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2-absorbing δ-semiprimary Ideals of Commutative Rings

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Abstract. Let R be a commutative ring with nonzero identity, $\mathcal{I}(\mathcal{R})$ the set of all ideals of R and $\delta : \mathcal{I}(\mathcal{R}) \to \mathcal{I}(\mathcal{R})$ an expansion of ideals of R. In this paper, we introduce the concept of 2-absorbing δ-semiprimary ideals in commutative rings which is an extension of 2-absorbing ideals. A proper ideal I of R is called 2-absorbing δ-semiprimary ideal if whenever $a, b, c \in R$ and $abc \in I$, then either $ab \in \delta(I)$ or $bc \in \delta(I)$ or $ac \in \delta(I)$. Many properties and characterizations of 2-absorbing δ-semiprimary ideals are obtained. Furthermore, 2-absorbing δ_1-semiprimary avoidance theorem is proved.

1. Introduction

In this paper, all rings are commutative with nonzero identity. Let I be a proper ideal of a ring R and let $\mathcal{I}(\mathcal{R})$ denote the set of all ideals of R. The radical of I is defined by $\{a \in R : a^n \in I \ for \ some \ n \in \mathbb{N}\}$, denoted by $\sqrt{I}$. Let J be an ideal of R. Then the ideal $(I : J)$ consists of $r \in R$ with $rJ \subseteq I$, that is, $(I : J) = \{r \in R : rJ \subseteq I\}$. For undefined notations and terminology refer to [10].

Various generalizations of prime and primary ideals are studied extensively in [1]-[3],[13],[14]. Recall from [4] and [5] that a proper ideal I of R is called a (weakly) 2-absorbing ideal if whenever $a, b, c \in R$ and $(0 \neq abc \in I)$ $abc \in I$, then either $ab \in I$ or $ac \in I$ or $bc \in I$. A proper ideal I of R is called a (weakly) 2-absorbing primary ideal as in [6] and [7] if whenever $a, b, c \in R$ and $(0 \neq abc \in I)$ $abc \in I$, then either $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. As in the recent study [13], a proper ideal I of R is said to be a 2-absorbing quasi-primary if $\sqrt{I}$ is a 2-absorbing ideal; or equivalently, if whenever $a, b, c \in R$ and $abc \in I$, then either $ab \in \sqrt{I}$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. D. Zhao [14] introduced the concept of expansions of ideals and extended many results of prime and primary ideals into the new concept. From [14], a function δ from $\mathcal{I}(\mathcal{R})$ to $\mathcal{I}(\mathcal{R})$ is an ideal expansion if it has the following properties: $I \subseteq \delta(I)$ and if

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$I \subseteq J$ for some ideals I, J of R, then $\delta(I) \subseteq \delta(J)$. For example, δ_0 is the identity function where $\delta_0(I) = I$ for all ideals I of R, and δ_1 is defined by $\delta_1(I) = \sqrt{I}$. For other examples, consider the functions δ_+ and δ_* of $\mathcal{I}(\mathcal{R})$ defined by $\delta_+(I) = I + J$ and $\delta_*(I) = (I : J)$ for all $I \in \mathcal{I}(\mathcal{R})$, where J is an ideal of R, respectively (see [3]). He called a δ-primary ideal I of R if $ab \in I$ and $a \notin I$ for some $a, b \in R$ imply $b \in \delta(I)$. Recently, (weakly) δ-semiprimary ideals are studied in [8]. A proper ideal I of R is called a (weakly) δ-semiprimary if whenever $a, b \in R$ and $(0 \neq ab \in I)$ $ab \in I$, then $a \in \delta(I)$ or $b \in \delta(I)$.

In this paper, we introduce and study 2-absorbing δ-semiprimary and weakly 2-absorbing δ-semiprimary ideals of commutative rings. Let $\delta : \mathcal{I}(\mathcal{R}) \to \mathcal{I}(\mathcal{R})$ be an expansion of ideals of a ring R. We call a proper ideal I of R a (weakly) 2-absorbing δ-semiprimary ideal if whenever $a, b, c \in R$ and $(0 \neq abc \in I)$ $abc \in I$, then either $ab \in \delta(I)$ or $bc \in \delta(I)$ or $ac \in \delta(I)$. From the definitions, we have the following implications: 2-absorbing δ-primary $\overset{(1)}{\to}$ 2-absorbing δ-semiprimary $\overset{(2)}{\to}$ weakly 2-absorbing δ-semiprimary. Among many results in this study, it is shown in Example 2.3. and Example 2.4. that the implications (1) and (2) are not reversible. Also it shown in Theorem 2.17. that if $I \nsubseteq \sqrt{0}$, then the implication (2) is reversible. It is shown (Theorem 2.24.) that if $\delta(0)$ a 2-absorbing δ-semiprimary ideal with $\delta(\delta(0)) = \delta(0)$ and I is a weakly 2-absorbing δ-semiprimary ideal, then either I is a 2-absorbing δ-semiprimary ideal of R or I^2 is a 2-absorbing δ-semiprimary ideal of R. It is shown (Theorem 2.8.) that if $I \subseteq I_1 \cup I_2 \cup \cdots \cup I_n$ where $I_1, I_2, ..., I_n$ $(n \geq 2)$ are ideals of R such that at most two of them are not 2-absorbing δ_1-semiprimary, then $\sqrt{I} \subseteq \sqrt{I_i}$ for some $1 \leq i \leq n$. From Theorem 2.30. to Theorem 2.34., we characterize 2-absorbing δ-semiprimary ideals and weakly 2-absorbing δ-semiprimary ideals of $R = R_1 \times \cdots \times R_n$ where $n \geq 2$. Moreover, we state and prove 2-absorbing δ_1-semiprimary avoidance theorem (Theorem 2.9.). In Section 3, we search an answer to the question that if I is a weakly 2-absorbing δ-semiprimary ideal of R and if $0 \neq JKL \subseteq I$ for some ideals J, K, L of R, then does it imply either $JK \subseteq \delta(I)$ or $KL \subseteq \delta(I)$ or $JL \subseteq \delta(I)$?

