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Extending and Refining Some Inequalities for the Beta Function in n Variables

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Abstract. The fundamental goal of this paper is to investigate some inequalities involving the special beta function in n variables. Our theoretical results obtained here are extensions and refinements for some inequalities already discussed in the literature.

1. Introduction

Let $\mathbb{C}_+ := \{z \in \mathbb{C} : \Re e(z) > 0\}$. The celebrated Euler's Beta and Gamma functions are, respectively, defined by

$$\forall x, y \in \mathbb{C}_+ \quad B(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}dt$$

$$\forall x \in \mathbb{C}_+ \quad \Gamma(x) := \int_0^{+\infty} e^{-t}t^{x-1}dt.$$

The functions B and Γ, which play a central and primordial place in some different areas of mathematics, are useful from the theoretical point of view as well as for practical purposes. In this introductive section, we will restrict ourselves to recall some properties of B and Γ that will be needed throughout this paper.

One of the most elementary properties, expressing a connection between B and

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Γ, is the following relationship

$$\forall x, y \in \mathbb{C}_+ \quad B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{1.1}$$

It is also well-known that

$$\forall x \in \mathbb{C}_+ \quad \Gamma(x+1) = x\Gamma(x). \tag{1.2}$$

It is worth mentioning that (1.2) has many consequences and applications. First, (1.2) implies that $\Gamma(n+1) = n!$ for any integer $n \geq 0$ and so, $\Gamma(x)$ can be considered as an extension of the factorial function from positive integers to $x \in \mathbb{C}_+$. Secondly, (1.2) implies that

$$\Gamma(x) = \frac{\Gamma(x+1)}{x} = \frac{\Gamma(x+2)}{x(x+1)} = \frac{\Gamma(x+3)}{x(x+1)(x+2)} = ...,$$

which tells us that $\Gamma(x)$, previously defined for $x \in \mathbb{C}_+$, can be extended for any $x \in \mathbb{C}$ such that $x \neq 0, -1, -2, ...$ This, with (1.1), implies that $B(x,y)$ previously defined for $x, y \in \mathbb{C}_+$ can be in its turn extended for any $x, y \in \mathbb{C}$ such that $x, y, x+y \neq 0, -1, -2, ...$ For more details and information about the preceding discussion as well as for further properties and applications of B and Γ, we refer the interested reader to [2, 3, 4, 5, 6, 10, 11, 12] for instance.

This paper will be organized as follows: after this short introduction, Section 2 is devoted to recall the definition as well as the elementary properties of the beta function in n variables that will be needed throughout this paper. Section 3 is focused to investigate some inequalities involving the beta function in n variables, denoted by $B_n(x_1, x_2, ..., x_n)$, and such section is divided into three subsections. The two first subsections are devoted to give some upper bounds for $B_n(x_1, x_2, ..., x_n)$ when $x_1, x_2, ..., x_n \geq 1$ and $x_1, x_2, ..., x_n \in (0,1]$, respectively. The third subsection displays with some lower bounds of $B_n(x_1, x_2, ..., x_n)$ for any $x_1, x_2, ..., x_n > 0$. Our obtained inequalities are extensions and refinements for some inequalities already discussed in the literature.

2. Beta Function in Three or More V ariables

For the sake of simplicity and clearness for the reader, we first present the beta function in three variables and then that of n variables. We collect from the literature the elementary properties of the beta function in n variables that will be needed throughout this manuscript. For more details we refer the interested reader to [1, 2, 6] and the related references cited therein.

2.1. Beta function with three variables

Let T be the standard triangle of $\mathbb{R}^2$ having $e_0 := (0,0), e_1 := (1,0), e_2 := (0,1)$ as vertices. In analytic form, T is defined by

$$T = \Big\{(t,s):\ t \geq 0,\ s \geq 0,\ t+s \leq 1\Big\}.$$

For any $x, y, z \in \mathbb{C}_+$, we set

$$B_3(x,y,z) := \int_T t^{x-1}s^{y-1}\big(1-t-s\big)^{z-1}dtds. \tag{2.1}$$

The integrand function of (2.1), namely $(t,s) \longmapsto t^{x-1}s^{y-1}(1-t-s)^{z-1}$ for $(t,s) \in T$, presents eventual singularities since it is not defined when at least one of the three conditions $\Re e(x) < 1$, $\Re e(y) < 1$, $\Re e(z) < 1$ holds. This means that, for this class of x, y, z the double integral in (2.1) is an improper integral and so, its existence should be justified. We then must mention the following result.

Proposition 2.1. *Let $x, y, z \in \mathbb{C}_+$. Then the integral of the right hand-side of* (2.1) *is convergent. That is,* (2.1) *defines the map B_3 from $\big(\mathbb{C}_+\big)^3$ into $\mathbb{C}$.*

It is obvious that $B_3(1,1,1) = area(T) = 1/2$. Now, we state the following result which reduces the computation of B_3 to that of B.

