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Characterization of Prime and Maximal Ideals of Product Rings by $\mathcal{F}-\lim$

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Abstract. Let $\{R_i\}_{i\in I}$ be an infinite family of rings and $R = \prod_{i\in I} R_i$ their product. In this paper, we investigate the prime spectrum of R by $\mathcal{F}$-limits. Special attention is paid to relationship between the elements of $Spec(R_i)$ and the elements of $Spec(\prod_{i\in I} R_i)$ use $\mathcal{F}-\lim$, also we give a new condition so that $\prod_{i\in I} R_i$ is a zero dimensional ring.

1. Introduction

Let R be a commutative ring with an identity element. We also denote by $Max(R)$, $\beta(I)$, respectively the collection of all Maximal ideals of R, the set of all ultrafilters on I. We recall that R is a zero-dimensional ring if all prime ideals are maximal.

Let $\{\mathbf{R}_i\}_{i\in I}$ be a nonempty family of rings and $R = \prod_{i\in I} R_i$ their product. Several articles in the literature have dealt with the problem of characterization of the prime ideals of R with purely algebraic methods [2]. Particularly, it investigated the ultraproducts of $\{\mathbf{R}_i\}_{i\in I}$. There has been a great deal of work concerning the relationship between ultrafilter on I and maximal ideals in products of $\{R_i\}_{i\in I}$ (see [3]). On the other hand, the notion of $\mathcal{F}-\lim$ of a sequence of prime ideals is related to a construction proposed by S. Garcia-Ferreira and L. M. Ruza-Montilla in [4] by giving some properties on the prime spectrum of a commutative ring. The purpose of the present work is to give a characterization of the ideals on $Spec(\prod_{i\in I} R_i)$ by $\mathcal{F}-\lim$, and give a relationship between the elements of $Spec(R_i)$ and the elements of $Spec(\prod_{i\in I} R_i)$ use $\mathcal{F}-\lim$, we finish by giving a new condition for a product of zero-dimensional rings to be a zero-dimensional ring.

The paper is organized as follows, in the first section, we define the $\mathcal{F}-\lim$ of a collection of ideals for a commutative ring R and give some basic properties.

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Thereby, we characterize the maximal ideals of $\prod_{i\in I} R_i$ with $\mathcal{F}-\lim$. While in the second section, we give a relation between the maximal and prime ideals of R_i and the maximal ideals of $\prod_{i\in I} R_i$ use $\mathcal{F}-\lim$.

Now we state without proof some well-known properties (see [3]).

2. Notation and Preliminaries

We begin by giving some notation and preliminary results. If $\{R_i\}_{i\in I}$ is a nonempty family of rings indexed by a set I, their product will be denoted by $\prod_{i\in I} R_i$. Elements of this product are frequently considered in two different ways. The most rigorous way is to consider $\prod_{i\in I} R_i$ as the set of all functions $f : I \to \bigcup_{i\in I} R_i$ such that $f(i) \in R_i$ for each $i \in I$. An alternate approach is to regard elements of $\prod_{i\in I} R_i$ as tuples $\{r_i\}_{i\in I}$, where $r_i \in R_i$ for each i. We are interested in the first perspective.

We will work in at least ZFC, that is, Zermelo-Frankel set theory with the axiom of choice. We will in certain cases use additional axioms. We recall that $\mathcal{F}$ is a filter on I if it is a subset of the power set of I that satisfies the following conditions:

1. $\emptyset \notin \mathcal{F}$ and $I \in \mathcal{F}$;
2. If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$;
3. If $A \in \mathcal{F}$ and $A \subset A' \subset I$, then $A' \in \mathcal{F}$;

A filter $\mathcal{F}$ on I is called an ultrafilter if $\mathcal{F}$ is maximal with respect to being a filter, or equivalently, if whenever $A \subset I$, then either $A \in \mathcal{F}$ or $I \setminus A \in \mathcal{F}$. An ultrafilter $\mathcal{F}$ is called principal if there exists an element $i_0 \in I$ such that $\mathcal{F}$ consist of all subsets of I that contain i_0. Other ultrafilters are called non-principal. We denote the collection of all ultrafilters on a set I by $\beta(I)$. Now we state without proof some well-known properties (see [3]).

