cdots.$

Furthermore, the sequence $\{w_j/\sqrt{\lambda_j}\}$ *is also the Hilbert orthonormal base of* V *with respect to the scalar product* $a\,(\cdot,\cdot)$.

On the other hand, we also have w_j *satisfying the following boundary value problem*

$$\begin{cases} Aw_j\equiv-\left(w_{jxx}+\frac{1}{x}w_{jx}\right)=-\frac{1}{x}\frac{\partial}{\partial x}\left(xw_{jx}\right)=\lambda_j w_j,\ \text{in } \Omega,\\ w_j\,(\rho)=w_{jx}(1)+\zeta w_j(1)=0,\ w_j\in C^{\infty}\left([\rho,1]\right). \end{cases} \tag{2.8}$$

The proof of Lemma 2.5 can be found in [[22], p.87, Theorem 7.7], with $H=L^2$, and $a(\cdot,\cdot)$ as defined by (2.7). □

We also note that the operator $A:V\longrightarrow V'$ in (2.8) is uniquely defined by the Lax-Milgram Lemma, i.e.,

$$a\,(u,v)=\langle Au,v\rangle\ \text{for all } u,\ v\in V. \tag{2.9}$$

Lemma 2.7. *On* $V\cap H^2$, *three norms* $v\mapsto\|v\|_{H^2}$, $v\mapsto\|v\|_2=\sqrt{\|v\|_0^2+\|v_x\|_0^2+\|v_{xx}\|_0^2}$ *and* $v\mapsto\|v\|_{2*}=\sqrt{\|v_x\|_0^2+\|Av\|_0^2}$ *are equivalent.*

The proof of Lemma 2.7 can be found in [17].

Remark 2.8. The weak formulation of the initial-boundary value problem (1.1)-(1.3) can be given in the following manner: Find $u\in\bar{W}_T=\{u\in L^{\infty}(0,T;V\cap H^2):u'\in L^{\infty}(0,T;V),\ u''\in L^{\infty}(0,T;L^2)\}$, such that u satisfies the following variational equation

$$\langle u''(t),v\rangle+\mu[u]\,(t)\,a(u(t),v)=\langle f[u]\,(t)\,,v\rangle\,, \tag{2.10}$$

for all $v \in V$, a.e., $t \in (0,T)$, together with the initial conditions

$$u(0) = \tilde{u}_0,\ u'(0) = \tilde{u}_1, \tag{2.11}$$

where $a(\cdot,\cdot)$ is the symmetric bilinear form on V defined by (2.7) and

$$\begin{aligned} \mu[u]\,(t) &= \mu(t, u(1,t), \|u(t)\|_0^2, \|u_x(t)\|_0^2), \\ f[u](x,t) &= f\left(x, t, u(x,t), u_x(x,t), u'(x,t)\right). \end{aligned} \tag{2.12}$$

3. The Existence and Uniqueness Theorem

Now, let $T^* > 0$. We shall consider Prob. (1.1)-(1.3) with the constant $\zeta \geq 0$ and make the following assumptions.

(H_1) $\tilde{u}_0 \in V \cap H^2$, $\tilde{u}_1 \in V$;

(H_2) $\mu \in C^1\left([0,T^*] \times \mathbb{R} \times \mathbb{R}_+^2\right)$ and there exists the constant $\mu_* > 0$ such that $\mu(t, y_1, y_2, y_3) \geq \mu_* > 0$, $\forall (t, y_1, y_2, y_3) \in [0,T^*] \times \mathbb{R} \times \mathbb{R}_+^2$;

(H_3) $f \in C^0\left(\bar{\Omega} \times [0,T^*] \times \mathbb{R}^3\right)$ such that $f(\rho, t, 0, y_2, 0) = 0$, $\forall (t, y_2) \in [0,T^*] \times \mathbb{R}$ and $D_i f \in C^0\left(\bar{\Omega} \times [0,T^*] \times \mathbb{R}^3\right)$, $i = 1,3,4,5$.

Let $M > 0$, we put

$$\begin{aligned} \tilde{K}_M(\mu) &= \sup_{(t,y_1,y_2,y_3)\in \tilde{A}_*(M)} \left(\mu(t,y_1,y_2,y_3) + \sum\nolimits_{i=1}^{4} |D_i \mu(t,y_1,y_2,y_3)| \right), \\ K_M(f) &= \|f\|_{C^0(A_*(M))} + \|D_1 f\|_{C^0(A_*(M))} + \sum_{3\leq i\leq 5} \|D_i f\|_{C^0(A_*(M))}, \\ \|f\|_{C^0(A_*(M))} &= \sup\left\{ |f(x,t,y_1,y_2,y_3)| : (x,t,y_1,y_2,y_3) \in A_*(M) \right\}, \end{aligned}$$

where

$$\tilde{A}_*(M) = \left\{ (t,y_1,y_2,y_3) \in [0,T^*] \times \mathbb{R} \times \mathbb{R}_+^2 : |y_1| \leq \sqrt{\frac{1-\rho}{\rho}} M,\ 0 \leq y_i \leq M^2,\ i = 2,3 \right\},$$

$$A_*(M) = \left\{ (x,t,y_1,y_2,y_3) \in \bar{\Omega} \times [0,T^*] \times \mathbb{R}^3 : |y_2| \leq \alpha_0 M,\ \ |y_i| \leq \sqrt{\frac{1-\rho}{\rho}} M,\ i = 1,3 \right\}.$$

Also for each $M > 0$ and $T \in (0,T^*]$, we set

$$\begin{aligned} W(M,T) &= \{ v \in L^\infty\left(0,T; V \cap H^2\right) : v' \in L^\infty(0,T;V), \\ &\quad\ v'' \in L^2(Q_T),\ \ \|v\|_T \leq M \}, \\ W_1(M,T) &= \{ v \in W(M,T) : v'' \in L^\infty\left(0,T;L^2\right) \}, \end{aligned}$$

where $\|v\|_T = \max\{\|v\|_{L^\infty(0,T;V\cap H^2)}, \|v'\|_{L^\infty(0,T;V)}, \|v''\|_{L^2(Q_T)}\}$.

We choose the first term $u_0 \equiv 0$. Suppose that

$$u_{m-1} \in W_1(M,T), \tag{3.1}$$

and associate with Prob. (1.1)-(1.3) the following variational problem: Find $u_m \in W_1(M,T)$ $(m \geq 1)$ so that

$$\begin{cases} \langle u_m''(t), v\rangle + \mu_m(t)\, a(u_m(t), v) = \langle F_m(t), v\rangle, \ \forall v \in V, \\ u_m(0) = \tilde{u}_0, \ u_m'(0) = \tilde{u}_1, \end{cases} \tag{3.2}$$

where

$$\begin{aligned} \mu_m(t) &= \mu[u_{m-1}](t) = \mu(t, u_{m-1}(1,t), \|u_{m-1}(t)\|_0^2, \|\nabla u_{m-1}(t)\|_0^2), \\ F_m(x,t) &= f[u_{m-1}](x,t) = f\left(x, t, u_{m-1}(x,t), \nabla u_{m-1}(x,t), u_{m-1}'(x,t)\right). \end{aligned} \tag{3.3}$$

Then we have the following result.

Theorem 3.1. *Let the assumptions* $(H_1)-(H_3)$ *hold. Then there exist positive constants* M, T *such that the problem* (3.2), (3.3) *has a solution* $u_m \in W_1(M,T)$.

Proof. The proof is similar to the arguments in [17]. It consists of three steps.

Step 1. *The Faedo-Galerkin approximation* (introduced by Lions [9]). Consider the basis $\{w_j\}$ for V as in Lemma 2.5. Put

$$u_m^{(k)}(t) = \sum\nolimits_{j=1}^{k} c_{mj}^{(k)}(t) w_j, \tag{3.4}$$

where the coefficients $c_{mj}^{(k)}$ satisfy the system of linear differential equations

$$\begin{cases} \left\langle \ddot{u}_m^{(k)}(t), w_j \right\rangle + \mu_m(t)\, a(u_m^{(k)}(t), w_j) = \langle F_m(t), w_j\rangle, \\ u_m^{(k)}(0) = u_{0k}, \ \dot{u}_m^{(k)}(0) = u_{1k}, \ j = 1, \cdots, k, \end{cases} \tag{3.5}$$

with

$$\begin{cases} u_{0k} = \sum_{j=1}^{k} \alpha_j^{(k)} w_j \to \tilde{u}_0 \text{ strongly in } V \cap H^2, \\ u_{1k} = \sum_{j=1}^{k} \beta_j^{(k)} w_j \to \tilde{u}_1 \text{ strongly in } V. \end{cases} \tag{3.6}$$

The system of the equations (3.5) can be rewritten in form

$$\begin{cases} \ddot{c}_{mj}^{(k)}(t) + \lambda_j \mu_m(t) c_{mj}^{(k)}(t) = F_{mj}(t), \\ c_{mj}^{(k)}(0) = \alpha_j^{(k)}, \ \ \dot{c}_{mj}^{(k)}(0) = \beta_j^{(k)}, \ 1 \leq j \leq k, \end{cases} \tag{3.7}$$

in which

$$F_{mj}(t) = \langle F_m(t), w_j\rangle, \ 1 \leq j \leq k. \tag{3.8}$$

Note that by (3.1), it is not difficult to prove that the system (3.7), (3.8) has a unique solution $c_{mj}^{(k)}(t)$, $1 \le j \le k$ on interval $[0,T]$, so let us omit the details.