Proposition 2.2. *For any $x, y, z \in \mathbb{C}_+$, the following relationship holds:*

$$B_3(x,y,z) = B(y,z)B(x,y+z). \tag{2.2}$$

The formula (2.2) when combined with (1.1) immediately yields the following result.

Corollary 2.3. *The following relation*

$$B_3(x,y,z) = \frac{\Gamma(x)\Gamma(y)\Gamma(z)}{\Gamma(x+y+z)} \tag{2.3}$$

holds for any $x, y, z \in \mathbb{C}_+$. As a consequence, $B_3(x,y,z)$ is symmetric in x, y, z.

2.2. Beta function in n variables

This subsection extends the preceding one. We need some notation. Let $n \geq 3$ be an integer and let E_{n-1} be the standard $(n-1)$-dimensional simplex of $\mathbb{R}^{n-1}$ defined by

$$E_{n-1} = \Big\{(t_1, t_2, ..., t_{n-1}) \in \mathbb{R}^{n-1} : \sum_{i=1}^{n-1} t_i \leq 1; \;\; t_i \geq 0, \text{ for } i = 1, 2, ..., n-1\Big\}.$$

Let $x_1, x_2, ..., x_n \in \mathbb{C}_+$, we define

$$B_n\big(x_1, x_2, ..., x_n\big) := \int_{E_{n-1}} \prod_{i=1}^{n} t_i^{x_i - 1} dt_1 dt_2 ... dt_{n-1}, \tag{2.4}$$

where we set $t_n := 1 - \sum_{i=1}^{n-1} t_i$.

For $n = 2$, the function B_2 is exactly the classical beta function usually denoted by B, notation that we conserve throughout the following. For $n = 3$ we are in the situation of the preceding subsection.

In a similar manner as for $n = 3$, we can prove a result that justifies the existence of the integral in (2.4) which is improper for a class of $x_1, x_2, ..., x_n$. Precisely, we have the following.

Proposition 2.4. *The map B_n given by* (2.4) *is well-defined from $(\mathbb{C}_+)^n$ into $\mathbb{C}$.*

It is well-known that $B_n(1, 1, ..., 1) = Vol(E_{n-1}) = 1/(n-1)!$, where $Vol(E_{n-1})$ refers to the volume of E_{n-1} for the $\mathbb{R}^{n-1}$- Lebesgue measure.

Now, we are in the position to state the following result which gives an interesting recursive relationship that reduces the computation of the beta function B_n to those of B_{n-1} and B.

Theorem 2.5. *For any $x_1, x_2, ..., x_n \in \mathbb{C}_+$ and $n \geq 3$ there holds*

$$B_n\big(x_1, x_2, ..., x_n\big) = B_{n-1}\big(x_2, x_3, ..., x_n\big) B\Big(x_1, \sum_{i=2}^{n} x_i\Big). \tag{2.5}$$

Proof. We apply the celebrated Fubini's theorem to the multiple integral in (2.4). For a similar way, see Remark 3.8 below. □

As we will see in the next section, the recursive relation (2.5) is useful for extending many properties from the beta function with two variables to that of n arguments. It can be also used for obtaining the following result.

Corollary 2.6. *The following formula*

$$B_n\big(x_1, x_2, ..., x_n\big) = \frac{\prod_{i=1}^{n} \Gamma(x_i)}{\Gamma\big(\sigma(x)\big)} \tag{2.6}$$

holds for any $x_1, x_2, ..., x_n \in \mathbb{C}_+$, where we set $\sigma(x) =: \sum_{i=1}^{n} x_i$. So, $B_n\big(x_1, x_2, ..., x_n\big)$ is symmetric in $x_1, x_2, ..., x_n$.

From (2.6) we can immediately deduce some other properties for the function B_n. As examples, the following relationships

$$\begin{aligned} B_n\big(x_1 + 1, x_2, ..., x_n\big) + B_n\big(x_1, x_2 + 1, ..., x_n\big) + ... \\ + B_n\big(x_1, x_2, ..., x_n + 1\big) = B\big(x_1, x_2, ..., x_n\big), \end{aligned}$$

$$B_n\big(x_1, x_2, ..., x_n\big) = B_{n-m}\big(x_{m+1}, x_{m+2}, ..., x_n\big) B_{m+1}\big(x_1, x_2, ..., x_m, \sum_{i=m+1}^{n} x_i\big)$$

hold for any $x_1, x_2, ..., x_n \in \mathbb{C}_+$ and any integers $n \geq 2, m \geq 1$ such that $n \geq m + 2$.

We end this section by stating the following remark which may be useful for the reader.

Remark 2.7. (i) As for $n = 2$, we can give an expression of $B_n(x_1, x_2, ..., x_n)$ in terms of the spherical coordinates in n-dimension. We omit all detail about this

latter point, since it will not be needed in the sequel.
(ii) From (2.5) we can deduce, by a simple mathematical induction, that the following equality

$$B_n(x_1, x_2, ..., x_n) = \prod_{k=2}^{n} B\Big(\sum_{i=1}^{k-1} x_i, x_k\Big)$$

holds for any $x_1, x_2, ..., x_n \in \mathbb{C}_+$ and $n \geq 3$.