Remark 2.1. Let $\{R_i\}_{i\in I}$ be a nonempty family of commutative unitary rings indexed by a set I and let $R = \prod_{i\in I} R_i$. For $f \in R$, let $\mathcal{Y}(f) := \{i \in I \ : \ f(i) \in R_i \setminus U(R_i)\}$, where $U(R_i)$ denotes the set of all unit elements of R_i. For an ideal J of R, let $\mathcal{Y}(J) = \{\mathcal{Y}(f) \ : \ f \in J\}$.

1. If J is a proper ideal of R, then $\mathcal{Y}(J)$ is a filter on I; if J is prime, $\mathcal{Y}(J)$ is an ultrafilter on I;
2. If each R_i is quasilocal and $\mathcal{F}$ is a filter on I, then $J_{\mathcal{F}} = \{f \in R : \mathcal{Y}(f) \in \mathcal{F}\}$ is an ideal of R and $\mathcal{Y}(J_{\mathcal{F}}) = \mathcal{F}$. Thus $J_{\mathcal{F}}$ is the unique largest ideal of R such that $\mathcal{Y}(J) = \mathcal{F}$;
3. If each R_i is quasilocal, then the map $M \to \mathcal{Y}(M)$ is a bijection between the set of maximal ideals of R and the set of ultrafilter on I, this map restricts to a bijection between the set of maximal ideals of R that contain an ideal P and the set of free ultrafilter on I. Moreover, each $P \in Spec(R)$ is contained in a unique maximal ideal of R.

In the following definition, we give a construction of the ideals of $\prod_{i\in I} R_i$ starting from the ideals of R_i using the ultrafilters.

Definition 2.2. Let $R = \prod_{i\in I} R_i$, where I is an infinite set and each R_i is nonzero rings, recall $\hat{i}$ denotes the principal ultrafilter on I generated by $\{i\}$, we define for any ideal N_i of R_i the set $(\hat{i}, N_i) := \{r \in R \ : \ r_i \in N_i\}$.

It is easy to show that $(\hat{i}, N_i)$ is also an ideal of $\prod_{i\in I} R_i$. Notice that

$$(\hat{i}, N_i) = \{r \in R \ : \ \{j \in I \ : \ r_j \in N_j\} \in \hat{i}\}.$$

Example 2.3. Let $f \in \prod_{i\in I} R_i$ such that $f(k) = 1$ if $k \neq i$ and $f(i) = 0$. Then $(\hat{i}, 0) = (f)$.

Definition 2.4. Let A be a set, $S(A)$ be the set of all subsets of A and let I be a set, $\mathcal{F}$ an ultrafilter on I, and $\{S_i \ : \ i \in I\} \subseteq S(A)$, we define

$$\mathcal{F} - \lim_{i\in I} S_i := \{a \in A : \{i \in I \ : \ a \in S_i\} \in \mathcal{F}\}.$$

We note that the set $\mathcal{F} - \lim_{i\in I} S_i$ is a subset of A.

Example 2.5.

1. Let R be ring and let I be an infinite set. If $P_i \in Spec(R)$, for each $i \in I$, and $\mathcal{F}$ is an ultrafilter on I, then

   $$\mathcal{F} - \lim_{i\in I} P_i := \{a \in \mathbf{R} : \{i \in I \ : \ a \in P_i\} \in \mathcal{F}\}.$$

   is a prime ideal. We check the prime property.
   Let $P_i \in Spec(R)$ and suppose that $ab \in \mathcal{F} - \lim_{i\in I} P_i$, then $A = \{i \in I \ : \ ab \in P_i\} \in \mathcal{F}$.
   Then $A = \{i \in I \ : \ a \in P_i\} \cup \{i \in I \ : \ b \in P_i\}$ and, since $\mathcal{F}$ is an ultrafilter, then $a \in \mathcal{F} - \lim_{i\in I} P_i$ or $b \in \mathcal{F} - \lim_{i\in I} P_i$, then $\mathcal{F} - \lim_{i\in I} P_i$ is a prime ideal.