2. Properties of 2-absorbing δ-semiprimary Ideals

Definition 2.1. Let $\delta : \mathcal{I}(\mathcal{R}) \to \mathcal{I}(\mathcal{R})$ be an expansion of ideals of R and I a proper ideal of R.

1. We call I a 2-absorbing δ-semiprimary ideal if whenever $a, b, c \in R$ and $abc \in I$, then either $ab \in \delta(I)$ or $bc \in \delta(I)$ or $ac \in \delta(I)$.
2. We call I a weakly 2-absorbing δ-semiprimary ideal if whenever $a, b, c \in R$ and $0 \neq abc \in I$, then either $ab \in \delta(I)$ or $bc \in \delta(I)$ or $ac \in \delta(I)$.

We start with trivial relations, hence we omit the proof.

Theorem 2.2. *Let I be a proper ideal of R. Then the following statements hold:*

1. *If I is a 2-absorbing δ-semiprimary ideal, then I is a weakly 2-absorbing δ-semiprimary ideal.*

2. *I is a (weakly) 2-absorbing δ_0-semiprimary ideal if and only if I is a (weakly) 2-absorbing ideal.*
3. *I is a (weakly) 2-absorbing δ_1-semiprimary ideal if and only if I is a (weakly) 2-absorbing quasi primary ideal.*
4. *If I is a (weakly) δ-semiprimary ideal, then I is a (weakly) 2-absorbing δ-semiprimary ideal.*
5. *If I is a (weakly) 2-absorbing δ-primary ideal, then I is a (weakly) 2-absorbing δ-semiprimary ideal.*
6. *Let δ and γ be two ideal expansions with $\delta(I) \subseteq \gamma(I)$. If I is a (weakly) 2-absorbing δ-semiprimary ideal of R, then I is a (weakly) 2-absorbing γ-semiprimary ideal of R.*

The following example shows that the concepts of weakly 2-absorbing δ-semiprimary and 2-absorbing δ-semiprimary are different.

Example 2.3. Let $R = \mathbb{Z}_{210}$ and $\delta : \mathcal{I}(\mathcal{R}) \to \mathcal{I}(\mathcal{R})$ an expansion of ideals of R defined by $\delta(I) = (I : J)$ where $J = 7R$. Then $I = 0$ is a weakly 2-absorbing δ-semiprimary ideal of R by definition. Observe that $\delta(I) = 30R$. However, I is not 2-absorbing δ-semiprimary since $3 \cdot 5 \cdot 14 \in I$ but neither $3 \cdot 5 \in \delta(I)$ nor $3 \cdot 14 \in \delta(I)$ nor $5 \cdot 14 \in \delta(I)$.

For a 2-absorbing δ-semiprimary ideal which is not 2-absorbing δ-primary, see the next example:

Example 2.4.([6, Example 2.9]) Let $R = Z[X, Y, Z]$ and consider an ideal $I = (XYZ, X^3Y^3)R$ of R. Then I is a 2-absorbing δ_1-semiprimary ideal of R but I is not a 2-absorbing δ_1-primary ideal of R since $XYZ \in I$ but neither $XY \in I$ nor $YZ \in \delta_1(I)$ nor $XZ \in \delta_1(I)$ where $\delta_1(I) = (XY)R$.

Theorem 2.5. *Let δ be an expansion function of $\mathcal{I}(\mathcal{R})$ and I a proper ideal of R.*

1. *If $\delta(I)$ is a (weakly) 2-absorbing ideal of R, then I is a (weakly) 2-absorbing δ-semiprimary ideal of R.*
2. *Let $\delta(\delta(I)) = \delta(I)$. Then $\delta(I)$ is a (weakly) 2-absorbing δ-semiprimary ideal of R if and only if $\delta(I)$ is a (weakly) 2-absorbing ideal of R. Moreover, if $\delta(I)$ is 2-absorbing δ-semiprimary, then $|Min(\delta(I))| \leq 2$.*

Proof. (1) Suppose that $(0 \neq abc \in I)$ $abc \in I$ and $ab \notin \delta(I)$. Since $I \subseteq \delta(I)$ and $\delta(I)$ is 2-absorbing, we have $ac \in \delta(I)$ or $bc \in \delta(I)$. Thus I is a (weakly) 2-absorbing δ-semiprimary ideal.

(2) Suppose that $\delta(I)$ is a (weakly) 2-absorbing δ-semiprimary ideal of R, since $\delta(\delta(I)) = \delta(I)$, $\delta(I)$ is (weakly) 2-absorbing by the definition. The converse part is clear from (1). Suppose that $\delta(I)$ is a 2-absorbing δ-semiprimary ideal of R. Then $\delta(I)$ is 2-absorbing; and so, $|Min(\delta(I))| \leq 2$ by [4, Theorem 2.3.]. □

The converse of Theorem 2.5. (1) is also true for $\delta = \delta_1$ by [13, Proposition 2.5].