3. Inequalities Involving B_n

This section deals with some inequalities for the beta function in n variables. In the literature we can find a lot of inequalities involving the beta function in two variables, see [1, 6, 7, 8, 9] and the related references cited therein. However, to find inequalities involving the function B_n for $n \geq 3$ seems to be difficult.

In [1], H. Alzer proved that the inequalities

$$0 < \frac{1}{\prod_{i=1}^{n} x_i} - B_n(x_1, x_2, ..., x_n) \leq 1 - \frac{1}{(n-1)!} \tag{3.1}$$

hold for any integer $n \geq 3$ and any $x_1, x_2, ..., x_n \geq 1$. Further he established there that the extreme bounds 0 and $1 - 1/(n-1)!$ of (3.1) can not be improved. The right inequality in (3.1) is not valid for $n = 2$. However, the left inequality of (3.1) is valid for $n = 2$ as already obtained by S.S. Dragomir et al. in [7], i.e.

$$\forall x, y \geq 1 \qquad B(x, y) \leq \frac{1}{xy}. \tag{3.2}$$

3.1. Upper bounds for $B_n(x_1, x_2, ..., x_n)$ when $x_1, x_2, ..., x_n \geq 1$.

In what follows we will give a refinement of the left inequality in (3.1) with a short proof. Precisely, the following theorem which is our first main result may be stated.

Theorem 3.1. *For any $n \geq 3$ and $x_1, x_2, ..., x_n \geq 1$ we have*

$$\begin{aligned} B_n(x_1, x_2, ..., x_n) &\leq \frac{1}{(n-2)!} \frac{1}{\big(\prod_{i=1}^{n} x_i\big)\big(\sum_{i=2}^{n} x_i\big)} \\ &\leq \frac{1}{(n-1)!\big(\prod_{i=1}^{n} x_i\big)} < \frac{1}{\prod_{i=1}^{n} x_i}. \end{aligned} \tag{3.3}$$

Further, the coefficients $1/(n-2)!$ and $1/(n-1)!$ in (3.3) are sharp.

Proof. The two right inequalities in (3.3) are obvious. To prove the left inequality in (3.3) we will use a mathematical induction. For $n = 3$, we have by (2.2) and (3.2)

$$B_3(x_1, x_2, x_3) = B(x_2, x_3)B(x_1, x_2+x_3) \leq \frac{1}{x_2x_3}\frac{1}{x_1(x_2 + x_3)} \leq \frac{1}{2x_1x_2x_3} < \frac{1}{x_1x_2x_3}.$$

Assume that (3.3) is true for $n-1$. According to (2.5), with (3.2) and the recurrence hypothesis, we have

$$B_n\big(x_1,x_2,...,x_n\big) = B_{n-1}\big(x_2,x_3,...,x_n\big)B\big(x_1,\sum_{i=2}^{n}x_i\big)$$
$$\leq \frac{1}{(n-3)!}\frac{1}{\big(\prod_{i=2}^{n}x_i\big)\big(\sum_{i=2}^{n-1}x_i\big)}\frac{1}{x_1\big(\sum_{i=2}^{n}x_i\big)}.$$

This, with the fact that $\sum_{i=2}^{n-1}x_i \geq n-2$, yields

$$B_n\big(x_1,x_2,...,x_n\big) \leq \frac{1}{\big(\prod_{i=1}^{n}x_i\big)\big(\sum_{i=2}^{n}x_i\big)}\frac{1}{(n-3)!\big(\sum_{i=2}^{n-1}x_i\big)}$$
$$\leq \frac{1}{(n-2)!\big(\prod_{i=1}^{n}x_i\big)\big(\sum_{i=2}^{n}x_i\big)}.$$

The desired result is so proved. If we take $x_1=x_2=...=x_n=1$ then the two left inequalities of (3.3) are reduced to an equality. This implies that the coefficients $1/(n-2)!$ and $1/(n-1)!$ in (3.3) can not be improved. □

As a consequence of the previous theorem we have the following corollary which, in its turn, gives a refinement of the left inequality in (3.3).

Corollary 3.2. *For any $n \geq 3$ and $x_1,x_2,...,x_n \geq 1$ there holds*

$$B_n(x_1,x_2,...,x_n) \leq \frac{1}{(n-2)!}\frac{1}{\big(\prod_{i=1}^{n}x_i\big)}\frac{1}{s_n(x_1,x_2,...,x_n)}, \tag{3.4}$$

where we set

$$s_n(x_1,x_2,...,x_n) := \max_{1\leq k\leq n}\Big(\sum_{i=1,i\neq k}^{n}x_i\Big).$$