2. Let K be a field, let A be a ring contained in K, and let $Zar(K|A)$ denote the set of valuation domains containing A with quotient field K, suppose that $Zar(K|A) = \{V_i \ : \ i \in I\}$ and that $\mathcal{F}$ is an ultrafilter on I, then

   $$\mathcal{F} - \lim_{i\in I} V_i := \{a \in \mathbf{R} : \{i \in I \ : \ a \in V_i\} \in \mathcal{F}\}.$$

   is a valuation ring.
   Indeed, let $x \in K$. If $x \notin \mathcal{F} - \lim_{i\in I} V_i$, then $\{i \in I \ : \ x \in V_i\} \notin \mathcal{F}$. Since $\mathcal{F}$ is an ultrafilter and each V_i is a valuation ring with quotient field K, then $I \backslash \{i \in I \ : \ x \in V_i\} = \{i \in I \ : \ x \notin V_i\} = \{i \in I \ : \ x^{-1} \in V_i\} \in \mathcal{F}$, then $x^{-1} \in \mathcal{F} - \lim_{i\in I} V_i$, form where $\mathcal{F} - \lim_{i\in I} V_i$ is a valuation ring.

Next, we start by the basic theorem of the $\mathcal{F}-\lim$ of a collection of subsets.
Theorem 2.6. *Let I be a set, $\mathcal{F}$ an ultrafilter on I, and $\{S_i : i \in I\} \subseteq S(A)$, then:*

1. *$\hat{k}-\lim_{i\in I} S_i = S_k$ for each $k \in I$, with $\hat{k} = \{X \subseteq I : k \in X\}$;*
2. *If $J \in \mathcal{F}$, then*
$$\mathcal{F}-\lim_{i\in I} S_i = \mathcal{F}\mid_J - \lim_{i\in J} S_i.$$
3. *Let Γ be an infinite set, and let $\sigma : \Delta \to \Gamma$ be a surjective function. For each $j \in \Gamma$ put $T_j = S_i$ if $\sigma(i) = j$. Then,*
$$\mathcal{F}-\lim_{i\in\Delta} S_i = \mathcal{C}-\lim_{j\in\Gamma} T_j.$$
where $\sigma(\mathcal{F}) = \{\sigma[F] : F \in \mathcal{F}\} = \mathcal{C}$.

Proof.

1. Is immediate form definition of principal ultrafilter;
2. $a \in \mathcal{F}-\lim_{i\in I} S_i \Rightarrow \{i \in I : a \in S_i\} \in \mathcal{F} \Rightarrow \{i \in I : a \in S_i\} \in \mathcal{F}\mid_J$.
$$\Rightarrow a \in \mathcal{F}\mid_J - \lim_{i\in J} S_i.$$
Thus, $\mathcal{F}-\lim_{i\in I} S_i \subseteq \mathcal{F}\mid_J - \lim_{i\in I} S_i$.
Suppose that $a \in \mathcal{F}\mid_J -lim_{i\in J}S_i$ and that $\{i \in I : a \notin S_i\} \in \mathcal{F}$.
Then $\{i \in J : a \in S_i\} \in \mathcal{F}\mid_J$ and $\{i \in J : a \notin S_i\} \in \mathcal{F}\mid_J$ which is impossible. Therefore,
$$\mathcal{F}\mid_J -lim_{i\in J}S_i \subseteq \mathcal{F} - lim_{i\in I}S_i;$$
3. $a \in \mathcal{F}-\lim_{i\in\Delta} S_i \Leftrightarrow \{i \in \Delta : a \in S_i\} \in \mathcal{F} \Leftrightarrow \{j \in \Gamma : a \in T_j\} \in \mathcal{C}$
$$\Leftrightarrow a \in \mathcal{C}-\lim_{j\in\Gamma} T_j.$$

□

In the following proprieties, we give a characterization of the ideals by ultrafilter that allows us to give results already known with a simpler method.

Definition 2.7. If $\mathcal{I}$ and $\mathcal{J}$ are ideals in R, their ideal quotient is $(\mathcal{I} : \mathcal{J}) = \{x \in R : x\mathcal{J} \subseteq \mathcal{I}\}$.
In particular, $(0 : \mathcal{J})$ is called the annihilator of $\mathcal{J}$ and it is also denoted by $Ann(\mathcal{J})$. In this notation, the set of all zero-divisors in R is $D = \bigcup_{x\neq 0} Ann(x)$(see [1, Chapter.1])

Proposition 2.8. *Let $R = \prod_{i\in I} R_i$, where I is infinite and each R_i is nonzero quasilocal rings and $\mathcal{F}$ is an ultrafilter on I, then*

1. *For* $(f_1, ..., f_n) \in R$, *we have*
$$((f_1, ..., f_n)R_{J_\mathcal{F}} : R_{J_\mathcal{F}}) = \mathcal{F} - \lim_{i\in I}(\hat{a}, (f_1(i), ..., f_n(i))R_i);$$
2. *An prime ideal* P *in* R *is contained in a unique maximal ideal* $J_\mathcal{F}$ *if only if* $\mathcal{F} - \lim_{i\in I}(\hat{i}, 0) \subset P$;
3. $(0 : R_{J_\mathcal{F}}) = \mathcal{F} - \lim_{i\in I}(\hat{i}, 0)$;
4. $J_\mathcal{F} = \mathcal{F} - \lim_{i\in I}(\hat{i}, R_i \setminus U(R_i))$.