Theorem 2.6. *Let δ be an expansion of $\mathfrak{I}(\mathcal{R})$ and I a 2-absorbing δ-semiprimary ideal of R. If $\sqrt{\delta(I)} \subseteq \delta(\sqrt{I})$, then $\sqrt{I}$ is a 2-absorbing δ-semiprimary ideal of R. In particular, if $\delta = \delta_1$, then $\sqrt{I}$ is a 2-absorbing ideal of R.*

Proof. Let $a, b, c \in R$ with $abc \in \sqrt{I}$ and $ab \notin \delta(\sqrt{I})$. Then $a^n b^n c^n \in I$ for some positive integer $n \geq 1$. Since $\sqrt{\delta(I)} \subseteq \delta(\sqrt{I})$, we conclude $a^n b^n \notin \delta(I)$. Since I is 2-absorbing δ-semiprimary, we have $b^n c^n \in \delta(I)$ or $a^n c^n \in \delta(I)$. Since $\sqrt{\delta(I)} \subseteq \delta(\sqrt{I})$, we conclude that $bc \in \delta(\sqrt{I})$ or $ac \in \delta(\sqrt{I})$, we are done. The particular case is clear from [13, Proposition 2.5]. □

The next example shows that the converse of Theorem 2.6. is not satisfied in general.

Example 2.7. Consider the ideal $I = (X^3)/(X^4)$ of $R = \mathbb{Z}_{12}[X]/(X^4)$. Then $\sqrt{I} = (6, X)/(X^4)$ is a 2-absorbing ideal (2-absorbing δ_0-semiprimary ideal). However, I is not a 2-absorbing δ_0-semiprimary ideal since $0 \neq (X+(X^4))(X+(X^4))(X+(X^4)) \in I$ but $X^2 + (X^4) \notin \delta_0(I) = I$.

Theorem 2.8. *Let $I_1, I_2, ..., I_n$ $(n \geq 2)$ be ideals of R such that at most two of them are not 2-absorbing δ_1-semiprimary. If $I \subseteq I_1 \cup I_2 \cup \cdots \cup I_n$, then $\sqrt{I} \subseteq \sqrt{I_i}$ for some $1 \leq i \leq n$.*

Proof. From the covering $I \subseteq I_1 \cup I_2 \cup \cdots \cup I_n$, we conclude that $\sqrt{I} \subseteq \sqrt{I_1} \cup \sqrt{I_2} \cup \cdots \cup \sqrt{I_n}$. By our hypothesis, we may assume that I_k is a 2-absorbing δ_1-semiprimary ideal for all $k \geq 3$. Hence $\sqrt{I_k}$ is 2-absorbing for all $k \geq 3$ by Theorem 2.6. Thus $\sqrt{I} = \sqrt{\sqrt{I}} \subseteq \sqrt{\sqrt{I_i}} = \sqrt{I_i}$ for some $1 \leq i \leq n$ by [12, Theorem 3.1]. □

Let $I, I_1, I_2, ..., I_n$ be ideals of R. Recall that an efficient covering of R is a covering $I \subseteq I_1 \cup I_2 \cup \cdots \cup I_n$ in which no I_j where $1 \leq j \leq n$ satisfies $I \subseteq I_j$ (i.e., no I_j is superfluous.) In the following result, we obtain 2-absorbing δ_1-semiprimary avoidance theorem.

Theorem 2.9. (2-absorbing δ_1-semiprimary avoidance theorem) *Let $I_1, I_2, ..., I_n$ $(n \geq 2)$ be ideals of R such that at most two of them not 2-absorbing δ_1-semiprimary. Suppose that $\sqrt{I_i} \nsubseteq (\sqrt{I_j} : x)$ for all $x \in R \backslash \sqrt{I_j}$ for all $i \neq j$. If $I \subseteq I_1 \cup I_2 \cup \cdots \cup I_n$, then $I \subseteq I_i$ for some $1 \leq i \leq n$.*

Proof. Assume on the contrary that $I \nsubseteq I_i$ for all $1 \leq i \leq n$. Hence $I \subseteq I_1 \cup I_2 \cup \cdots \cup I_n$ is an efficient covering of ideals of R. So, it is clear that $I = (I \cap I_1) \cup (I \cap I_2) \cup \cdots \cup (I \cap I_n)$ is an efficient union. From [11, Lemma 2.1], $\bigcap_{i \neq k} (I \cap I_i) \subseteq I \cap I_k$. Since at most two of the I_i are not 2-absorbing δ_1-semiprimary ideals, we may assume that I_k is a 2-absorbing δ_1-semiprimary ideal. From Theorem 2.6., we conclude that $\sqrt{I_k}$ is 2-absorbing; and so $(\sqrt{I_k} : x)$ is a prime ideal of R for all $x \notin \sqrt{I_i}$ by [4, Theorem 2.5.]. On the other hand, our hypotesis implies that there exist $a_i \in \sqrt{I_i} \backslash (\sqrt{I_k} : x)$

for all $i \neq k$. Hence there are positive integers m_i such that $a_i^{m_i} \in I_i$. Put $a = \prod_{i\neq k} a_i$ and $m = \max\{m_1, m_2, ..., m_n\}$. It is clear that $a^m x \in \bigcap_{i\neq k}(I \cap I_i)$. We show that $a^m x \notin I \cap I_k$. Assume that $a^m x \in I \cap I_k$. Hence $a^m x \in \sqrt{I_k}$, and so $a^m \in (\sqrt{I_k} : x)$. Since $(\sqrt{I_k} : x)$ is prime, we conclude that $a_i \in (\sqrt{I_k} : x)$ for some $i \neq k$, a contradiction. Thus $\left(\bigcap_{i\neq k}(I \cap I_i)\right) \backslash (I \cap I_k)$ is nonempty which contradicts with $\bigcap_{i\neq k}(I \cap I_i) \subseteq I \cap I_k$. Thus $I \subseteq I_i$ for some $1 \leq i \leq n$. □

Recall from [14] that an ideal expansion δ of $\mathcal{I}(\mathcal{R})$ is said to be intersection preserving if it satisfies $\delta(I_1 \cap I_2 \cap \cdots \cap I_n) = \delta(I_1) \cap \delta(I_2) \cap \cdots \cap \delta(I_n)$ for any ideals $I_1, I_2, ..., I_n$ of R.