In particular, for any $x \geq 1$ real number and $n \geq 3$ we have

$$B_n(x,x,...,x) \leq \frac{1}{(n-1)!}\frac{1}{x^{n+1}}. \tag{3.5}$$

Proof. By (3.3) with the fact that $B(x_1,x_2,...,x_n)$ is symmetric in $x_1,x_2,..,x_n$ we immediately deduce (3.4) after simple manipulations. The fact that the left inequality in (3.4) gives a refinement of that in (3.3) is immediate. The details are straightforward and therefore omitted here. □

Remark 3.3. (i) The left inequality in (3.3), and so that in (3.4), is not valid for $n=2$. That is, the inequality $B(x,y) \leq 1/xy^2$ does not hold for any $x,y \geq 1$. Indeed, if we take $x=1, y=2$ we obtain $B(1,2)=1/2 > 1/4$.
(ii) Throughout the following, we will adopt as usual the equality $0^0=1$, for the

sake of convenience and simplicity. Such convention is legitime by virtue of the equality $\lim_{x\downarrow 0} x^x = 1$.

In order to give more inequalities about the beta function B_n, we need to state the following lemma.

Lemma 3.4. *Let $a, b, c \geq 0$ with $c \leq 1$. Then one has*

$$\sup_{0\leq t\leq 1-c} t^a\big(1-t-c\big)^b = \frac{a^a b^b (1-c)^{a+b}}{(a+b)^{a+b}}. \tag{3.6}$$

Consequently, we have

$$\int_0^{1-c} t^a(1-t-c)^b dt \leq \frac{a^a b^b (1-c)^{a+b+1}}{(a+b)^{a+b}}. \tag{3.7}$$

Proof. If $a = 0$ or $b = 0$ or $c = 1$, (3.6) is trivial. Assume that $a > 0,\ b > 0$ and $0 \leq c < 1$. It is easy to see that

$$t^a\big(1-t-c\big)^b = (1-c)^{a+b}u^a(1-u)^b, \ \text{ with } u := \frac{t}{1-c}, 0 \leq u < 1.$$

It is then enough to study the following function

$$\Phi(u) := u^a(1-u)^b, \ \ 0 \leq u < 1.$$

Clearly, $\Phi(0) = \Phi(1) = 0$. Simple computation leads to

$$\Phi^{'}(u) = u^{a-1}(1-u)^{b-1}\big(a-(a+b)u\big).$$

We then deduce that Φ is increasing for $u \in [0, a/(a+b)]$ and decreasing for $u \in [a/(a+b), 1)$. It follows that Φ presents a maximum at $u = a/(a+b) \in (0,1)$, with

$$\sup_{0\leq u<1} \Phi(u) = \Phi(a/(a+b)) = \Big(\frac{a}{a+b}\Big)^a\Big(\frac{b}{a+b}\Big)^b = \frac{a^a b^b}{(a+b)^{a+b}}.$$

The proof of (3.6) is finished and then (3.7) follows. □

After this, the following result may be stated.

Proposition 3.5. *Let $x, y \geq 1$. Then we have*

$$B(x,y) \leq \frac{(x-1)^{x-1}(y-1)^{y-1}}{(x+y-2)^{x+y-2}}. \tag{3.8}$$

Proof. Follows from (3.6) with $a = x-1,\ b = y-1,\ c = 0$, and the definition of the beta function B. □

Remark 3.6. (i) Numeric experiments show that neither (3.2) nor (3.8) is uniformly better than the other. The following items explain this claim:

- If we take $x = y = 3/2$ then (3.2) and (3.8) give $B(3/2, 3/2) \leq 4/9$ and $B(3/2, 3/2) \leq 1/2$, respectively, with $4/9 < 1/2$.
- If we take $x = y = 3$ then (3.2) and (3.8) imply that $B(3, 3) \leq 1/9$ and $B(3, 3) \leq 1/16$, respectively, with $1/9 > 1/16$.

(ii) It is easy to check that if $x = y \geq 2$ then (3.8) is better than (3.2). We conjecture that, if $x, y \geq 2$ then (3.8) refines (3.2).

The following result gives an inequality involving B_n in a recursive manner.

Proposition 3.7. *Let $x_1, x_2, ..., x_n \geq 1$ and $n \geq 3$. Then the following inequality*

$$(3.9)\quad B_n\big(x_1, x_2, ..., x_{n-1}, x_n\big) \leq \frac{(x_i-1)^{x_i-1}(x_n-1)^{x_n-1}}{\big(x_i+x_n-2\big)^{x_i+x_n-2}} B_{n-1}\big(x_1, x_2, ..., x_{i-1}, x_{i+1}, ..., x_{n-1}, x_i+x_n\big)$$

holds for any $i = 1, 2, ..., n-1$. In particular, for any real number $x \geq 1$ and $n \geq 3$ we have

$$(3.10)\quad B_n(x, x, ..., x) \leq \frac{(x-1)^{2x-2}}{4^{(n-3)(x-1)}} \frac{\big((n-2)x-1\big)^{(n-2)x-1}}{\big((n-1)x-2\big)^{(n-1)x-2}} \frac{\big((n-1)x-1\big)^{(n-1)x-1}}{\big(nx-2\big)^{nx-2}}.$$