Proof.

1.
$$h \notin J_\mathcal{F} \Leftrightarrow \{i \in I,\, h(i) \textit{ is a nonunit of } R_i\} \notin \mathcal{F}$$
$$\Leftrightarrow \{i \in I,\, h(i) \textit{ is a unit of } R_i\} \in \mathcal{F}$$
On the other hand,
$g \in ((f_1, ..., f_n)R_{J_\mathcal{F}} : R_{J_\mathcal{F}}) \Leftrightarrow \exists r \in R\backslash J_\mathcal{F}$ such that $gr \in (f_1, ..., f_n)R$. Then $g(i) \in (f_1(i), ..., f_n(i))R_i$ for each i such that $r(i)$ is a unit of R_i and $X = \{i \in I; g(i) \in (f_1(i), ..., f_n(i))R_i\} \in \mathcal{F}$, form where $g \in \mathcal{F} - \lim_{i\in I}(\hat{i}, (f_1(i), ..., f_n(i))R_i)$.
Conversely, if $g \in \mathcal{F} - \lim_{i\in I}(\hat{i}, (f_1(i), ..., f_n(i))R_i)$, then $X \in \mathcal{F}$ with $X = \{i \in I; g(i) \in (f_1(i), ..., f_n(i))R_i\}$. Let $r(i) = 1$ if $i \in X$ and $r(i) = 0$ if $i \notin X$. Then $r \in R\backslash J_\mathcal{F}$ and $(gr)(i) = 0$ for each $i \in I\backslash X$. Therefore $g(i)r(i) \in (f_1(i), ..., f_n(i))R_i$ for each $i \in I$, which means $gr \in (f_1, ..., f_n)R$. Then $g \in ((f_1, ..., f_n)R_{J_\mathcal{F}} : R_{J_\mathcal{F}})$.
2. If $P \in Spec(R)$ is not contained in $J_\mathcal{F}$, then exist an ultrafilter $\mathcal{G}$ on I distinct from $\mathcal{F}$ ($\mathcal{G} = \mathcal{Y}(P)$). Hence there exist a subset X of I such that $X \in \mathcal{F}$ and $I\backslash X \in \mathcal{G}$. The characteristic function $e_X \in \mathcal{F} - \lim_{i\in I}(\hat{i}, 0)\backslash P$, so $\mathcal{F} - \lim_{i\in I}(\hat{i}, 0)$ is not contained in P.
3. Is immediate form (1) following by taking $(f_1, ..., f_n)R = (0)$.
4. Is immediate from the Definition 2.2, Definition 2.4 and Remark 2.1.

□

Corollary 2.9. *Let* $\mathbb{Q}$ *the field of rational numbers and* $\mathbb{N}$ *the natural number, then* $\mathbb{Q}^\mathbb{N}$ *is a zero dimensional ring.*

Proof. In Proposition 2.8(4) Let $R_i = \mathbb{Q}$ and $I = \mathbb{N}$, we have that $R_i \setminus U(R_i) = 0$. Then
$$J_\mathcal{F} = \mathcal{F} - \lim_{i\in I}(\hat{i}, 0).$$
According to Proposition 2.8(2) and [5, Theorem 1.3] the ring $\mathbb{Q}^\mathbb{N}$ is a zero dimensional ring □

Corollary 2.10. *Let $R = \prod_{i \in I} R_i$, where A is infinite and R_i is nonzero quasilocal ring and $\mathcal{F}$ is an ultrafilter on I. If $B \in \mathcal{F}$ such that R_b is a fields for each $b \in B$, then*

$$J_{\mathcal{F}} = \mathcal{F} - \lim_{i \in I}(\hat{\imath}, 0).$$

Proof. Let $\mathcal{F}$ an ultrafilter on I and $B \in \mathcal{F}$, then by Theorem 2.6 we have :

$$\mathcal{F} - \lim_{i \in I}(\hat{\imath}, 0) = \mathcal{F} \mid_B - \lim_{b \in B}(\hat{b}, 0).$$