Theorem 2.10. *Let δ be an intersection preserving expansion function of $\mathcal{I}(\mathcal{R})$. If $I_1, I_2, ..., I_n$ are 2-absorbing δ-semiprimary ideals of R with $\delta(I_i) = K$ for all $i \in \{1, 2, ..., n\}$, then $I = \bigcap_{i=1}^n I_i$ is a 2-absorbing δ-semiprimary ideal of R.*

Proof. Suppose that $abc \in I$, $ab \notin \delta(I)$ and $bc \notin \delta(I)$ for some $a, b, c \in R$. Since $\delta(I) = \delta(\bigcap_{i=1}^n I_i) = \bigcap_{i=1}^n \delta(I_i) = K$, we have $ab \notin K$ and $bc \notin K$. Since $abc \in I_i$ and I_i is 2-absorbing δ-semiprimary, $ac \in \delta(I_i) = K = \delta(I)$; so we are done. □

Theorem 2.11. *Let δ be an expansion function of $\mathcal{I}(\mathcal{R})$ and I a (weakly) 2-absorbing δ-semiprimary ideal of R. Then the following hold:*

1. *If $J \subseteq I$ and $\delta(J) = \delta(I)$, then J is a (weakly) 2-absorbing δ-semiprimary ideal of R.*
2. *If $J \subseteq I \subseteq \sqrt{0}$, then J is a (weakly) 2-absorbing quasi-primary ideal of R.*
3. *Let $I \subseteq \sqrt{0}$. If K is an ideal of R, then IK, $I \cap K$ and I^n are (weakly) 2-absorbing quasi-primary ideals of R for all positive integers $n \geq 1$*

Proof. (1) Suppose that $(0 \neq abc \in J)$ $abc \in J$ for some $a, b, c \in R$. Since $J \subseteq I$, we have $(0 \neq abc \in I)$ $abc \in I$. Since I is (weakly) 2-absorbing δ-semiprimary, we have either $ab \in \delta(I)$ or $bc \in \delta(I)$ or $ac \in \delta(I)$. Since $\delta(I) = \delta(J)$, J is a (weakly) 2-absorbing δ-semiprimary ideal of R.

(2) Since $J \subseteq I \subseteq \sqrt{0}$, we have $\delta_1(J) = \sqrt{J} = \sqrt{I} = \delta_1(I) = \sqrt{0}$. Thus the result is clear from (1).

(3) is a particular result of (2). □

Theorem 2.12. *Let δ be an expansion of $\mathcal{I}(\mathcal{R})$. Every proper principal ideal is a 2-absorbing δ-semiprimary ideal of R if and only if every proper ideal is a 2-absorbing δ-semiprimary ideal of R.*

Proof. Suppose that every proper principal ideal is a 2-absorbing δ-semiprimary ideal of R. Let I be a proper ideal of R and $a, b, c \in R$ with $abc \in I$. Then $abc \in (abc)$

and since (abc) is 2-absorbing δ-semiprimary ideal of R by our assumption, we have either $ab \in \delta(abc) \subseteq \delta(I)$ or $bc \in \delta(abc) \subseteq \delta(I)$ or $ac \in \delta(abc) \subseteq \delta(I)$ or Thus I is a 2-absorbing δ-semiprimary ideal of R. The converse part is obvious. □

Theorem 2.13. *Let δ be an ideal expansion such that $\delta(I) \subseteq \sqrt{I}$ and $\delta(I)$ a semiprime ideal of R for every ideal I. If I is a 2-absorbing δ-semiprimary ideal, then $\delta(I) = \sqrt{I}$.*

Proof. Suppose that $a \in \sqrt{I}$. Then there exists n which is the least positive integer n with $a^n \in I$. If $n = 1$, then $a \in I \subseteq \delta(I)$. For $n \geq 2$, $a^n = a^{n-2}aa \in I$. Since $a^{n-1} \notin I$, we have $a^2 \in \delta(I)$. Since $\delta(I)$ is semiprime, $a \in \delta(I)$. Thus $\delta(I) = \sqrt{I}$. □

Recall that a ring R is said to be a Boolean ring if $x^2 = x$ for every $x \in R$. Since $\sqrt{I} = I$ for every proper ideal I of R, we have the following result.

Theorem 2.14. *Let R be a Boolean ring and I a proper ideal of R. Then the following are equivalent:*

1. *I is a (weakly) 2-absorbing quasi-primary ideal of R.*
2. *I is a (weakly) 2-absorbing primary ideal of R.*
3. *I is a (weakly) 2-absorbing ideal of R.*

Proof. Since $\sqrt{I} = I$, the claim is clear. □

Definition 2.15. Let δ be an expansion function of $\mathcal{I}(\mathcal{R})$ and I a weakly 2-absorbing δ-semiprimary ideal of R. We call a triple (a, b, c) a δ–triple zero of I if $abc = 0$ for some elements a, b, c of R (not necessarily distinct) and neither $ab \in \delta(I)$ nor $bc \in \delta(I)$ nor $ac \in \delta(I)$.