Proof. By virtue of the symmetric character of $B_n(x_1, x_2, ..., x_n)$, it is enough to prove (3.9) for $i = 1$. According to (2.6) we can easily check that

$$B_n(x_1, x_2, ..., x_n) = B_{n-1}(x_2, x_3, ..., x_{n-1}, x_1 + x_n)B(x_1, x_n).$$

This, with (3.8), immediately yields (3.9). The inequality (3.10) follows from (3.9) after a simple mathematical induction. □

Remark 3.8. The previous result can be also shown in the following way. By (2.4) with the Fubini's theorem we can write

$$(3.11)\quad B_n\big(x_1, x_2, ..., x_n\big) = \int_{E_{n-2}} \Big(\int_0^{1-\overline{t}_n} t_1^{x_1-1} t_n^{x_n-1} dt_1\Big) t_2^{x_2-1} t_3^{x_3-1} ... t_{n-1}^{x_{n-1}-1} dt_2 dt_3 ... dt_{n-1},$$

where we set $t_n := 1 - t_1 - t_2 - ... - t_{n-1}$ and $\overline{t}_n := t_2 + t_3 + ... + t_{n-1}$ for the sake of simplicity. According to (3.7) we have

$$\int_0^{1-\overline{t}_n} t_1^{x_1-1} t_n^{x_n-1} dt_1 \leq \frac{(x_1-1)^{x_1-1}(x_n-1)^{x_n-1}}{\big(x_1+x_n-2\big)^{x_1+x_n-2}} \big(1-\overline{t}_n\big)^{x_1+x_n-1}.$$

Substituting this in (3.11), with $\overline{t}_n = t_2+t_3+...+t_{n-1}$ and again by (2.4) we obtain (3.9).

The previous theorem gives a recursive inequality involving B_n and B_{n-1}. The following example explains the use of such inequality.

Example 3.9. Take $n = 3$. Simple manipulations lead to

$$B_3(x_1,x_2,x_3) \leq \frac{(x_1-1)^{x_1-1}(x_2-1)^{x_2-1}(x_3-1)^{x_3-1}}{(x_1+x_2+x_3-2)^{x_1+x_2+x_3-2}} \frac{(x_1+x_3-1)^{x_1+x_3-1}}{(x_1+x_3-2)^{x_1+x_3-2}}.$$

This, with the fact that $B_3(x_1,x_2,x_3)$ is symmetric in x_1,x_2,x_3 allows us to obtain

$$B_3(x_1,x_2,x_3) \leq \frac{(x_1-1)^{x_1-1}(x_2-1)^{x_2-1}(x_3-1)^{x_3-1}}{(x_1+x_2+x_3-2)^{x_1+x_2+x_3-2}} C(x_1,x_2,x_3), \tag{3.12}$$

where we set

$$C(x_1,x_2,x_3) := \min\left\{\frac{(x_1+x_2-1)^{x_1+x_2-1}}{(x_1+x_2-2)^{x_1+x_2-2}}, \frac{(x_1+x_3-1)^{x_1+x_3-1}}{(x_1+x_3-2)^{x_1+x_3-2}}\right.$$
$$\left., \frac{(x_2+x_3-1)^{x_2+x_3-1}}{(x_2+x_3-2)^{x_2+x_3-2}}\right\}.$$

In particular, for any $x \geq 1$ one has

$$B_3(x,x,x) \leq \frac{(x-1)^{x-1}}{4^{x-1}} \frac{(2x-1)^{2x-1}}{(3x-2)^{3x-2}}. \tag{3.13}$$

Now, a question arises from the above: does (3.8) have an analog for n variables. The following result answers positively this latter question.

Theorem 3.10. *Let $x_1,x_2,...,x_n \geq 1$ and $n \geq 2$. Then we have*

$$B_n\big(x_1,x_2,...,x_n\big) \leq \frac{1}{(n-1)!} \frac{\prod_{i=1}^n \big(x_i-1\big)^{x_i-1}}{\Big(\sum_{i=1}^n x_i - n\Big)^{\sum_{i=1}^n x_i - n}}. \tag{3.14}$$

Further, the coefficient $1/(n-1)!$ *in* (3.14) *is sharp. In particular, for any real number $x \geq 1$ one has*

$$B_n(x,x,...,x) \leq \frac{1}{(n-1)!} \frac{1}{n^{n(x-1)}}. \tag{3.15}$$

Proof. Let $x_1,x_2,...,x_n \geq 1$ be fixed. The map

$$(t_1,t_2,...,t_{n-1}) \longmapsto t_1^{x_1-1}t_2^{x_2-1}...t_{n-1}^{x_{n-1}-1}\big(1-t_1-t_2-...-t_{n-1}\big)^{x_n-1}$$

is continuous on the compact E_{n-1} of $\mathbb{R}^{n-1}$. So, it is bounded in E_{n-1} and its supremum, i.e.