Step 2. *A priori estimates.* We put

$$S_m^{(k)}(t) = \left\|\dot{u}_m^{(k)}(t)\right\|_0^2 + \left\|\dot{u}_m^{(k)}(t)\right\|_a^2 + \mu_m(t)\left(\left\|u_m^{(k)}(t)\right\|_a^2 + \left\|Au_m^{(k)}(t)\right\|_0^2\right) + \int_0^t \left\|\ddot{u}_m^{(k)}(s)\right\|_0^2 ds, \tag{3.9}$$

where we denote $\|v\|_a = \sqrt{a(v,v)}$, $\forall v \in V$.

Then, it follows from (3.5), (3.9), that

$$\begin{aligned} S_m^{(k)}(t) &= S_m^{(k)}(0) + \int_0^t \mu_m'(s)\left[\left\|u_m^{(k)}(s)\right\|_a^2 + \left\|Au_m^{(k)}(s)\right\|_0^2\right] ds \\ &+ 2\int_0^t \left\langle F_m(s), \dot{u}_m^{(k)}(s)\right\rangle ds + 2\int_0^t a\left(F_m(s), \dot{u}_m^{(k)}(s)\right) ds \\ &+ \int_0^t \left\|\ddot{u}_m^{(k)}(s)\right\|_0^2 ds \equiv S_m^{(k)}(0) + \sum\nolimits_{j=1}^4 I_j. \end{aligned} \tag{3.10}$$

We shall estimate the terms I_j on the right-hand side of (3.10) as follows.

First term I_1. It is known that

$$\begin{aligned} \mu_m'(t) &= D_1\mu[u_{m-1}](t) \\ &+ D_2\mu[u_{m-1}](t)u_{m-1}'(1,t) + 2D_3\mu[u_{m-1}](t)\langle u_{m-1}(t), u_{m-1}'(t)\rangle \\ &+ 2D_4\mu[u_{m-1}](t)\langle \nabla u_{m-1}(t), \nabla u_{m-1}'(t)\rangle, \end{aligned}$$

with $D_i\mu[u_{m-1}](t) = D_i\mu(t, u_{m-1}(1,t), \|u_{m-1}(t)\|_0^2, \|\nabla u_{m-1}(t)\|_0^2)$, $i = 1, \cdots, 4$, it implies from (3.1) that

$$\begin{aligned} &\left|\mu_m'(t)\right| \\ \le\ & \tilde{K}_M(\mu)\left[1 + \left|u_{m-1}'(1,t)\right| + 2\|u_{m-1}(t)\|_0 \left\|u_{m-1}'(t)\right\|_0 + 2\|\nabla u_{m-1}(t)\|_0 \left\|\nabla u_{m-1}'(t)\right\|_0\right] \\ \le\ & \tilde{K}_M(\mu)\left[1 + \sqrt{\frac{1-\rho}{\rho}}M + 4M^2\right] \equiv q_M \tilde{K}_M(\mu), \end{aligned} \tag{3.11}$$

where $q_M = 1 + \sqrt{\dfrac{1-\rho}{\rho}}M + 4M^2$.

By the following inequality

$$S_m^{(k)}(t) \ge \mu_m(t)\left[\left\|u_m^{(k)}(t)\right\|_a^2 + \left\|Au_m^{(k)}(t)\right\|_0^2\right] \ge \mu_*\left[\left\|u_m^{(k)}(t)\right\|_a^2 + \left\|Au_m^{(k)}(t)\right\|_0^2\right],$$

we have

$$I_1 = \int_0^t \mu_m'(s)\left[\left\|u_m^{(k)}(s)\right\|_a^2 + \left\|Au_m^{(k)}(s)\right\|_0^2\right] ds \le \frac{1}{\mu_*} q_M \tilde{K}_M(\mu) \int_0^t S_m^{(k)}(s) ds. \tag{3.12}$$

Second term I_2. By the Cauchy-Schwartz inequality, we have

$$(3.13)\qquad |I_2| = 2\left|\int_0^t \left\langle F_m(s), \dot{u}_m^{(k)}(s)\right\rangle ds\right| \le \frac{1-\rho^2}{2} T K_M^2(f) + \int_0^t S_m^{(k)}(s) ds.$$

Third term I_3. Similarly, we have

$$(3.14)\qquad |I_3| = 2\left|\int_0^t a\left(F_m(s), \dot{u}_m^{(k)}(s)\right) ds\right| \le \int_0^t \|F_m(s)\|_a^2 ds + \int_0^t S_m^{(k)}(s) ds.$$

Note that

$$(3.15)\qquad \begin{aligned} \|v\|_a^2 &\le C_1 \|v\|_1^2 \le C_1\left(\frac{(1-\rho)^2}{2\rho}\|v_x\|_0^2 + \|v_x\|_0^2\right) \\ &= C_1 \frac{(1+\rho^2)}{2\rho}\|v_x\|_0^2 \text{ for all } v \in V, \end{aligned}$$

so

$$(3.16)\qquad \|F_m(s)\|_a^2 \le C_1 \frac{(1+\rho^2)}{2\rho}\|F_{mx}(s)\|_0^2.$$

We also have

$$(3.17)\qquad \begin{aligned} F_{mx}(t) &= D_1 f[u_{m-1}] + D_3 f[u_{m-1}]\nabla u_{m-1}(t) \\ &\quad + D_4 f[u_{m-1}]\Delta u_{m-1}(t) + D_5 f[u_{m-1}]\nabla u'_{m-1}(t), \end{aligned}$$

where $D_i f[u_{m-1}] = D_i f\left(x, t, u_{m-1}(t), \nabla u_{m-1}(t), u'_{m-1}(t)\right)$, $i = 1, \cdots, 5$.
From (3.1) and (3.17) we get

$$(3.18)\qquad \|F_{mx}(t)\|_0 \le \left(\sqrt{\frac{1-\rho^2}{2}} + 3M\right) K_M(f).$$

Combining (3.14), (3.16) and (3.18), we obtain

$$(3.19)\qquad |I_3| \le C_1 \frac{(1+\rho^2)}{2\rho}\left(\sqrt{\frac{1-\rho^2}{2}} + 3M\right)^2 T K_M^2(f) + \int_0^t S_m^{(k)}(s) ds.$$

Fourth term I_4. Eq. $(3.5)_1$ can be rewritten as follows

$$(3.20)\qquad \left\langle \ddot{u}_m^{(k)}(t), w_j\right\rangle + \mu_m(t)\left\langle A u_m^{(k)}(t), w_j\right\rangle = \langle F_m(t), w_j\rangle, \ j = 1, \cdots, k.$$

Hence, it follows after replacing w_j with $\ddot{u}_m^{(k)}(t)$, that

$$(3.21)\qquad \begin{aligned} \left\|\ddot{u}_m^{(k)}(t)\right\|_0^2 &= -\mu_m(t)\left\langle A u_m^{(k)}(t), \ddot{u}_m^{(k)}(t)\right\rangle + \left\langle F_m(t), \ddot{u}_m^{(k)}(t)\right\rangle \\ &\le \left[\mu_m(t)\left\|A u_m^{(k)}(t)\right\|_0 + \|F_m(t)\|_0\right]\left\|\ddot{u}_m^{(k)}(t)\right\|_0 \\ &\le 2\mu_m^2(t)\left\|A u_m^{(k)}(t)\right\|_0^2 + 2\|F_m(t)\|_0^2 \\ &\le 2\tilde{K}_M(\mu) S_m^{(k)}(t) + \left(1-\rho^2\right) K_M^2(f). \end{aligned}$$

Integrating in t to get

$$I_4 = \int_0^t \left\|\ddot{u}_m^{(k)}(s)\right\|_0^2 ds \leq (1-\rho^2)TK_M^2(f) + 2\tilde{K}_M(\mu)\int_0^t S_m^{(k)}(s)ds. \tag{3.22}$$

The term $S_m^{(k)}(0)$. By means of the convergences in (3.6), we can deduce the existence of a constant $M > 0$ independent of k and m such that

$$S_m^{(k)}(0) = \|u_{1k}\|_0^2 + \|u_{1k}\|_a^2 + \mu(0, \tilde{u}_0(1), \|\tilde{u}_0\|_0^2, \|\tilde{u}_{0x}\|_0^2)\left[\|u_{0k}\|_a^2 + \|Au_{0k}\|_0^2\right] \leq \frac{1}{2}M^2, \tag{3.23}$$

for all m, $k \in \mathbb{N}$

It follows from (3.10), (3.12), (3.13), (3.19), (3.22) and (3.23), that

$$S_m^{(k)}(t) \leq \frac{1}{2}M^2 + TD_1(M) + D_2(M)\int_0^t S_m^{(k)}(s)ds, \tag{3.24}$$

where

$$\begin{cases} D_1(M) = \left[\frac{3}{2}(1-\rho^2) + C_{1\frac{(1+\rho^2)}{2\rho}}\left(\sqrt{\frac{1-\rho^2}{2}} + 3M\right)^2\right]K_M^2(f), \\ D_2(M) = 2\left[1 + \left(1 + \frac{1}{2\mu_*}q_M\right)\tilde{K}_M(\mu)\right]. \end{cases} \tag{3.25}$$