Remark 2.16. Let δ be an expansion function of $\mathcal{I}(\mathcal{R})$ and I a weakly 2-absorbing δ-semiprimary ideal of R. Then I is a not 2-absorbing δ-semiprimary ideal of R if and only if there exists at least one δ–triple zero of I.

Theorem 2.17. *Let δ be an expansion function of $\mathcal{I}(\mathcal{R})$ and I a weakly 2-absorbing δ-semiprimary ideal of R. If I is not a 2-absorbing δ-semiprimary ideal of R, then $I^3 = 0$; that is $I \subseteq \sqrt{0}$.*

Proof. Suppose that I is a weakly 2-absorbing δ-semiprimary ideal of R which is not 2-absorbing δ-semiprimary. Hence there exists a δ-triple zero (a, b, c) of I by Remark 2.16. Then $abc = 0$ and neither $ab \in \delta(I)$ nor $bc \in \delta(I)$ nor $ac \in \delta(I)$. for some $b, c \in R$. First, we show that $abI = 0$. Assume that $abi \neq 0$ for some $i \in I$. Since $0 \neq ab(c + i) \in I$ and I is a weakly 2-absorbing δ-semiprimary, we conclude $ab \in I$ or $a(c + i) \in I$ or $b(c + i) \in I$, a contradiction. Similarly, it is easy to show that $bcI = acI = 0$. Now, we show that $aI^2 = 0$. Assume that $ai_1i_2 \neq 0$ for some $i_1, i_2 \in I$. Since $abI = bcI = acI = 0$, we have $0 \neq a(b + i_1)(c + i_2) = ai_1i_2 \in I$. Since I is a weakly 2-absorbing δ-semiprimary, this contradicts our assumption that $ab \in \delta(I)$ nor $bc \in \delta(I)$ nor $ac \in \delta(I)$. Thus $aI^2 = 0$. One can easily show symmetrically that $bI^2 = cI^2 = 0$. Lastly, we show that $I^3 = 0$. Assume that $i_1 i_2 i_3 \neq 0$ for some $i_1, i_2, i_3 \in I$. Since $abI = bcI = acI = aI^2 = bI^2 = cI^2 = 0$, observe that $0 \neq (a+i_1)(b+i_2)(c+i_2) = i_1 i_2 i_3 \in I$. Since I is a weakly 2-absorbing δ-semiprimary, again we conclude a contradiction. Thus $I^3 = 0$. □

The following example shows that for an ideal I of R with $I^3 = 0$, I needs not to be a weakly 2-absorbing δ-semiprimary ideal.

Example 2.18. Let $R = \mathbb{Z}_{60}$ and $I = 30R$. Then $I^3 = 0$. However, I is not a weakly 2-absorbing δ-semiprimary ideal (for $\delta = \delta_0$ or $\delta = \delta_1$) since $2 \cdot 3 \cdot 5 \in I$ but neither $2 \cdot 3 \in \delta(I)$ nor $3 \cdot 5 \in \delta(I)$ nor $2 \cdot 5 \in \delta(I)$.

As a conclusion of Theorem 2.17., we have the following two results.

Corollary 2.19. *Let R be a reduced ring. Then every nonzero weakly 2-absorbing δ-semiprimary ideal of R is a 2-absorbing δ-semiprimary ideal of R.*

Corollary 2.20. *Let M be a finitely generated R-module. Let I be a weakly 2-absorbing δ-semiprimary ideal of R that is not 2-absorbing δ-semiprimary. If $IM = M$, then $M = 0$.*

Theorem 2.21 *Let δ be an expansion function of $\mathcal{I}(\mathcal{R})$ and I a weakly 2-absorbing δ-semiprimary ideal of R with $\delta(I) = \delta(0)$. Then I is not 2-absorbing δ-semiprimary ideal of R if and only if there exists a δ-triple zero of 0.*

Proof. Suppose that I is not a 2-absorbing δ-semiprimary ideal. Hence $abc = 0$ but $ab \notin \delta(I)$, $ac \notin \delta(I)$ and $bc \notin \delta(I)$ for some $a, b, c \in R$. Since $\delta(I) = \delta(0)$, (a, b, c) is a δ-triple zero of 0. The converse part is obvious. □

In particular, suppose that $I \subseteq \sqrt{0}$ is a weakly 2-absorbing quasi primary ideal of R. A a consequence of Theorem 2.21., there exists a δ-triple zero of 0 if and only if I is not a 2-absorbing quasi primary ideal of R. The following example shows that if $\delta(I) \neq \delta(0)$ and there exists a δ-triple zero of 0, then I may be a 2-absorbing δ-semiprimary ideal of R.

Example 2.22. Consider $R = \mathbb{Z}_{24}$, $\delta : \mathcal{I}(\mathcal{R}) \to \mathcal{I}(\mathcal{R})$ is defined by $\delta(I) = \delta_1(I)$ for every nonzero proper ideal I of R, and $\delta(0) = \delta_0(0) = 0$. Let $I = 12R$. Then $\delta(I) = 6R$ is 2-absorbing by [4, Theorem 3.15], I is a 2-absorbing δ-semiprimary ideal by Theorem 2.5. Since $2 \cdot 3 \cdot 4 = 0$ but neither $2 \cdot 3 \in \delta(0)$ nor $2 \cdot 4 \in \delta(0)$ nor $3 \cdot 4 \in \delta(0)$, we conclude that $(2, 3, 4)$ is δ–triple zero of 0.