$$S_n(x_1,x_2,...,x_n) := \sup_{(t_1,...,t_{n-1})\in E_{n-1}} t_1^{x_1-1}t_2^{x_2-1}...t_{n-1}^{x_{n-1}-1}\big(1-t_1-t_2-...-t_{n-1}\big)^{x_n-1}$$

exists. We will compute $S_n(x_1,x_2,...,x_n)$ in a recursive manner. If we put

$$S_n^1\big(t_2,t_3,...,t_{n-1}\big) := \sup_{0\le t_1\le 1-t_2-...-t_{n-1}} t_1^{x_1-1}\big(1-t_1-...-t_{n-1}\big)^{x_n-1}$$

then it is clear that

$$S_n(x_1,x_2,...,x_n) = \sup_{(t_2,...,t_{n-1})\in E_{n-2}} t_2^{x_2-1}...t_{n-1}^{x_{n-1}-1}S_n^1\big(t_2,t_3,...,t_{n-1}\big).$$

Following (3.6) we have

$$S_n^1\big(t_2,t_3,...,t_{n-1}\big) = \frac{(x_1-1)^{x_1-1}(x_n-1)^{x_n-1}\big(1-t_2-...-t_{n-1}\big)^{x_1+x_n-2}}{(x_1+x_n-2)^{x_1+x_n-2}}.$$

Summarizing, the following recursive relationship

$$S_n(x_1,x_2,...,x_n) = \frac{(x_1-1)^{x_1-1}(x_n-1)^{x_n-1}}{(x_1+x_n-2)^{x_1+x_n-2}}S_{n-1}(x_2,...,x_{n-1},x_1+x_n-1)$$

holds for any $x_1,x_2,...,x_n\ge 1$ and $n\ge 2$. By a simple mathematical induction we show that

$$S_n(x_1,x_2,...,x_n) = \frac{\prod_{i=1}^n (x_i-1)^{x_i-1}}{\Big(\sum_{i=1}^n x_i - n\Big)^{\sum_{i=1}^n x_i - n}}.$$

This, with the definition of $S_n(x_1,x_2,...,x_n)$, implies that the inequality

$$t_1^{x_1-1}t_2^{x_2-1}...t_{n-1}^{x_{n-1}-1}\big(1-t_1-t_2-...-t_{n-1}\big)^{x_n-1} \le S_n(x_1,x_2,...,x_n) \tag{3.16}$$

holds for any $(t_1,t_2,...,t_{n-1})\in E_{n-1}$ and $x_1,x_2,...,x_n\ge 1$. Integrating (3.16) over $(t_1,t_2,...,t_{n-1})\in E_{n-1}$, with the definition of $B_n(x_1,x_2,...,x_n)$ and the fact that $Vol(E_{n-1}) = 1/(n-1)!$, we obtain (3.14).

To justify that the coefficient $1/(n-1)!$ in (3.14) can not be improved it is enough to remark that (3.14) is an equality for $x_1 = x_2 = ... = x_n = 1$, always with the usual convention $0^0 = 1$. (3.15) is immediate from (3.14) after a simple reduction. □

It is worth mentioning that, numerical experiments show that there is no general comparison between (3.4), (3.9) and (3.14). That is, neither one among (3.4), (3.9) and (3.14) is uniformly better than the other. In fact, for $x_1 = x_2 = ... = x_n = x \ge 1$, it is easy to check that (3.5) is better than (3.15) if and only if $x^{n+1} \ge n^{n(x-1)}$.

Table 1:

x	$x = 4/3$	$x = 3/2$	$x = 3$
(3.5)	$U_1 = 81/512 \simeq 0.158$	$U_1 = 8/81 \simeq 0.099$	$U_1 = 1/162 \simeq 0.006$
(3.13)	$U_2 = 5\sqrt[3]{50}/72 \simeq 0.256$	$U_2 = 8/25\sqrt{5} \simeq 0.143$	$U_2 = 5^5/2^2.7^7 \simeq 0.0009$
(3.15)	$U_3 = 1/6 \simeq 0.162$	$U_3 = 1/6\sqrt{3} \simeq 0.288$	$U_3 = 1/486 \simeq 0.002$
Bounds	$U_1 < U_3 < U_2$	$U_1 < U_2 < U_3$	$U_2 < U_3 < U_1$

However, a full comparison between (3.15) and one of the inequalities (3.5) and (3.15) seems to be generally difficult, by virtue of the complicated expression of (3.15). For particular values of x_1, x_2, x_3, the following example explains this latter situation.

Example 3.11. Let us choose $n = 3$. For (3.9) we use (3.13). Elementary computations lead to TABLE 1 which gives a comparison between the upper bounds for $B_3(x, x, x)$, denoted by U_1, U_2 and U_3, respectively, when applying (3.5), (3.13) and (3.15) with particular values for $x > 1$.