On the other hand,

$$\begin{aligned} J_{\mathcal{F}} &= \mathcal{F} - \lim_{i \in I}(\hat{\imath}, R_i \setminus U(R_i)) \\ &= \mathcal{F} \mid_B - \lim_{b \in B}(\hat{b}, R_b \setminus U(R_b)) \\ &= \mathcal{F} \mid_B - \lim_{b \in B}(\hat{b}, 0) \end{aligned}$$

Then:

$$J_{\mathcal{F}} = \mathcal{F} - \lim_{i \in I}(\hat{\imath}, 0).$$

□

Remark 2.11. We do not necessarily have that an $\mathcal{F}-\lim$ of maximal ideals will be a maximal ideal. For example, for each $\mathcal{F}$ on $Spec(\mathbb{Z})$, one has $\mathcal{F} - \lim_{p \in \mathbb{Z}}(p) = (0)$ is a prime ideal but not maximal.

In the following property, we give a type of maximal ideal which is stable by the $\mathcal{F} - \lim$.

Proposition 2.12. *Let $R = \prod_{i \in I} R_i$, where I is infinite set and each R_i is quasilocal rings and let $\mathcal{F}$ be an ultrafilter on I. If $(J_{\mathcal{F}_i})_{i \in I}$ is a collection of maximal ideal of R, then*

$$\mathcal{F} - \lim_{i \in I} J_{\mathcal{F}_i} = J_{\mathcal{F} - \lim_{i \in I} \mathcal{F}_i}$$

Proof. If $a \in \mathcal{F} - lim_{i \in I}\ J_{\mathcal{F}_i}$, then:

$$A = \{i \in I \,:\, a \in J_{\mathcal{F}_i}\} = \{i \in I \,:\, \{j \in I \ :\ a(j) \in R_j \setminus U(R_j)\} \in \mathcal{F}_i\} \in \mathcal{F}.$$

If $a \notin J_{\mathcal{F}_i}$, then $\{j \in I \ :\ a(j) \in R_j \setminus U(R_j)\} \notin \mathcal{F} - \lim_{i \in I} \mathcal{F}_i$, hence $B = \{i \in I \ :\ \{j \in I \ :\ f(j) \in R_j \setminus U(R_j)\} \notin \mathcal{F}_i\} \in \mathcal{F}$; but if $i \in A \bigcap B$, then $\{j \in I \ :\ a(j) \in R_j \setminus U(R_j)\} \in \mathcal{F}_i$ and $\{j \in I \ :\ a(j) \in R_j \setminus U(R_j)\} \notin \mathcal{F}_i$, which is a contradiction. Thus, we must have that $a \in J_{\mathcal{F} - \lim_{i \in I} \mathcal{F}_i}$. Therefore, $\mathcal{F} - lim_{i \in I}\ J_{\mathcal{F}_i} \subseteq J_{\mathcal{F} - \lim_{i \in I} \mathcal{F}_i}$. Now, we suppose that $f \in J_{\mathcal{F}_i}$. Then, $\{j \in I \ :\ a(j) \in R_j \setminus U(R_j)\} \in \mathcal{F}$. By the previous definition of maximal ideal, we have that:

$$\{i \in I \,:\, \{j \in I \ :\ a(j) \in R_j \setminus U(R_j)\} \in \mathcal{F}_i\} = \{i \in I \ :\ a \in J_{\mathcal{F}_i}\} \in \mathcal{F}$$

That is $a \in \mathcal{F} - limJ_{\mathcal{F}_i}$. Thus, we get that $J_{\mathcal{F}-\lim_{i\in I}\mathcal{F}_i} \subseteq \mathcal{F} - lim_{i\in I}\ J_{\mathcal{F}_i}$. □

3. Relationship Between $Max(R_i)$ and $Max(\prod_{i\in I} R_i)$

In this section, we will give some theorems that allow you to find the maximum ideals of R_i from the maximum ideals of $\prod_{i\in I} R_i$ and vice versa using the notion of $\mathcal{F} - \lim$

Theorem 3.1. *Let $R = \prod_{i\in I} R_i$, where I is infinite and each R_i is nonzero commutative ring and P is a prime ideal of R, then there is an infinite set K and an ultrafilter $\mathcal{E}$ on K such that for each $k \in K$, there are prime ideals of R of the form $J_i = \mathcal{F}_k - \lim_{i\in I}(\hat{i}, J_{i,k})$ where $\mathcal{F}_k$ is an ultrafilter on I and $J_{i,k}$ are prime ideals in R_i, such that:*

$$P = \mathcal{E} - \lim_{k\in K} J_k.$$

To prove this result, we need the following Lemma.