Therefore, we can choose $T \in (0, T^*]$, such that

$$\left(\frac{1}{2}M^2 + TD_1(M)\right)\exp\left(TD_2(M)\right) \leq M^2, \tag{3.26}$$

and

$$k_T = \left(1 + \frac{1}{\sqrt{\mu_* C_0}}\right)\sqrt{\bar{q}_M}\sqrt{T}\exp\left[T\left(1 + \frac{q_M\tilde{K}_M(\mu)}{2\mu_*}\right)\right] < 1, \tag{3.27}$$

where

$$\bar{q}_M = \left(\sqrt{\frac{1-\rho}{\rho}} + 2M\right)^2 M^2\tilde{K}_M^2(\mu) + K_M^2(f)\left(1 + \frac{1-\rho}{\sqrt{2\rho}}\right)^2. \tag{3.28}$$

Finally, it follows from (3.24) and (3.26) that

$$S_m^{(k)}(t) \leq M^2\exp\left(-TD_2(M)\right) + D_2(M)\int_0^t S_m^{(k)}(s)ds. \tag{3.29}$$

Using Gronwall's Lemma, (3.29) yields

$$S_m^{(k)}(t) \leq M^2\exp\left(-TD_2(M)\right)\exp\left(tD_2(M)\right) \leq M^2, \tag{3.30}$$

for all $t \in [0, T]$, for all m and k. Therefore, we have

$$u_m^{(k)} \in W(M,T), \text{ for all } m \text{ and } k \in \mathbb{N}. \tag{3.31}$$

Step 3. *Limiting process.* From (3.31), there exists a subsequence of $\{u_m^{(k)}\}$, still so denoted, such that

$$\begin{cases} u_m^{(k)} \to u_m & \text{in} \quad L^\infty(0,T;V\cap H^2) \text{ weakly*}, \\ \dot{u}_m^{(k)} \to u_m' & \text{in} \quad L^\infty(0,T;V) \text{ weakly*}, \\ \ddot{u}_m^{(k)} \to u_m'' & \text{in} \quad L^2(Q_T) \text{ weakly}, \\ u_m \in W(M,T). \end{cases} \tag{3.32}$$

Passing to the limit in (3.5), we have u_m satisfying (3.2), (3.3) in $L^2(0,T)$. On the other hand, it follows from $(3.2)_1$ and $(3.32)_4$ that $u_m'' = -\mu_m(t)\,Au_m + F_m \in L^\infty(0,T;L^2)$, hence $u_m \in W_1(M,T)$ and the proof of Theorem 3.1 is complete. □

In order to get the existence and uniqueness, we introduce the Banach space (see [9])

$$W_1(T) = \{v \in L^\infty(0,T;V) : v' \in L^\infty(0,T;L^2)\},$$

with respect to the norm $\|v\|_{W_1(T)} = \|v\|_{L^\infty(0,T;V)} + \|v'\|_{L^\infty(0,T;L^2)}$. □

By the result given in Theorem 3.1 and by the compact imbedding theorems, we prove the main results in this section as follows

Theorem 3.2. *Let* $(H_1)-(H_3)$ *hold. Then, there exist positive constants* M, T *satisfying* (3.23), (3.26)-(3.28) *such that Prob.* (1.1)-(1.3) *has a unique weak solution* $u \in W_1(M,T)$. *Furthermore, the linear recurrent sequence* $\{u_m\}$ *defined by* (3.2), (3.3) *converges to the solution* u *strongly in the space* $W_1(T)$ *with the estimate*

$$\|u_m - u\|_{W_1(T)} \le \frac{M}{1-k_T} k_T^m, \text{ for all } m \in \mathbb{N}. \tag{3.33}$$

Proof.
(a) **The existence.** First, we shall prove that $\{u_m\}$ is a Cauchy sequence in $W_1(T)$. Let $w_m = u_{m+1} - u_m$. Then w_m satisfies the variational problem

$$\begin{cases} \langle w_m''(t), v\rangle + \mu_{m+1}(t)\,a(w_m(t),v) + [\mu_{m+1}(t) - \mu_m(t)]\,\langle Au_m(t), v\rangle \\ \qquad\qquad = \langle F_{m+1}(t) - F_m(t), v\rangle\,, \forall v \in V, \\ w_m(0) = w_m'(0) = 0. \end{cases} \tag{3.34}$$

Taking $v = w_m'$ in $(3.34)_1$, after integrating in t, we get

$$\begin{aligned} Z_m(t) &= \int_0^t \mu_{m+1}'(s)\,\|w_m(s)\|_a^2\,ds - 2\int_0^t [\mu_{m+1}(s) - \mu_m(s)]\,\langle Au_m(s), w_m'(s)\rangle ds \\ &\quad +2\int_0^t \langle F_{m+1}(s) - F_m(s), w_m'(s)\rangle\,ds \\ &\equiv J_1 + J_2 + J_3, \end{aligned} \tag{3.35}$$

where

$$Z_m(t) = \|w_m'(t)\|_0^2 + \mu_{m+1}(t)\|w_m(t)\|_a^2 \geq \|w_m'(t)\|_0^2 + \mu_* C_0 \|w_m(t)\|_1^2. \tag{3.36}$$

All integrals on the right – hand side of (3.35) will be estimated as below.
First integral J_1. By (3.11) and (3.36), we have

$$|J_1| \leq \int_0^t |\mu_{m+1}'(s)| \|w_m(s)\|_a^2 ds \leq \frac{1}{\mu_*} q_M \tilde{K}_M(\mu) \int_0^t Z_m(s)ds. \tag{3.37}$$

Second integral J_2. By (H_2), it is clear to see that

$$\begin{aligned} &|\mu_{m+1}(t) - \mu_m(t)| \\ \leq\ & \tilde{K}_M(\mu)\left[|w_{m-1}(1,t)| + \left|\|u_m(t)\|_0^2 - \|u_{m-1}(t)\|_0^2\right| + \left|\|\nabla u_m(t)\|_0^2 - \|\nabla u_{m-1}(t)\|_0^2\right|\right] \\ \leq\ & \left(\sqrt{\frac{1-\rho}{\rho}} + 2M\right)\tilde{K}_M(\mu)\|w_{m-1}\|_{W_1(T)}. \end{aligned} \tag{3.38}$$

Hence

$$\begin{aligned} |J_2| &= 2\left|\int_0^t [\mu_{m+1}(s) - \mu_m(s)] \langle Au_m(s), w_m'(s)\rangle ds\right| \\ &\leq T\left(\sqrt{\frac{1-\rho}{\rho}} + 2M\right)^2 M^2 \tilde{K}_M^2(\mu)\|w_{m-1}\|_{W_1(T)}^2 + \int_0^t Z_m(s)ds. \end{aligned} \tag{3.39}$$

Third integral J_3. By (H_3) it yields

$$\begin{aligned} \|F_{m+1}(t) - F_m(t)\|_0 &\leq K_M(f)\left(\|w_{m-1}(t)\|_0 + \|\nabla w_{m-1}(t)\|_0 + \left\|w_{m-1}'(t)\right\|_0\right) \\ &\leq K_M(f)\left(\frac{1-\rho}{\sqrt{2\rho}}\|\nabla w_{m-1}(t)\|_0 + \|\nabla w_{m-1}(t)\|_0 + \left\|w_{m-1}'(t)\right\|_0\right) \\ &\leq K_M(f)\left(1 + \frac{1-\rho}{\sqrt{2\rho}}\right)\|w_{m-1}\|_{W_1(T)}. \end{aligned} \tag{3.40}$$

Hence

$$\begin{aligned} |J_3| &= 2\left|\int_0^t \langle F_{m+1}(s) - F_m(s), w_m'(s)\rangle ds\right| \\ &\leq T^* K_M^2(f)\left(1 + \frac{1-\rho}{\sqrt{2\rho}}\right)^2 \|w_{m-1}\|_{W_1(T)}^2 + \int_0^t Z_m(s)ds. \end{aligned} \tag{3.41}$$

Combining (3.35), (3.37), (3.39) and (3.41), we obtain

$$Z_m(t) \leq \bar{q}_M T \|w_{m-1}\|_{W_1(T)}^2 + 2\left(1 + \frac{q_M \tilde{K}_M(\mu)}{2\mu_*}\right)\int_0^t Z_m(s)ds, \tag{3.42}$$

where $\bar{q}_M$ as in (3.28). Using Gronwall's Lemma, we deduce from (3.42) that

$$\|w_m\|_{W_1(T)} \le k_T \|w_{m-1}\|_{W_1(T)} \quad \forall m \in \mathbb{N}, \tag{3.43}$$

where k_T as in (3.27). It implies that

$$\begin{aligned} \|u_m - u_{m+p}\|_{W_1(T)} &\le \|u_0 - u_1\|_{W_1(T)} (1 - k_{T^*})^{-1} k_T^m \\ &\le \frac{M}{1 - k_T} k_T^m \quad \forall m, p \in \mathbb{N}. \end{aligned} \tag{3.44}$$

It follows that $\{u_m\}$ is a Cauchy sequence in $W_1(T)$. Then there exists $u \in W_1(T)$ such that

$$u_m \to u \text{ strongly in } W_1(T). \tag{3.45}$$

Note that $u_m \in W_1(M, T)$, then there exists a subsequence $\{u_{m_j}\}$ of $\{u_m\}$ such that

$$\begin{cases} u_{m_j} \to u & \text{in} \quad L^\infty(0, T; V \cap H^2) \text{ weakly*}, \\ u'_{m_j} \to u' & \text{in} \quad L^\infty(0, T; V) \text{ weakly*}, \\ u''_{m_j} \to u'' & \text{in} \quad L^2(Q_T) \text{ weakly}, \\ u \in W(M, T). \end{cases} \tag{3.46}$$