3.2. Upper bounds for B_n when $x_1, x_2, ..., x_n \in (0, 1]$.

In [7], Dragomir et al. proved the following result:

$$\forall x, y > 0, \text{ with } (x-1)(y-1) \geq 0, \quad B(x, y) \leq \frac{1}{xy}, \tag{3.17}$$

which in fact includes (3.2). Also, (3.17) implies that the inequality $B(x, y) \leq 1/xy$ is valid for $x, y \in (0, 1]$. In [9], P. Ivády gave the following inequality

$$\forall x, y \in (0, 1] \quad B(x, y) \leq \frac{1}{xy} \frac{x + y}{1 + xy}, \tag{3.18}$$

which refines $B(x, y) \leq 1/xy$ for $x, y \in (0, 1]$. The following result, which gives an extension of (3.18) for n variables, tells us that the left inequality in (3.1) is still valid when $(x_1, x_2, ..., x_n) \in E_n$ and $n \geq 3$, with more information and improvement.

Theorem 3.12. *Let $x_1, x_2, ..., x_n \in (0, 1]$. For any $n \geq 3$ we have*

$$B_n(x_1, x_2, ..., x_n) \leq \frac{\sum_{i=1}^n x_i}{\prod_{i=1}^n x_i} \frac{1}{1 + p_n(x_1, x_2, ..., x_n)} \frac{1}{1 + q_n(x_1, x_2, ..., x_n)}, \tag{3.19}$$

where, for the sake of simplicity, we set

$$p_n(x_1, x_2, ..., x_n) := \max_{1 \leq k \leq n} \prod_{i=1, i \neq k}^{n} x_i,$$

$$q_n(x_1, x_2, ..., x_n) := \max_{1 \leq k \leq n} x_k \Big(\sum_{i=1, i \neq k}^{n} x_i \Big).$$

If moreover $(x_1, x_2, ..., x_n) \in E_n$ *with* $x_1, x_2, ..., x_n \neq 0$ *then we have*

$$B_n\big(x_1, x_2, ..., x_n\big) < \frac{1}{\prod_{i=1}^n x_i} \frac{\sum_{i=1}^n x_i}{1+\prod_{i=1}^n x_i} < \frac{1}{\prod_{i=1}^n x_i}. \tag{3.20}$$

Proof. The right inequality in (3.20) is obvious, since

$$\forall (x_1, x_2, ..., x_n) \in E_n, \quad \text{with} \;\; x_1, x_2, ..., x_n \neq 0, \qquad \sum_{i=1}^n x_i \leq 1 < 1 + \prod_{i=1}^n x_i.$$

We mention that the condition $(x_1, x_2, ..., x_n) \in E_n$, with $x_1, x_2, ..., x_n \neq 0$ and $n \geq 2$, implies that $0 < x_i < 1$ for any $i = 1, 2, ..., n$.

To prove (3.19), we use a mathematical induction by utilizing (2.5). For $n = 3$, (2.2) with (3.18) gives

$$\begin{aligned} B_3(x_1, x_2, x_3) &= B(x_2, x_3) B(x_1, x_2 + x_3) \\ &\leq \frac{1}{x_2 x_3} \frac{x_2 + x_3}{1 + x_2 x_3} \frac{1}{x_1(x_2 + x_3)} \frac{x_1 + x_2 + x_3}{1 + x_1(x_2 + x_3)} \\ &= \frac{x_1 + x_2 + x_3}{x_1 x_2 x_3} \frac{1}{1 + x_2 x_3} \frac{1}{1 + x_1(x_2 + x_3)}. \end{aligned}$$

This, with the symmetric character of $B_3(x_1, x_2, x_3)$ immediately implies that (3.19) is true for $n = 3$.

Assume that (3.19) is true for $n - 1$. The recursive relation (2.5), with the recurrence hypothesis and (3.18), allows us to write

$$\begin{aligned} B_n\big(x_1, x_2, ..., x_n\big) &= B_{n-1}\big(x_2, x_3, ..., x_n\big) B\big(x_1, \sum_{i=2}^n x_i\big) \\ &\leq \frac{1}{\prod_{i=2}^n x_i} \frac{\sum_{i=2}^n x_i}{1 + \prod_{i=2}^n x_i} \frac{1}{x_1 \sum_{i=2}^n x_i} \frac{x_1 + \sum_{i=2}^n x_i}{1 + x_1 \sum_{i=2}^n x_i} \\ &= \frac{1}{\prod_{i=1}^n x_i} \frac{\sum_{i=1}^n x_i}{\big(1 + \prod_{i=2}^n x_i\big)\big(1 + x_1 \sum_{i=2}^n x_i\big)}. \end{aligned}$$

This, with the fact that $B_n(x_1, x_2, ..., x_n)$ is symmetric in $x_1, x_2, ..., x_n$ immediately yields (3.19).