Theorem 2.23. *δ be an intersection preserving expansion function of $\mathcal{I}(\mathcal{R})$. If $I_1, I_2, ..., I_n$ are weakly 2-absorbing quasi-primary (weakly 2-absorbing δ_1-semiprimary) ideals of R that are not 2-absorbing quasi-primary, then $I = \cap_{i=1}^{n} I_i$ is a weakly 2-absorbing quasi-primary ideal of R.*

Proof. Since each I_i is a 2-absorbing quasi-primary ideal of R that is not 2-absorbing quasi-primary, we have $\delta(I_i) = \sqrt{I_i} = \sqrt{0}$ by Theorem 2.17. Thus, remain of the proof is easily concluded similar to the proof of Theorem 2.10. □

Theorem 2.24. *Let δ be an expansion function of $\mathcal{I}(\mathcal{R})$ and $\delta(0)$ a 2-absorbing δ-semiprimary ideal such that $\delta(\delta(0)) = \delta(0)$. Suppose that I is a weakly 2-absorbing δ-semiprimary ideal. Then I is a 2-absorbing δ-semiprimary ideal of R or I^2 is a 2-absorbing δ-semiprimary ideal of R.*

Proof. Suppose that I is a weakly 2-absorbing δ-semiprimary ideal that is not 2-absorbing δ-semiprimary. Hence $I^3 \subseteq 0 \subseteq \delta(0)$ by Theorem 2.17. Since $\delta(0)$ is a 2-absorbing ideal of R by Theorem , we conclude that $I^2 \subseteq \delta(0)$ by [4, Theorem 2.13.]. Since $0 \subseteq I^2 \subseteq \delta(0)$ and $\delta(\delta(0)) = \delta(0)$, we have $\delta(I^2) = \delta(0)$. Since $\delta(I^2)$ is a 2-absorbing ideal of R, I^2 is a 2-absorbing δ-semiprimary ideal of R by Theorem 2.5. □

Let R and S be commutative rings with $1 \neq 0$, and let δ, γ be two expansion functions of $\mathcal{I}(\mathcal{R})$ and $\mathcal{I}(\mathcal{S})$, respectively. Then a ring homomorphism $f : R \to S$ is called a $\delta\gamma$-homomorphism if $\delta(f^{-1}(I)) = f^{-1}(\gamma(I))$ for all ideals I of S. For example, if γ_1 is a radical operation on ideals of S and δ_1 is a radical operation on ideals of R. Then every homomorphism from R to S is a $\delta_1\gamma_1$-homomorphism. Additionally, if f is a $\delta\gamma$-epimorphism and I is an ideal of R containing $\ker(f)$, then $\gamma(f(I)) = f(\delta(I))$ [3]

Theorem 2.25. *Let $f : R \to S$ be a $\delta\gamma$-homomorphism, where δ and γ are expansion functions of $\mathcal{I}(\mathcal{R})$ and $\mathcal{I}(\mathcal{S})$, respectively. Then the following statements hold:*

1. *If J is a 2-absorbing γ-semiprimary ideal of S, then $f^{-1}(J)$ is a 2-absorbing δ-semiprimary ideal of R.*
2. *If J is a weakly 2-absorbing γ-semiprimary ideal of S, and $\ker(f)$ is a weakly 2-absorbing δ-semiprimary ideal of R, then $f^{-1}(J)$ is a weakly 2-absorbing δ-semiprimary ideal of R.*
3. *Let f be an epimorphism and I a proper ideal of R with $\ker(f) \subseteq I$. Then I is (weakly) 2-absorbing δ-semiprimary ideal of R if and only if $f(I)$ is a (weakly) 2-absorbing γ-semiprimary ideal of S.*

Proof. (1) Let $abc \in f^{-1}(J)$ for some $a, b, c \in R$. Then $f(abc) = f(a)f(b)f(c) \in J$, which implies $f(a)f(b) = f(ab) \in \gamma(J)$ or $f(b)f(c) = f(bc) \in \gamma(J)$ or $f(a)f(c) = f(bc) \in \gamma(J)$. Thus we have $ab \in f^{-1}(\gamma(J))$ or $bc \in f^{-1}(\gamma(J))$ or $ac \in f^{-1}(\gamma(J))$. Since $f^{-1}(\gamma(J)) = \delta(f^{-1}(J))$, $f^{-1}(J)$ is a 2-absorbing δ-semiprimary ideal of R.

(2) Let $0 \neq abc \in f^{-1}(J)$ for some $a, b, c \in R$. Then $f(abc) = f(a)f(b)f(c) \in J$. If $f(abc) \neq 0$, it can be easily proved similar to (1) that $f^{-1}(J)$ is a weakly 2-absorbing δ-semiprimary ideal of R. Assume that $f(abc) = 0$. Hence $abc \in \ker(f)$. Since $\ker(f)$ is weakly 2-absorbing δ-semiprimary, we have $ab \in \delta(\ker(f))$ or $bc \in \delta(\ker(f))$ or $ac \in \delta(\ker(f))$. Since $\delta(\ker(f)) = \delta(f^{-1}(0)) \subseteq \delta(f^{-1}(J))$, we are done.