We also note that

$$\|F_m(t) - f(\cdot, t, u, u_x, u')\|_{L^\infty(0,T;L^2)} \le K_M(f) \left(1 + \frac{1-\rho}{\sqrt{2\rho}}\right) \|u_{m-1} - u\|_{W_1(T)}. \tag{3.47}$$

Hence, from (3.45) and (3.47), we obtain

$$F_m(t) \to f(\cdot, t, u, u_x, u') \text{ strongly in } L^\infty(0, T; L^2). \tag{3.48}$$

On the other hand, we have

$$|\mu_m(t) - \mu[u](t)| \le \left(\sqrt{\frac{1-\rho}{\rho}} + 2M\right) \tilde{K}_M(\mu) \|u_{m-1} - u\|_{W_1(T)}. \tag{3.49}$$

Hence, it follows from (3.45) and (3.49) that

$$\mu_m \to \mu[u] \text{ strongly in } L^\infty(0, T). \tag{3.50}$$

Finally, passing to limit in (3.2), (3.3) as $m = m_j \to \infty$, it implies from (3.45), $(3.46)_{1,3}$ (3.48) and (3.50) that there exists $u \in W(M, T)$ satisfying the equation

$$\langle u''(t), v\rangle + \mu[u](t) a(u(t), v) = \langle f[u](t), v\rangle, \tag{3.51}$$

for all $v \in V$ and the initial conditions

$$u(0) = \tilde{u}_0, \; u'(0) = \tilde{u}_1. \tag{3.52}$$

Furthermore, from the assumptions (H_2), (H_3) we obtain from $(3.46)_4$, (3.48), (3.50) and (3.51), that

$$u'' = -\mu[u](t)Au(t) + f[u](t) \in L^\infty(0,T;L^2), \tag{3.53}$$

thus we have $u \in W_1(M,T)$. The existence of a weak solution of Prob. (1.1) - (1.3) is proved.

(b) **The uniqueness.** Let u_1, $u_2 \in W_1(M,T)$ be two weak solutions of Prob. (1.1) - (1.3). Then $u = u_1 - u_2$ satisfies the variational problem

$$\begin{cases} \langle u''(t), v\rangle + \mu_1(t)\, a(u(t),v) + [\mu_1(t) - \mu_2(t)]\, \langle Au_2(t), v\rangle \\ \qquad\qquad = \langle F_1(t) - F_2(t), v\rangle\,,\ \forall v \in V, \\ u(0) = u'(0) = 0, \end{cases} \tag{3.54}$$

where $F_i(x,t) = f(x,t,u_i,u_{ix},u_i')\,,\ \mu_i(t) = \mu[u_i](t),\ i = 1,\,2$. We take $w = u'$ in $(3.54)_1$ and integrate in t to get

$$Z(t) \le K_M^* \int_0^t Z(s)ds, \text{ for all } t \in [0,T],$$

where

$$\begin{aligned} Z(t) &= \|u'(t)\|_0^2 + \mu_1(t)\,\|u(t)\|_a^2\,, \\ K_M^* &= 2\left(1 + \frac{1-\rho}{\sqrt{2\rho}}\right)\left(1 + \frac{1}{\sqrt{\mu_*}}\right) K_M(f) \\ &\quad + \left[\frac{q_M}{\mu_*} + 2\left(\sqrt{\frac{1-\rho}{\rho}} + 2M\right)\frac{M}{\sqrt{C_0\mu_*}}\right]\tilde{K}_M(\mu)\,. \end{aligned}$$

Using Gronwall's Lemma, it follows that $Z = u_1 - u_2 \equiv 0$. Therefore, Theorem 3.2 is proved. □

4. Blow Up Result

In this section, Prob. (1.1) – (1.3) is considered with $\zeta = 0$, $f(x,t,u,u_x,u_t) = -\lambda u_t + f(u) + F(x,t)$, as follows

$$\begin{cases} u_{tt} - \mu\left(\|u_x(t)\|_0^2\right)(u_{xx} + \frac{1}{x}u_x) + \lambda u_t = f(u) + F(x,t), \\ \qquad\qquad \rho < x < 1,\ 0 < t < T, \\ u(\rho,t) = u_x(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x),\ u_t(x,0) = \tilde{u}_1(x), \end{cases} \tag{4.1}$$

where $\lambda > 0$, $0 < \rho < 1$ are given constants and $\tilde{u}_0$, $\tilde{u}_1$, μ, f, F are given functions satisfying conditions specified later.

In this section, we assume that

(H_2^*) $\mu \in C^1(\mathbb{R})$ and there exists the constant $\mu_* > 0$ such that
$$\mu(y) \geq \mu_* > 0, \forall y \in \mathbb{R};$$

(H_3^*) $f \in C^1(\mathbb{R}),\ f(0) = 0;$

(H_4^*) $F,\ F' \in L^2(\Omega \times (0, T^*)).$

Then we obtain the following theorem about the existence of a weak solution.

Theorem 4.1. *Suppose that* (H_1), $(H_2^*) - (H_4^*)$ *hold. Then Prob.* (4.1) *has a unique local solution*

$$(4.2)\qquad \begin{aligned} u &\in C([0,T];V) \cap C^1\left([0,T];L^2\right) \cap L^\infty\left(0,T;V \cap H^2\right), \\ u' &\in L^\infty(0,T;V),\ u'' \in L^\infty\left(0,T;L^2\right), \end{aligned}$$

for $T > 0$ *small enough.*

Proof. The proof is similar to those of Theorem 3.1 and Theorem 3.2. □

First, in order to obtain a blow up result, we make the following assumptions.

$(\hat{H}_2^*)$ $\mu \in C^1(\mathbb{R}_+)$, and there exist the constants $\mu_* > 0$, $\bar{\mu}_1 > 0$ such that
(i) $\mu(y) \geq \mu_* > 0, \forall y \geq 0,$
(ii) $y\mu(y) \leq \bar{\mu}_1 \int_0^y \mu(z)dz, \forall y \geq 0;$

$(\hat{H}_3^*)$ $f \in C^1(\mathbb{R})$, $f(0) = 0$ and there exist the constants $p > 2$, $d_1 > 2$, $\bar{d}_1 > 0$ such that
(i) $yf(y) \geq d_1 \int_0^y f(z)dz, \forall y \in \mathbb{R},$
(ii) $\int_0^y f(z)dz \geq \bar{d}_1 |y|^p, \forall y \in \mathbb{R};$

$(\hat{H}_4^*)$ $F(x,t) \equiv 0;$

$(\hat{H}_5^*)$ $d_1 > 2\bar{\mu}_1$, with d_1, $\bar{\mu}_1$ as in $(\hat{H}_2^*)(ii)$, $(\hat{H}_3^*)(i)$.

We present an example of the functions μ, f satisfying $(\hat{H}_2^*)$, $(\hat{H}_3^*)$, $(\hat{H}_5^*)$ as below

$$(4.3)\qquad \begin{aligned} \mu(y) &= \mu_* + y^q,\ \forall y \geq 0, \\ f(u) &= d\,|u|^{p-2}\,u \ln^n(e + u^2), \end{aligned}$$

where $\mu_* > 0$, $d > 0$, $q > 1$, $p > 2q + 2$, $n > 1$ are constants.

It is obvious that $\mu \in C^1(\mathbb{R}_+)$ and $(\hat{H}_2^*)(i)$ holds.

Moreover

$$\int_0^y \mu(z)dz = \mu_* y + \frac{y^{q+1}}{q+1} \geq \frac{1}{q+1}\left(\mu_* y + y^{q+1}\right) = \frac{1}{q+1} y\mu(y),\ \forall y \geq 0.$$

Hence, $(\hat{H}_2^*)(ii)$ holds with $\bar{\mu}_1 = q + 1$.

With the function f, it is clearly that $f \in C^1(\mathbb{R})$, $f(0) = 0$.
Using integration by parts, it gives

$$\int_0^y f(z)dz = \frac{1}{p} y f(y) - \frac{2nd}{p} \int_0^y \frac{|z|^p z}{e+z^2} \ln^{n-1}(e+z^2)dz. \tag{4.4}$$

Note that $\int_0^y \frac{|z|^p z}{e+z^2} \ln^{n-1}(e+z^2)dz \geq 0$, $\forall y \in \mathbb{R}$, so we obtain

$$yf(y) \geq p \int_0^y f(z)dz, \ \forall y \in \mathbb{R}. \tag{4.5}$$

Hence, $(\hat{H}_3^*)(i)$ holds, with $d_1 = p$.
With the condition $(\hat{H}_3^*)(ii)$:
Note that the function $\bar{f}(y) = \int_0^y f(z)dz = d \int_0^y |z|^{p-2} z \ln^n(e+z^2)dz$ satisfies

$$\bar{f}(-y) = \bar{f}(y) \geq 0, \ \forall y \in \mathbb{R}. \tag{4.6}$$

Let $y \geq 0$, by $\ln^n(e+z^2) \geq 1$, $\forall z \in [0, y]$, we have

$$\bar{f}(y) = \int_0^y f(z)dz = d \int_0^y |z|^{p-2} z \ln^n(e+z^2)dz \geq d \int_0^y |z|^{p-2} z dz = \frac{d}{p} |y|^p. \tag{4.7}$$