Now, to prove the left inequality in (3.20) it is enough to remark that

$$\begin{aligned} &\big(1 + \prod_{i=2}^n x_i\big)\big(1 + x_1 \sum_{i=2}^n x_i\big) \\ &\qquad = 1 + \big(\prod_{i=2}^n x_i\big)\big(1 + x_1 \sum_{i=2}^n x_i\big) + x_1 \sum_{i=2}^n x_i > 1 + \prod_{i=2}^n x_i > 1 + \prod_{i=1}^n x_i, \end{aligned}$$

since $0 < x_1, x_2, ..., x_n < 1$. □

The following corollary is an immediate consequence of Theorem 3.12.

Corollary 3.13. *For any $x \in (0, 1]$ and $n \geq 3$ we have*

$$B_n(x, x, ..., x) \leq \frac{nx}{x^n(1 + x^{n-1})(1 + (n-1)x^2)}.$$

Remark 3.14. The inequality (3.19) does not persist for $n = 2$. That is, the following inequality

$$B(x, y) \leq \frac{x + y}{xy} \frac{1}{1 + xy} \frac{1}{1 + \max(x, y)} \tag{3.21}$$

does not hold for any $x, y > 0$ with $x + y \leq 1$. In fact, if we take $x = y = 1/2$ then (3.21) yields

$$B(1/2, 1/2) = \big(\Gamma(1/2)\big)^2 = \pi \leq (4)(4/5)(2/3) = 32/15,$$

which is false, so justifying our claim.

3.3. Lower bounds for $B_n(x_1, x_2, ..., x_n)$ when $x_1, x_2, ..., x_n > 0$.

Here, we will be interested by giving some lower bounds for the function B_n. For the beta function in two variables, the following inequality

$$\forall x, y > 0 \quad B(x, y) \geq \frac{x^{x-1} y^{y-1}}{(x + y)^{x+y-1}} \tag{3.22}$$

is well-known in the literature. See [8] for instance and the related references cited therein. An extension of (3.22) for n variables is our aim in this subsection and it is recited in what follows.

Theorem 3.15. *Let $x_1, x_2, ..., x_n > 0$ and $n \geq 2$. Then the following inequality holds:*

$$B_n\big(x_1, x_2, ..., x_n\big) \geq \frac{\prod_{i=1}^n (x_i)^{x_i - 1}}{\big(\sum_{i=1}^n x_i\big)^{\sum_{i=1}^n x_i - 1}}. \tag{3.23}$$

In particular, for any $x > 0$ we have

$$B_n(x, x, ..., x) \geq \frac{1}{n^{nx-1} x^{n-1}}.$$

Proof. For $n=2$, (3.23) is (3.22). By similar way as in the proof of Theorem 3.12 we have

$$B_n\big(x_1,x_2,...,x_n\big)=B_{n-1}\big(x_2,x_3,...,x_n\big)B\big(x_1,\sum_{i=2}^{n}x_i\big)$$
$$\geq \frac{\prod_{i=2}^{n}(x_i)^{x_i-1}}{\big(\sum_{i=2}^{n}x_i\big)^{\sum_{i=2}^{n}x_i-1}}\,\frac{(x_1)^{x_1-1}\big(\sum_{i=2}^{n}x_i\big)^{\sum_{i=2}^{n}x_i-1}}{\big(\sum_{i=1}^{n}x_i\big)^{\sum_{i=1}^{n}x_i-1}},$$

from which the desired result follows after a simple reduction. □

Another inequality concerning the lower bound of $B(x,y)$ is given by, see [8, 12]

$$\forall x,y>0 \quad B(x,y)>\sqrt{2\pi}\frac{x^{x-1/2}y^{y-1/2}}{\big(x+y\big)^{x+y-1/2}}. \tag{3.24}$$

For a discussion about comparison between (3.22) and (3.24) see [8]. Now, we will state an extension of (3.24) for n variables as recited in the following.

Theorem 3.16. *Let $x_1,x_2,..,x_n>0$ and $n\geq 2$. Then we have*

$$B_n\big(x_1,x_2,...,x_n\big)>\big(2\pi\big)^{(n-1)/2}\frac{\prod_{i=1}^{n}(x_i)^{x_i-1/2}}{\big(\sum_{i=1}^{n}x_i\big)^{\sum_{i=1}^{n}x_i-1/2}}. \tag{3.25}$$

In particular, for any $x>0$ one has

$$B_n(x,x,...,x)>\big(2\pi\big)^{(n-1)/2}\frac{1}{n^{nx-1/2}x^{(n-1)/2}}.$$

Proof. For $n=2$, (3.25) is exactly (3.24). We proceed by a mathematical induction as previous. The details are similar to those of the proof of Theorem 3.15 and therefore omitted here for the reader. □

Finally, we state the following remark which concerns a full comparison between (3.23) and (3.25).

Remark 3.17. For $x_1,x_2,...,x_n>0$ and $n\geq 2$, let us consider the following condition:

$$\sum_{i=1}^{n}x_i>\big(2\pi\big)^{n-1}\prod_{i=1}^{n}x_i. \tag{3.26}$$

Then it is easy to check that, (3.23) is strictly stronger than (3.25) if and only if (3.26) holds. If (3.26) is an equality then (3.23) and (3.25) coincide.

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