(3) Let $(0 \neq xyz \in f(I))$ $xyz \in f(I)$ for some $x, y, z \in S$. Then there are some elements $a, b, c \in I$ such that $x = f(a)$, $y = f(b)$ and $z = f(c)$. Then $f(a)f(b)f(c) = f(abc) \in f(I)$ and since $\ker(f) \subseteq I$, we conclude $(0 \neq abc \in I)$ $abc \in I$. Since I is (weakly) 2-absorbing δ-semiprimary, we have either $ab \in \delta(I)$ or $bc \in \delta(I)$

or $ac \in \delta(I)$. Thus $xy \in f(\delta(I))$ or $yz \in f(\delta(I))$ or $yz \in f(\delta(I))$. Since $f(\delta(I)) = \delta(f(I))$, we are done. □

Remark 2.26. Let δ be an expansion function of $\mathcal{I}(\mathcal{R})$ and I a proper ideal of R. Then the function $\delta_q : R/I \to R/I$ defined by $\delta_q(J/I) = \delta(J)/I$ for all ideals $I \subseteq J$, becomes an expansion function of R/I [3]. Consider the natural homomorphism $\pi : R \to R/J$. Then for ideals I of R with $ker(\pi) \subseteq I$, we have $\delta_q(\pi(I)) = \pi(\delta(I))$.

From Theorem 2.25. and Remark 2.26., we have the following result.

Corollary 2.27. *Let δ be an expansion function of $\mathcal{I}(\mathcal{R})$.*

1. *Let I and J be ideals of R with $I \subseteq J$. Then J is a 2-absorbing δ-semiprimary ideal of R if and only if J/I is a 2-absorbing δ_q-semiprimary ideal of R/I.*
2. *If I is a 2-absorbing δ-semiprimary ideal of R and R' is a subring with $R' \nsubseteq I$, then $I \cap R'$ is a 2-absorbing δ-semiprimary ideal of R'.*

Let δ be an expansion function of ideals of a polynomial ring $R[X]$ where X is an indeterminate. Observe that the function as in Remark 2.26., $\delta_q : R[X]/(X) \to R[X]/(X)$ defined by $\delta_q(J/(X)) = \delta(J)/(X)$ for all ideals J of $R[X]$ with $(X) \subseteq J$, is an expansion function of ideals of R as $R[X]/(X) \cong R$. According to these expansions, we have the following equivalent situations:

Theorem 2.28. *Let δ be an expansion function of $\mathcal{I}(\mathcal{R})$ and I a proper ideal of R. Then the following are equivalent:*

1. *I is a 2-absorbing δ_q-semiprimary ideal of R.*
2. *(I, X) is a 2-absorbing δ-semiprimary ideal of $R[X]$.*

Proof. From Corollary 2.27., we conclude that (I, X) is a 2-absorbing δ-semiprimary ideal of $R[X]$ if and only if $(I, X)/(X)$ is a 2-absorbing δ_q-semiprimary ideal of $R[X]/(X)$. Since $(I, X)/(X) \cong I$ and $R[X]/(X) \cong R$, the result is obtained. □

Let S be a multiplicatively closed subset of a ring R and let δ be an expansion function of $\mathcal{I}(\mathcal{R})$. Note that δ_S is an expansion function of $\mathcal{I}(\mathcal{R}_S)$ such that $\delta_S(I_S) = (\delta(I))_S$. In the next theorem, we investigate 2-absorbing δ_S–semiprimary ideals of the localization R_S.

Theorem 2.29. *Let δ be an expansion function of $\mathcal{I}(\mathcal{R})$ and S a multiplicatively closed subset of R. If I is a (weakly) 2-absorbing δ-semiprimary ideal of R with $I \cap S = \emptyset$, then I_S is a (weakly) 2-absorbing δ_S-semiprimary ideal of R_S.*

Proof. Let $(0 \neq \frac{x}{s_1}\frac{y}{s_2}\frac{z}{s_3} \in I_S)$ $\frac{x}{s_1}\frac{y}{s_2}\frac{z}{s_3} \in I_S$ for some $x, y, z \in R$; $s_1, s_2, s_3 \in S$. Then we have $(0 \neq sxyz \in I)$ $sxyz \in I$ for some $s \in S$. Then $sxy \in \delta(I)$ or $yz \in \delta(I)$ or $sxz \in \delta(I)$. Hence $\frac{sxy}{ss_1s_2} \in \delta(I)_S$ or $\frac{yz}{s_2s_3} \in \delta(I)_S$ or $\frac{sxz}{ss_1s_3} \in \delta(I)_S$. Since $(\delta(I))_S = \delta_S(I_S)$, I_S is a (weakly) 2-absorbing δ_S-semiprimary ideal of R_S. □

Let $R = R_1 \times \cdots \times R_n$ $(n \geq 2)$ where $R_1, R_2, ..., R_n$ are commutative rings with nonzero identity, let δ_i be an expansion function of $\mathcal{I}(\mathcal{R}_i)$ for each $i \in \{1, 2, ..., n\}$.

For a proper ideal $I_1 \times \cdots \times I_n$, the function $\delta_\times$ defined by $\delta_\times(I_1 \times I_2 \times ... \times I_n) = \delta_1(I_1) \times \delta_2(I_2) \times ... \times \delta_n(I_n)$ is an expansion function of $\mathcal{I}(\mathcal{R})$. In the next four theorems, we characterize 2-absorbing δ-semiprimary ideals and weakly 2-absorbing δ-semiprimary ideals of $R_1 \times \cdots \times R_n$.