By (4.6), we have

$$\bar{f}(y) = \int_0^y f(z)dz \geq \frac{d}{p} |y|^p, \ \forall y \in \mathbb{R}. \tag{4.8}$$

Hence, $(\hat{H}_3^*)(ii)$ holds, with $\bar{d}_1 = \frac{d}{p}$.
It is obvious that $(\hat{H}_5^*)$ holds, because $d_1 - 2\bar{\mu}_1 = p - 2q - 2 > 0$.
Put

$$H(0) = -\frac{1}{2} \|\tilde{u}_1\|_0^2 - \frac{1}{2} \int_0^{\|\tilde{u}_{0x}\|_0^2} \mu(z)dz + \int_\rho^1 xdx \int_0^{\tilde{u}_0(x)} f(z)dz. \tag{4.9}$$

Theorem 4.2. *Let $(\hat{H}_2^*) - (\hat{H}_5^*)$ hold. Then, for any $(\tilde{u}_0, \tilde{u}_1) \in (V \cap H^2) \times V$ such that $H(0) > 0$, the weak solution $u = u(x,t)$ of Prob. (4.1) blows up in finite time.*

Proof. It consists of two steps, in which, the Lyapunov functional $L(t)$ is constructed in step 1 and then the blow up is proved in step 2.
Step 1. We define the energy associated with (4.1) by

$$E(t) = \frac{1}{2} \|u'(t)\|_0^2 + \frac{1}{2} \int_0^{\|u_x(t)\|_0^2} \mu(z)dz - \int_\rho^1 xdx \int_0^{u(x,t)} f(z)dz, \tag{4.10}$$

and we put $H(t) = -E(t)$, $\forall t \in [0, T^*)$. Multiplying $(4.1)_1$ by $xu'(x,t)$ and integrating the resulting equation over $[\rho, 1]$, we have

$$H'(t) = \lambda \|u'(t)\|_0^2 \geq 0. \tag{4.11}$$

Hence, we can deduce from (4.11) and $H(0) > 0$ that

$$H(t) \geq H(0) > 0, \ \forall t \in [0, T), \tag{4.12}$$

so

$$\begin{cases} 0 < H(0) \leq H(t) \leq \displaystyle\int_\rho^1 xdx \int_0^{u(x,t)} f(z)dz; \\ \|u'(t)\|_0^2 + \displaystyle\int_0^{\|u_x(t)\|_0^2} \mu(z)dz \leq 2 \int_\rho^1 xdx \int_0^{u(x,t)} f(z)dz, \\ \qquad \forall t \in [0, T). \end{cases} \tag{4.13}$$

Now, we define the functional

$$L(t) = H^{1-\eta}(t) + \varepsilon \Psi(t), \tag{4.14}$$

where

$$\Psi(t) = \langle u'(t), u(t) \rangle + \frac{\lambda}{2} \|u(t)\|_0^2, \tag{4.15}$$

for ε small enough and

$$0 < 2\eta < 1, \ 2/(1 - 2\eta) \leq p. \tag{4.16}$$

In what follows, we show that, there exists a constant $\bar{\theta}_1 > 0$ such that

$$L'(t) \geq \bar{\theta}_1 \left[H(t) + \|u'(t)\|_0^2 + \|u(t)\|_{L^p}^p + \|u_x(t)\|_0^2 \right]. \tag{4.17}$$

Multiplying $(4.1)_1$ by $xu(x,t)$ and integrating over $[\rho, 1]$, it leads to

$$\Psi'(t) = \|u'(t)\|_0^2 - \|u_x(t)\|_0^2 \mu\left(\|u_x(t)\|_0^2\right) + \langle f(u(t)), u(t) \rangle. \tag{4.18}$$

Therefore

$$L'(t) = (1 - \eta) H^{-\eta}(t) H'(t) + \varepsilon \Psi'(t) \geq \varepsilon \Psi'(t). \tag{4.19}$$

By $(\hat{H}_2^*)$, $(\hat{H}_3^*)$, we obtain

$$\begin{cases} \|u_x(t)\|_0^2 \mu\left(\|u_x(t)\|_0^2\right) \leq \bar{\mu}_1 \int_0^{\|u_x(t)\|_0^2} \mu(z)dz, \\ \langle f(u(t)), u(t) \rangle \geq d_1 \int_\rho^1 xdx \int_0^{u(x,t)} f(z)dz, \\ \int_\rho^1 xdx \int_0^{u(x,t)} f(z)dz \geq \bar{d}_1 \rho \|u(t)\|_{L^p}^p. \end{cases} \tag{4.20}$$

Hence, combining (4.10), (4.18) and (4.20) give

$$
\begin{aligned}
(4.21)\Psi'(t) &= \|u'(t)\|_0^2 - \|u_x(t)\|_0^2 \mu\left(\|u_x(t)\|_0^2\right) + \langle f(u(t)), u(t)\rangle \\
&\geq \|u'(t)\|_0^2 - \bar{\mu}_1 \int_0^{\|u_x(t)\|_0^2} \mu(z)dz + d_1 \int_\rho^1 xdx \int_0^{u(x,t)} f(z)dz \\
&= \|u'(t)\|_0^2 - \bar{\mu}_1 \int_0^{\|u_x(t)\|_0^2} \mu(z)dz + d_1\delta_1 \int_\rho^1 xdx \int_0^{u(x,t)} f(z)dz \\
&\quad + d_1(1-\delta_1)\int_\rho^1 xdx \int_0^{u(x,t)} f(z)dz
\end{aligned}
$$

$$
\begin{aligned}
&= \|u'(t)\|_0^2 - \bar{\mu}_1 \int_0^{\|u_x(t)\|_0^2} \mu(z)dz + d_1\delta_1 \int_\rho^1 xdx \int_0^{u(x,t)} f(z)dz \\
&\quad + d_1(1-\delta_1)\left[H(t) + \frac{1}{2}\|u'(t)\|_0^2 + \frac{1}{2}\int_0^{\|u_x(t)\|_0^2} \mu(z)dz\right] \\
&= d_1(1-\delta_1)H(t) + \left(1 + \frac{d_1}{2}(1-\delta_1)\right)\|u'(t)\|_0^2 \\
&\quad + d_1\delta_1 \int_\rho^1 xdx \int_0^{u(x,t)} f(z)dz + \frac{1}{2}\left[d_1 - 2\bar{\mu}_1 - \delta_1 d_1\right]\int_0^{\|u_x(t)\|_0^2} \mu(z)dz \\
&\geq d_1(1-\delta_1)H(t) + \left(1 + \frac{d_1}{2}(1-\delta_1)\right)\|u'(t)\|_0^2 \\
&\quad + d_1\delta_1\bar{d}_1\rho\|u(t)\|_{L^p}^p + \frac{1}{2}\left[d_1 - 2\bar{\mu}_1 - \delta_1 d_1\right]\int_0^{\|u_x(t)\|_0^2} \mu(z)dz.
\end{aligned}
$$

By $d_1 > 2\bar{\mu}_1$, we can choose $\delta_1 \in (0,1)$ such that

$$
(4.22) \qquad d_1 - 2\bar{\mu}_1 - \delta_1 d_1 > 0.
$$

By using the inequalities (4.19), (4.21), (4.22), we obtain (4.17) with choosing $\bar{\theta}_1 > 0$ small enough.

From the formula of $L(t)$ and (4.14), we can choose ε small enough that

$$
(4.23) \qquad L(t) \geq L(0) > 0, \ \forall t \in [0,T).
$$

Using the inequality $\left(\sum_{i=1}^3 x_i\right)^r \leq 3^{r-1}\sum_{i=1}^3 x_i^r$, for all $r > 1$ and $x_1, \cdots, x_3 \geq 0$, we deduce from (4.14) - (4.16) that

$$
(4.24) \qquad L^{1/(1-\eta)}(t) \leq Const\left(H(t) + |\langle u(t), u'(t)\rangle|^{1/(1-\eta)} + \|u(t)\|_0^{2/(1-\eta)}\right).
$$

Step 2. *The estimates.* Using Young's inequality, we have

$$
(4.25) \qquad |\langle u(t), u'(t)\rangle|^{1/(1-\eta)} \leq Const\left(\|u(t)\|_{L^p}^s + \|u'(t)\|_0^2\right),
$$

where $s = 2/(1-2\eta) \le p$ by (4.16). □

Now, we shall need the following lemma

Lemma 4.3. *Let $s = 2/(1-2\eta) \le p$, we obtain*

$$\|v\|_{L^p}^s + \|v\|_0^{2/(1-\eta)} \le \frac{2}{\rho}\left(\|v_x\|_0^2 + \|v\|_{L^p}^p\right), \text{ for any } v \in V. \tag{4.26}$$

Proof of Lemma 4.3 is straightforward, so we omit the details.