Lemma 3.2. *Let $R = \prod_{i\in I} R_i$. For $a_1, ..., a_p \in R$ and $x \in R$, if $x \notin (a_1, ..., a_p)$, then there is an ultrafilter $\mathcal{F}$ on I and ideals $\{P_i\}_{i\in X}$ for some $X \in \mathcal{F}$ such that $(a_1, ..., a_p) \subseteq \mathcal{F} - \lim_{i\in I}(\hat{i}, P_i)$ and $x \notin \mathcal{F} - \lim_{i\in I}(\hat{i}, P_i)$.*

Proof. Clearly, $x \in (a_1, ..., a_p)$ if and only if there are $r_1, ..., r_p \in R$ such that $\{i \in I \ : \ x(i) = r_1a_1(i) + ... + a_1y_1(i)\} = I$ if and only if $\{i \in I \ : \ x(i) \in (a_1(i), ..., a_p(i))\} = I$. Thus, by hypothesis $X = \{i \in I \ : \ x(i) \in (a_1(i), ..., a_p(i))\} \neq I$. On the other hand, $\mathcal{F}$ is an ultrafilter, then $Y = I \setminus X \in \mathcal{F}$. Now for $i \in Y$ we can by our remark find ideals P_i of R_i maximal with respect to the property that $x(i) \notin P_i$ and $(a_1(i), ..., a_p(i)) \subseteq P_i$. So $\mathcal{F} - \lim_{i\in I}(\hat{i}, P_i)$ is an ideal of R and $(a_1, ..., a_p) \subseteq \mathcal{F} - \lim_{i\in I}(\hat{i}, P_i)$ and $x \notin \mathcal{F} - \lim_{i\in I}(\hat{i}, P_i)$. □

Proof of Theorem 3.1.

Let P be a prime ideal of R and define $T = P \times (R \setminus P)$. Let K be the set of all nonempty finite of T. For $k \in K$, say $k = \{(y_1, x_1), .., (y_p, x_p)\}$, where $y_i \in P$ and $x_i \notin P$ for each $i = 1, ..., p$. Clearly, $x = x_1...x_p \notin (y_1, ..., y_p) \subseteq P$ and by Lemma 3.2 we can find an ideal $J_i = \mathcal{F}_k - \lim_{i\in I}(\hat{i}, J_{i,k})$ where $\mathcal{F}_k$ is an ultrafilter on I and $J_{i,k}$ are ideals of R_i for all i in I such that $(y_1, ..., y_p) \subseteq J_i$ and $x \notin J_i$. On the other hand, for $t \in T$ let $\hat{t} = \{k \in K \ : \ t \in k\}$ and let $G = \{\hat{t} \ : \ t \in T\}$. Obviously, G has finite intersection property. Thus, G can be extended to a nonprincipal ultrafilter $\mathcal{F}$ on J. We claim that $P = \mathcal{E} - \lim_{k\in K} J_k$. For let $y \in P$. So $(y, 1) \in T$ and hence $(\hat{y}, 1) \in G$ with $(\hat{y}, 1) = \{k \in K \ : \ (y, 1) \in k\}$. So for $k \in (y, 1)$ we have that $y \in J_k$. Thus, $\{k \in K \ : \ y \in J_k\} \in \mathcal{F}$. Hence $y \in \mathcal{E} - \lim_{k\in K} J_k$. Then $P \subseteq \mathcal{E} - \lim_{k\in K} J_k$. Now, suppose $x \notin P$ and so $(0, x) \in T$. Let $k \in (0, x)$ then $x \notin J_k$. Thus, $\{k \in K \ : \ x \in J_k\} \bigcap (0, x) = \varnothing$. But $(0, x) \in \mathcal{F}$ and so $\{k \in K \ : \ y \in J_k\} \notin \mathcal{F}$. Then, $x \notin \mathcal{E} - \lim_{k\in K} J_k$. Thus $\mathcal{E} - \lim_{k\in K} J_k \subseteq P$. □