Step 3. *Blow up.* It follows from (4.24)-(4.26) that

$$L^{1/(1-\eta)}(t) \le Const\ \left(H(t) + \|u'(t)\|_0^2 + \|u_x(t)\|_0^2 + \|u(t)\|_{L^p}^p\right),\ \forall t \in [0,T). \tag{4.27}$$

Using (4.17), (4.27) yields

$$L'(t) \ge \bar{\theta}_2 L^{1/(1-\eta)}(t),\ \forall t \in [0,T), \tag{4.28}$$

where $\bar{\theta}_2$ is a positive constant. By integrating (4.28) over $(0,t)$, it gives

$$L^{\eta/(1-\eta)}(t) \ge \frac{1}{L^{-\eta/(1-\eta)}(0) - \frac{\bar{\theta}_2\eta}{1-\eta}t},\ 0 \le t < \frac{1}{\bar{\theta}_2\eta}(1-\eta)\,L^{-\eta/(1-\eta)}(0). \tag{4.29}$$

Therefore, $L(t)$ blows up in a finite time given by $T_* = \frac{1}{\bar{\theta}_2\eta}(1-\eta)\,L^{-\eta/(1-\eta)}(0)$. Theorem 4.2 is proved completely. □

5. Exponential Decay of Solutions

This section investigates the decay of the solution of Prob. (1.1) – (1.3) corresponding to with $\zeta = 0$, $f(x,t,u,u_x,u_t) = -\lambda u_t + f(u) + F(x,t)$, as follows

$$\begin{cases} u_{tt} - \mu\left(\|u_x(t)\|_0^2\right)(u_{xx} + \frac{1}{x}u_x) + \lambda u_t = f(u) + F(x,t), \\ \qquad\qquad \rho < x < 1,\ t > 0, \\ u(\rho,t) = u_x(1,t) = 0 \\ u(x,0) = \tilde{u}_0(x),\ u_t(x,0) = \tilde{u}_1(x), \end{cases} \tag{5.1}$$

where μ, f, F, $\tilde{u}_0$, $\tilde{u}_1$ are given functions and $\lambda > 0$, $0 < \rho < 1$ are the given constants. We prove that if $\int_0^{\|\tilde{u}_{0x}\|_0^2}\mu(z)dz - p\int_\rho^1 xdx\int_0^{\tilde{u}_0(x)} f(z)dz > 0$ and if the initial energy, $\|F(t)\|_0$ are small enough, then the energy of the solution decays exponentially as $t \to +\infty$. For this purpose, we make the following assumption

(H_3^∞) $f \in C^1(\mathbb{R})$, $f(0) = 0$ and there exist the constant $d_2 > 0$ and a nondecreasing function $F^* : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{z\to 0_+} F^*(z) = 0$ such that

(i) $yf(y) \leq d_2 \int_0^y f(z)dz$, for all $y \in \mathbb{R}$,

(ii) $\int_0^y f(z)dz \leq y^2 F^*(|y|)$, for all $y \in \mathbb{R}$;

(H_4^∞) $F \in L^\infty\left(\mathbb{R}_+; L^2\right) \cap L^1\left(\mathbb{R}_+; L^2\right)$, $F' \in L^2\left(\mathbb{R}_+; L^2\right)$ and there exist two constants $\bar{C}_0 > 0$, $\bar{\gamma}_0 > 0$ such that $\|F(t)\|_0 \leq \bar{C}_0 e^{-\bar{\gamma}_0 t}$, for all $t \geq 0$.

We will show that the example of f in Section 4 also satisfies (H_3^∞).

We consider the following function

$$f(u) = d\,|u|^{p-2}\, u \ln^n(e + u^2), \tag{5.2}$$

where $d > 0$, $p > 2$, $n > 1$ are constants, with $p > \max\{2q + 2, 2n\}$, with $q > 1$ as in (4.3).

With the condition $(H_3^\infty)(i)$:

By (5.4), we have

$$\int_0^y f(z)dz = \frac{1}{p} yf(y) - \frac{2nd}{p} G(y), \tag{5.3}$$

where $G(y) = \int_0^y \frac{|z|^p\, z}{e + z^2} \ln^{n-1}(e + z^2)dz$. Note that the function $G(y)$ satisfies

$$G(-y) = G(y) \geq 0,\ \forall y \in \mathbb{R}. \tag{5.4}$$

Let $y \geq 0$, by

$$\begin{aligned} 1 &\leq \ln^{n-1}(e + z^2) \leq \ln^n(e + z^2) \leq \ln^n(e + y^2), \\ 0 &\leq \frac{|z|^p\, z}{e + z^2} \leq |z|^{p-2}\, z,\ \forall z \in [0, y], \end{aligned}$$

we have

$$\begin{aligned} G(y) &= \int_0^y \frac{|z|^p\, z}{e + z^2} \ln^{n-1}(e + z^2)dz \\ &\leq \ln^n(e + y^2) \int_0^y |z|^{p-2}\, zdz \\ &= \frac{1}{p} |y|^p \ln^n(e + y^2),\ \forall y \geq 0. \end{aligned}$$

By (5.4), we have

$$G(y) \leq \frac{1}{p} |y|^p \ln^n(e + y^2) = \frac{1}{pd} yf(y),\ \forall y \in \mathbb{R}. \tag{5.5}$$

It follows from (5.3) and (5.5) that

$$\begin{aligned} \int_0^y f(z)dz &\geq \frac{1}{p}yf(y) - \frac{2nd}{p}\frac{1}{pd}yf(y) \\ &= \frac{p-2n}{p^2}yf(y), \ \forall y \in \mathbb{R}. \end{aligned} \tag{5.6}$$

Hence, $(H_3^\infty)(i)$ holds, with $d_2 = \dfrac{p^2}{p-2n}$.

With the condition $(H_3^\infty)(ii)$:

By $G(y) = \int_0^y \frac{|z|^p z}{e+z^2} \ln^{n-1}(e+z^2)dz \geq 0$ for all $y \in \mathbb{R}$ and (5.3), it yealds

$$\int_0^y f(z)dz \leq \frac{1}{p}yf(y) = \frac{d}{p}|y|^p \ln^n(e+y^2) = y^2 F^*(|y|), \tag{5.7}$$

where $F^*(z) = \frac{d}{p}z^{p-2}\ln^n(e+z^2) \to 0$ as $z \to 0_+$.

Thus, $(H_3^\infty)(ii)$ holds.

First, we construct the following Lyapunov functional

$$L_1(t) = E(t) + \delta\Psi(t), \tag{5.8}$$

where $\delta > 0$ is chosen later and

$$\begin{aligned} E(t) &= \frac{1}{2}\|u'(t)\|_0^2 + \frac{1}{2}\int_0^{\|u_x(t)\|_0^2} \mu(z)dz - \int_\rho^1 xdx \int_0^{u(x,t)} f(z)dz \\ &= \frac{1}{2}\|u'(t)\|_0^2 + \left(\frac{1}{2} - \frac{1}{p}\right)\int_0^{\|u_x(t)\|_0^2} \mu(z)dz + \frac{1}{p}I(t), \end{aligned} \tag{5.9}$$

$$\Psi(t) = \langle u'(t), u(t)\rangle + \frac{\lambda}{2}\|u(t)\|_0^2, \tag{5.10}$$

where

$$I(t) = I(u(t)) = \int_0^{\|u_x(t)\|_0^2} \mu(z)dz - p\int_\rho^1 xdx \int_0^{u(x,t)} f(z)dz. \tag{5.11}$$

Then we have the following theorem.

Theorem 5.1. *Assume that* (H_2^*), (H_3^∞), (H_4^∞) *hold. Let* $(\tilde{u}_0, \tilde{u}_1) \in (V \cap H^2) \times V$ *such that* $I(0) > 0$ *and the initial energy* $E(0)$ *satisfy*

$$\eta^* = \mu_* - \frac{1}{2\rho}p(1-\rho^2)(1-\rho)F^*\left(\sqrt{\frac{1-\rho}{\rho}}R_*\right) > 0, \tag{5.12}$$

where

$$\begin{aligned} R_* &= \left(\frac{2pE_*}{(p-2)\mu_*}\right)^{1/2}, \\ E_* &= \left(E(0)+\frac{1}{2}\rho_*\right)\exp\left(\rho_*\right), \\ \rho_* &= \|F\|_{L^1(\mathbb{R}_+;L^2)}. \end{aligned}$$

Let $\mu_{R_*} \equiv \max_{0\leq z\leq R_*^2}\mu(z) < \eta^* + \dfrac{p\mu_*}{d_2}$, *with* μ_*, d_2 *as in* (H_2^*), $(H_3^\infty)(i)$.

Then, for all the global weak solution of Prob. (1.1)-(1.3) is exponential decaying, i.e., there exist positive constants $\bar{C}$, $\bar{\gamma}$ *such that*

$$\|u'(t)\|_0^2 + \|u_x(t)\|_0^2 \leq \bar{C}\exp(-\bar{\gamma}t), \text{ for all } t\geq 0. \tag{5.13}$$

Proof.

First, we need the following lemmas.

Lemma 5.2. *The energy functional* $E(t)$ *defined by* (5.9) *satisfies*

$$\begin{aligned} \text{(i)}\quad E'(t) &\leq \frac{1}{2}\|F(t)\|_0 + \frac{1}{2}\|F(t)\|_0\|u'(t)\|_0^2, \\ \text{(ii)}\quad E'(t) &\leq -\left(\lambda-\frac{\varepsilon_1}{2}\right)\|u'(t)\|_0^2 + \frac{1}{2\varepsilon_1}\|F(t)\|_0^2, \end{aligned} \tag{5.14}$$

for all $\varepsilon_1 > 0$.

Proof. Multiplying $(5.1)_1$ by $xu'(x,t)$ and integrating over $[\rho,1]$, we get

$$E'(t) = -\lambda\|u'(t)\|_0^2 + \langle F(t), u'(t)\rangle. \tag{5.15}$$

On the other hand

$$\langle F(t), u'(t)\rangle \leq \frac{1}{2}\|F(t)\|_0 + \frac{1}{2}\|F(t)\|_0\|u'(t)\|_0^2. \tag{5.16}$$

It follows from (5.15) and (5.16), it is easy to see $(5.14)_{(i)}$ holds.

Similarly,

$$\langle F(t), u'(t)\rangle \leq \frac{1}{2\varepsilon_1}\|F(t)\|_0^2 + \frac{\varepsilon_1}{2}\|u'(t)\|_0^2, \text{ for all } \varepsilon_1>0. \tag{5.17}$$

It follows from (5.15) and (5.17), it is easy to see $(5.14)_{(ii)}$ holds.

Lemma 5.2 is proved completely. □

Lemma 5.3. *Assume that* (H_2^*), (H_3^∞), (H_4^∞) *hold. Then, if we have* $I(0)>0$ *and* (5.12) *holds. Then* $I(t)>0$, $\forall t\geq 0$.

Proof. By the continuity of $I(t)$ and $I(0)>0$, there exists $\tilde{T}_1>0$ such that

$$I(t) = I(u(t)) > 0, \ \forall t\in[0,\tilde{T}_1], \tag{5.18}$$

this implies

$$E(t) \geq \frac{1}{2}\|u'(t)\|_0^2 + \left(\frac{1}{2} - \frac{1}{p}\right)\int_0^{\|u_x(t)\|_0^2}\mu(z)dz \geq \frac{1}{2}\|u'(t)\|_0^2 + \frac{(p-2)\mu_*}{2p}\|u_x(t)\|_0^2, \ \forall t \in [0, \tilde{T}_1]. \tag{5.19}$$

Combining $(5.14)_i$, (5.19) and using Gronwall's inequality to obtain

$$\|u_x(t)\|_0^2 \leq \frac{2p}{(p-2)\mu_*}E(t) \leq \frac{2pE_*}{(p-2)\mu_*} \equiv R_*^2, \ \forall t \in [0, \tilde{T}_1], \tag{5.20}$$

where E_* as in (5.12).

Hence, it follows from $(H_3^\infty, (ii))$ and (5.20) that

$$\begin{aligned} p\int_\rho^1 xdx\int_0^{u(x,t)} f(z)dz &\leq p\int_\rho^1 xu^2(x,t)F^*(|u(x,t)|)dx \\ &\leq \frac{1}{2}p(1-\rho^2)\|u(t)\|_{C^0(\bar{\Omega})}^2 F^*\left(\|u(t)\|_{C^0(\bar{\Omega})}\right) \\ &\leq \frac{1}{2}p(1-\rho^2)\left(\frac{1-\rho}{\rho}\right)\|u_x(t)\|_0^2 F^*\left(\sqrt{\frac{1-\rho}{\rho}}\|u_x(t)\|_0^2\right) \\ &\leq \frac{1}{2\rho}p(1-\rho^2)(1-\rho)F^*\left(\sqrt{\frac{1-\rho}{\rho}}R_*\right)\|u_x(t)\|_0^2. \end{aligned} \tag{5.21}$$

Therefore $I(t) \geq \eta^* \|u_x(t)\|_0^2$, $\forall t \in [0, \tilde{T}_1]$, where η^* as in (5.12).

Now, we put $T_\infty = \sup\{T > 0 : I(t) > 0, \ \forall t \in [0, T]\}$. If $T_\infty < +\infty$ then, by the continuity of $I(t)$, we have $I(T_\infty) \geq 0$. By the same arguments as above, we can deduce that there exists $\tilde{T}_\infty > T_\infty$ such that $I(t) > 0$, $\forall t \in [0, \tilde{T}_\infty]$. Hence, we conclude that $I(t) > 0$, $\forall t \geq 0$.

Lemma 5.3 is proved completely. □

Lemma 5.4. *Assume that* (H_2^*), (H_3^∞), (H_4^∞) *hold. Let* $I(0) > 0$ *and* (5.12) *hold. Put*

$$E_1(t) = \|u'(t)\|_0^2 + \int_0^{\|u_x(t)\|_0^2}\mu(z)dz + I(t). \tag{5.22}$$

Then there exist the positive constants $\bar{\beta}_1$, $\bar{\beta}_2$ *such that*

$$\bar{\beta}_1 E_1(t) \leq L_1(t) \leq \bar{\beta}_2 E_1(t), \ \forall t \geq 0, \tag{5.23}$$

for δ *is small enough.*

Proof. It is easy to see that

$$\begin{aligned} L_1(t) = &\frac{1}{2}\|u'(t)\|_0^2 + \left(\frac{1}{2} - \frac{1}{p}\right)\int_0^{\|u_x(t)\|_0^2}\mu(z)dz + \frac{1}{p}I(t) \\ &+\delta\langle u'(t), u(t)\rangle + \frac{\delta\lambda}{2}\|u(t)\|_0^2. \end{aligned} \tag{5.24}$$

From the following inequalities

$$
\begin{aligned}
(5.25)\qquad \delta\left|\langle u'(t),u(t)\rangle\right| &\leq \frac{\delta}{2}\left\|u'(t)\right\|_0^2+\delta\frac{(1-\rho)^2}{4\rho}\left\|u_x(t)\right\|_0^2,\\
\int_0^{\|u_x(t)\|_0^2}\mu(z)dz &\geq \mu_*\left\|u_x(t)\right\|_0^2,
\end{aligned}
$$

we deduce that

$$
\begin{aligned}
L_1(t) &\geq \frac{1}{2}\left\|u'(t)\right\|_0^2+\left(\frac{1}{2}-\frac{1}{p}\right)\int_0^{\|u_x(t)\|_0^2}\mu(z)dz\\
&\quad+\frac{1}{p}I(t)-\frac{\delta}{2}\left\|u'(t)\right\|_0^2-\delta\frac{(1-\rho)^2}{4\rho}\left\|u_x(t)\right\|_0^2\\
&\geq \frac{1}{2}\left\|u'(t)\right\|_0^2+\left(\frac{1}{2}-\frac{1}{p}\right)\int_0^{\|u_x(t)\|_0^2}\mu(z)dz\\
&\quad+\frac{1}{p}I(t)-\frac{\delta}{2}\left\|u'(t)\right\|_0^2-\delta\frac{(1-\rho)^2}{4\rho\mu_*}\int_0^{\|u_x(t)\|_0^2}\mu(z)dz\\
&= \frac{1-\delta}{2}\left\|u'(t)\right\|_0^2\\
&+ \left(\frac{1}{2}-\frac{1}{p}-\delta\frac{(1-\rho)^2}{4\rho\mu_*}\right)\int_0^{\|u_x(t)\|_0^2}\mu(z)dz+\frac{1}{p}I(t)\\
(5.26)\qquad &\geq \bar{\beta}_1E_1(t),
\end{aligned}
$$

where we choose

$$
(5.27)\qquad \bar{\beta}_1=\min\{\frac{1-\delta}{2},\ \frac{1}{2}-\frac{1}{p}-\delta\frac{(1-\rho)^2}{4\rho\mu_*},\ \frac{1}{p}\},
$$

with δ small enough that $0<\delta<\min\{1;\dfrac{2\rho\mu_*\left(1-\frac{2}{p}\right)}{(1-\rho)^2}\}$.

Similarly, we can prove that

$$
\begin{aligned}
(5.28)\quad L_1(t) &\leq \frac{1}{2}\left\|u'(t)\right\|_0^2+\left(\frac{1}{2}-\frac{1}{p}\right)\int_0^{\|u_x(t)\|_0^2}\mu(z)dz+\frac{1}{p}I(t)\\
&\quad+\frac{1}{2}\delta\left\|u'(t)\right\|_0^2+\delta\frac{(1-\rho)^2}{4\rho}\left\|u_x(t)\right\|_0^2+\delta\lambda\frac{(1-\rho)^2}{4\rho}\left\|u_x(t)\right\|_0^2
\end{aligned}
$$

$$\begin{aligned}
&\le \frac{1+\delta}{2}\left\|u'(t)\right\|_0^2+\left(\frac{1}{2}-\frac{1}{p}\right)\int_0^{\|u_x(t)\|_0^2}\mu(z)dz+\frac{1}{p}I(t)\\
&+ \frac{\delta\left(1+\lambda\right)\left(1-\rho\right)^2}{4\rho}\left\|u_x(t)\right\|_0^2\\
&\le \frac{1+\delta}{2}\left\|u'(t)\right\|_0^2\\
&+ \left(\frac{1}{2}-\frac{1}{p}+\frac{\delta\left(1+\lambda\right)\left(1-\rho\right)^2}{4\rho\mu_*}\right)\int_0^{\|u_x(t)\|_0^2}\mu(z)dz+\frac{1}{p}I(t)\le\bar{\beta}_2E_1(t),
\end{aligned}$$

where $\bar{\beta}_2=\max\left\{\frac{1+\delta}{2},\ \frac{1}{2}-\frac{1}{p}+\frac{\delta(1+\lambda)(1-\rho)^2}{4\rho\mu_*}\right\}$.