Theorem 3.3. *Let $R = \prod_{i\in I} R_i$, where I is infinite and each R_i is nonzero commutative rings. If $J_{i,k}$ are maximal ideals for all $i \in I$ and $k \in K$ such that* *for some $Y \subseteq I$, for $i \in Y$ and all $a \in R_i$, if $O_i = \{M \in Max(R_i) \ : \ a_i \notin M\} \neq \emptyset$. Then, for some $t_i \in \bigcap O_i$ and $r_i \in R_i$, $r_i a_i + t_i = 1$ in R_i, and let $\mathcal{F}_k$ are nonprincipal ultrafilters on I for each $k \in K$, $\mathcal{E}$ is a nonprincipal ultrafilter on K, then there are ideals of R_i of the form $J_k = \mathcal{F}_k - \lim_{i \in I} J_{i,k}$ such that $\mathcal{E} - \lim_{k \in K}(\hat{k}, J_k)$ is maximal ideal of R.*

Proof. According to the above results, $\mathcal{E} - \lim_{k \in K}(\hat{k}, J_k)$ is an ideal of R. Suppose $a \notin \mathcal{E} - \lim_{k \in K}(\hat{k}, J_k)$. Then $\{k \in K \ : \ a \in \mathcal{F}_k - \lim_{i \in I} J_{i,k}\} \notin \mathcal{E}$ and so $Y = \{k \in K \ : \ a \notin \mathcal{F}_k - \lim_{i \in I} J_{i,k}\} \in \mathcal{E}$. Now, for $k \in Y$, $\{i \in Y \ : \ a_i \in J_{i,k}\} \notin \mathcal{F}_k$. So, for $k \in Y$, $\mathcal{F}_k \setminus \{i \in Y \ : \ a_i \in J_{i,k}\} \in \mathcal{F}_k$ (because $\mathcal{F}_k$ is an ultrafilter), then $\{i \in Y \ : \ a_i \notin J_{i,k}\} \in \mathcal{F}_k$. We choose $Z \subseteq I$, then for $i \in Z$, we can find a $t_i \in \cap\{J_{i,k} \ : \ a_i \notin J_{i,k} \wedge k \in K\}$ if $\{J_{i,k} \ : \ a_i \notin J_{i,k} \wedge k \in K\} \neq \emptyset$ and $r_i \in R_i$ such that $r_i a_i + t_i = 1$. If $i \notin Z$ or $\{J_{i,k} \ : \ a_i \notin J_{i,k} \wedge k \in K\} = \emptyset$, then we set $t_i = 1$ and $r_i = 0$. Let $k \in Y$. So for $i \in \{i \in Y \ : \ a_i \notin J_{i,k}\} \bigcap Z$, $t_i \in J_{i,k}$ and hence $t_i \in \mathcal{F}_k - \lim_{i \in I} J_{i,k}$ since $r_i a_i + t_i = 1$ for all $i \in I$. Then $(ra-1) \in \mathcal{E} - \lim_{k \in K}(\hat{k}, J_k)$ and $\mathcal{E} - \lim_{k \in K}(\hat{k}, J_k)$ is a maximal ideal of $\prod_{i \in I} R_i$. □

Corollary 3.4. *Suppose $\{R_i\}_{i \in I}$ is a nonempty family of zero dimensional rings and let $R = \prod_{i \in I} R_i$ such that for some $Y \subseteq I$, for $i \in Y$ and all $a_i \in R_i$, if $O_i = \{M \in Max(R_i) \ : \ a_i \notin M\} \neq \emptyset$, then for some $t_i \in \bigcap O_i$ and $r_i \in R_i$, $r_i a_i + t_i = 1$ in R_i, then $dim(R) = 0$.*

Proof. Let P be a prime ideal of R, then by Theorem 3.1 $P = \mathcal{E} - \lim_{k \in K} J_k$ such that $J_i = \mathcal{F}_k - \lim_{i \in I}(\hat{i}, J_{i,k})$ and $J_{i,k}$ are prime ideals of R_i, as $R_i' s$ are zero dimensional rings $J_{i,k}$ are maximal ideals, according to the Theorem 3.3 P is a maximal ideal of R. From where, the ring R is a zero dimensional ring.

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