Lemma 5.4 is proved completely. □

Lemma 5.5. *Assume that* (H_2^*), (H_3^∞), (H_4^∞) *hold. Let* $I(0)>0$ *and* (5.12) *hold. The functional* $\Psi(t)$ *defined by* (5.10) *satisfies*

$$\begin{aligned}
\Psi'(t) &\le \left\|u'(t)\right\|_0^2+\frac{1}{2\varepsilon_2}\left\|F(t)\right\|_0^2-\frac{\delta_1d_2}{p}I(t)\\
&- \left[\frac{d_2}{p}\left(\frac{p\mu_*}{d_2}+\eta^*-\mu_{R_*}\right)-\frac{\delta_1d_2\eta^*}{p}-\varepsilon_2\frac{(1-\rho)^2}{4\rho}\right]\frac{1}{\mu_{R_*}}\int_0^{\|u_x(t)\|_0^2}\mu(z)dz, \qquad (5.29)
\end{aligned}$$

for all $\varepsilon_2>0$, $\delta_1\in(0,1)$.

Proof. By multiplying $(5.1)_1$ by $xu(x,t)$ and integrating over $[\rho,1]$, we obtain

$$\Psi'(t)=\left\|u'(t)\right\|_0^2-\left\|u_x(t)\right\|_0^2\mu\left(\left\|u_x(t)\right\|_0^2\right)+\langle f(u(t)),u(t)\rangle+\langle F(t),u(t)\rangle. \qquad (5.30)$$

By the following inequalities

$$\begin{aligned}
-\left\|u_x(t)\right\|_0^2\mu\left(\left\|u_x(t)\right\|_0^2\right) &\le -\mu_*\left\|u_x(t)\right\|_0^2,\\
\langle f(u(t)),u(t)\rangle &\le d_2\int_\rho^1 xdx\int_0^{u(x,t)}f(z)dz\\
&= \frac{d_2}{p}\left[\int_0^{\|u_x(t)\|_0^2}\mu(z)dz-I(t)\right],\\
I(t) &\ge \eta^*\left\|u_x(t)\right\|_0^2,\\
\langle F(t),u(t)\rangle &\le \varepsilon_2\frac{(1-\rho)^2}{4\rho}\left\|u_x(t)\right\|_0^2+\frac{1}{2\varepsilon_2}\left\|F(t)\right\|_0^2,\ \forall\varepsilon_2>0, \qquad (5.31)
\end{aligned}$$

we deduce that

$$
\begin{aligned}
\Psi'(t) &= \|u'(t)\|_0^2 - \|u_x(t)\|_0^2 \mu\left(\|u_x(t)\|_0^2\right) + \langle f(u(t)), u(t)\rangle + \langle F(t), u(t)\rangle \\
&\leq \|u'(t)\|_0^2 - \mu_* \|u_x(t)\|_0^2 + \frac{d_2}{p}\left[\int_0^{\|u_x(t)\|_0^2} \mu(z)dz - I(t)\right] \\
&\quad + \varepsilon_2 \frac{(1-\rho)^2}{4\rho}\|u_x(t)\|_0^2 + \frac{1}{2\varepsilon_2}\|F(t)\|_0^2 \\
&= \|u'(t)\|_0^2 - \left(\mu_* - \varepsilon_2\frac{(1-\rho)^2}{4\rho}\right)\|u_x(t)\|_0^2 + \frac{d_2}{p}\int_0^{\|u_x(t)\|_0^2}\mu(z)dz \\
&\quad - \frac{\delta_1 d_2}{p} I(t) - \frac{(1-\delta_1)d_2}{p} I(t) + \frac{1}{2\varepsilon_2}\|F(t)\|_0^2 \\
&\leq \|u'(t)\|_0^2 - \left(\mu_* - \varepsilon_2\frac{(1-\rho)^2}{4\rho}\right)\|u_x(t)\|_0^2 + \frac{d_2}{p}\int_0^{\|u_x(t)\|_0^2}\mu(z)dz \\
&\quad - \frac{\delta_1 d_2}{p} I(t) - \frac{(1-\delta_1)d_2}{p}\eta^* \|u_x(t)\|_0^2 + \frac{1}{2\varepsilon_2}\|F(t)\|_0^2 \\
&= \|u'(t)\|_0^2 - \left(\mu_* + \frac{(1-\delta_1)d_2}{p}\eta^* - \varepsilon_2\frac{(1-\rho)^2}{4\rho}\right)\|u_x(t)\|_0^2 \\
&\quad + \frac{d_2}{p}\int_0^{\|u_x(t)\|_0^2}\mu(z)dz - \frac{\delta_1 d_2}{p} I(t) + \frac{1}{2\varepsilon_2}\|F(t)\|_0^2
\end{aligned}
$$

$$
\begin{aligned}
&\leq \|u'(t)\|_0^2 - \left(\mu_* + \frac{(1-\delta_1)d_2}{p}\eta^* - \varepsilon_2\frac{(1-\rho)^2}{4\rho}\right)\frac{1}{\mu_{R_*}}\int_0^{\|u_x(t)\|_0^2}\mu(z)dz \\
&\quad + \frac{d_2}{p}\int_0^{\|u_x(t)\|_0^2}\mu(z)dz - \frac{\delta_1 d_2}{p} I(t) + \frac{1}{2\varepsilon_2}\|F(t)\|_0^2 \\
&= \|u'(t)\|_0^2 \\
&\quad - \left[\frac{d_2}{p}\left(\frac{p\mu_*}{d_2} + \eta^* - \mu_{R_*}\right) - \frac{\delta_1 d_2 \eta^*}{p} - \varepsilon_2\frac{(1-\rho)^2}{4\rho}\right]\frac{1}{\mu_{R_*}}\int_0^{\|u_x(t)\|_0^2}\mu(z)dz \\
&\quad - \frac{\delta_1 d_2}{p} I(t) + \frac{1}{2\varepsilon_2}\|F(t)\|_0^2. \qquad (5.32)
\end{aligned}
$$

Hence, the Lemma 5.5 is proved by using some simple estimates. □

Now we continue to prove Theorem 5.1.

Then, we deduce from (5.8), (5.14)(*ii*) and (5.29) that

$$\begin{aligned} L_1'(t) &\leq -\left(\lambda - \frac{\varepsilon_1}{2} - \delta\right) \|u'(t)\|_0^2 \\ &- \delta\left[\frac{d_2}{p}\left(\frac{p\mu_*}{d_2} + \eta^* - \mu_{R_*}\right) - \frac{\delta_1 d_2 \eta^*}{p} - \varepsilon_2 \frac{(1-\rho)^2}{4\rho}\right] \frac{1}{\mu_{R_*}} \int_0^{\|u_x(t)\|_0^2} \mu(z)dz \\ &- \frac{\delta\delta_1 d_2}{p} I(t) + \frac{1}{2}\left(\frac{1}{\varepsilon_1} + \frac{\delta}{\varepsilon_2}\right) \|F(t)\|_0^2 \end{aligned} \tag{5.33}$$

for all δ, ε_1, $\varepsilon_2 > 0$, $\delta_1 \in (0,1)$.

Because of $\mu_{R_*} < \eta^* + \frac{p\mu_*}{d_2}$ and

$$\begin{aligned} &\lim_{\delta_1 \to 0_+,\, \varepsilon_2 \to 0_+} \left[\frac{d_2}{p}\left(\frac{p\mu_*}{d_2} + \eta^* - \mu_{R_*}\right) - \frac{\delta_1 d_2 \eta^*}{p} - \varepsilon_2 \frac{(1-\rho)^2}{4\rho}\right] \frac{1}{\mu_{R_*}} \\ &= \frac{d_2}{p\mu_{R_*}}\left(\frac{p\mu_*}{d_2} + \eta^* - \mu_{R_*}\right) > 0, \end{aligned}$$

we can choose $\delta_1 \in (0,1)$ and $\varepsilon_2 > 0$ such that

$$\sigma_1 = \left[\frac{d_2}{p}\left(\frac{p\mu_*}{d_2} + \eta^* - \mu_{R_*}\right) - \frac{\delta_1 d_2 \eta^*}{p} - \varepsilon_2 \frac{(1-\rho)^2}{4\rho}\right] \frac{1}{\mu_{R_*}} > 0. \tag{5.34}$$

Then, for ε_1 small enough such that $0 < \frac{\varepsilon_1}{2} < \lambda$ and if $\delta > 0$ such that

$$\sigma_2 = \lambda - \frac{\varepsilon_1}{2} - \delta > 0,\ 0 < \delta < \min\left\{1; \frac{2\rho\mu_*}{(1-\rho)^2}\left(1 - \frac{2}{p}\right)\right\}, \tag{5.35}$$

it follows from (5.23), (5.33)-(5.35) that

$$L_1'(t) \leq -\bar{\gamma} L_1(t) + \tilde{C}_0 e^{-2\bar{\gamma}_0 t}, \tag{5.36}$$

where $0 < \bar{\gamma} < \min\{\frac{\bar{\beta}_3}{\bar{\beta}_2},\ 2\bar{\gamma}_0\}$ with $\bar{\beta}_3 = \min\{\delta\sigma_1,\ \sigma_2,\ \frac{\delta\delta_1 d_2}{p}\}$ and $\tilde{C}_0 = \frac{1}{2}\left(\frac{1}{\varepsilon_1} + \frac{\delta}{\varepsilon_2}\right)\bar{C}_0^2$.

On the other hand, we have

$$L_1(t) \geq \bar{\beta}_1 \min\{1, \mu_*\} \left(\|u'(t)\|_0^2 + \|u_x(t)\|_0^2\right). \tag{5.37}$$

Combining (5.36) and (5.37) we get (5.13). Theorem 5.1 is proved completely. $\square$

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