Theorem 2.30. *Let R_1 and R_2 be commutative rings with $1 \neq 0$ and $R = R_1 \times R_2$, and let δ_1, δ_2 be expansion functions of $\mathcal{I}(\mathcal{R}_1)$ and $\mathcal{I}(\mathcal{R}_2)$, respectively. Suppose that $\delta_\times(I)$ is a proper ideal of R for any proper ideal of R. Then the following statements are equivalent:*

1. *$I = I_1 \times I_2$ is a 2-absorbing $\delta_\times$-semiprimary ideal of R.*
2. *Either I_1 is a 2-absorbing δ_1-semiprimary ideal of R_1 and $\delta_2(I_2) = R_2$ or I_2 is a 2-absorbing δ_2-semiprimary ideal of R_2 and $\delta_1(I_1) = R_1$ or I_1, I_2 are $\delta_{1,2}$-semiprimary ideals of R_1, R_2, respectively.*

Proof. (1) $\Rightarrow$ (2) : Suppose that $I = I_1 \times I_2$ is a 2-absorbing $\delta_\times$-semiprimary ideal of R. Since I is proper, $\delta_\times(I) = \delta_1(I_1) \times \delta_2(I_2)$ is a proper ideal of R from the hypothesis. Hence we have three cases:
Case 1: Let $\delta_1(I_1) \neq R_1$ and $\delta_2(I_2) = R_2$. We show that I_1 is a 2-absorbing δ_1-semiprimary ideal of R_1. Suppose that $abc \in I_1$ and $ab \notin \delta_1(I_1)$.Then $(a,0)(b,0)(c,0) \in I$ and $(a,0)(b,0) \notin \delta_\times(I)$ implies that $(b,0)(c,0) \in \delta_\times(I)$ or $(a,0)(c,0) \in \delta_\times(I)$. Thus $bc \in \delta_1(I_1)$ or $ac \in \delta_1(I_1)$, we are done.
Case 2: Let $\delta_2(I_2) \neq R_2$ and $\delta_1(I_1) = R_1$. One can easily obtain similar to Case 1 that I_2 is a 2-absorbing δ_2-semiprimary ideal of R_2.
Case 3: Let $\delta_1(I_1) \neq R_1$ and $\delta_2(I_2) \neq R_2$. Suppose that $ab \in I_1$ and $a \notin \delta_1(I_1)$ for some $a, b \in R_1$. Observe that $(a,1)(b,1)(1,0) \in I$, $(a,1)(b,1) \notin \delta_\times(I)$, and $(a,1)(1,0) \notin \delta_\times(I)$. Since I is 2-absorbing $\delta_\times$-semiprimary, we conclude $(b,1)(1,0) \in \delta_\times(I)$. Thus $b \in \delta_1(I_1)$; and so I_1 is δ_1-semiprimary ideal of R_1. It can be shown by a symmetric way that I_2 is a 2-absorbing δ_2-semiprimary ideal of R_2.

(2) $\Rightarrow$ (1) : If I_1 is a 2-absorbing δ_1-semiprimary ideal of R_1 and $\delta_2(I_2) = R_2$ or I_2 is a 2-absorbing δ_2-semiprimary ideal of R_2 and $\delta_1(I_1) = R_1$, then clearly I is a 2-absorbing $\delta_\times$-semiprimary ideal of R. Now, suppose that I_1 and I_2 are $\delta_{1,2}$-semiprimary ideals of R_1, R_2, respectively. Suppose that $(a_1,a_2)(b_1,b_2)(c_1,c_2) \in I = I_1 \times I_2$, $(a_1,a_2)(b_1,b_2) \notin \delta_\times(I)$ and $(a_1,a_2)(c_1,c_2) \notin \delta_\times(I)$. Here we have four cases.
Case 1: Let $a_1b_1 \notin \delta_1(I_1)$ and $a_1c_1 \notin \delta_1(I_1)$.Since $a_1b_1c_1 \in I_1$, it contradicts with the assumption that I_1 is a δ_1-semiprimary ideal.
Case 2: Let $a_2b_2 \notin \delta_2(I_2)$ and $a_2c_2 \notin \delta_2(I_2)$. Since $a_2b_2c_2 \in I_2$, it contradicts with the assumption that I_2 is a δ_2-semiprimary ideal.
Case 3: Let $a_1b_1 \notin \delta_1(I_1)$ and $a_2c_2 \notin \delta_2(I_2)$. Since $a_1b_1c_1 \in I_1$ and I_1 is δ_1-semiprimary, we have $c_1 \in \delta_1(I_1)$. Since $a_2b_2c_2 \in I_2$ and I_2 is δ_2-semiprimary, we have $b_2 \in \delta_2(I_2)$. Thus $(b_1,b_2)(c_1,c_2) \in \delta_\times(I)$. **Case 4:** Let $a_1c_1 \notin \delta_1(I_1)$ and $a_2b_2 \notin \delta_2(I_2)$. Since $a_1b_1c_1 \in I_1$ and I_1 is δ_1-semiprimary, we have $b_1 \in \delta_1(I_1)$.Since $a_2b_2c_2 \in I_2$ and I_2 is δ_2-semiprimary, we have $c_2 \in \delta_2(I_2)$. Thus $(b_1,b_2)(c_1,c_2) \in \delta_\times(I)$. Therefore, I is a 2-absorbing $\delta_\times$-semiprimary ideal of R. □

Theorem 2.31. *Let $R_1, R_2, ..., R_n$ be commutative rings with nonzero identity and $R = R_1 \times \cdots \times R_n$ where $n \geq 2$. Let δ_i be an expansion function of $\mathcal{I}(\mathcal{R}_i)$ for each $i = 1, ..., n$. Then the following statements are equivalent:*

1. *I is a 2-absorbing $\delta_\times$-semiprimary ideal of R.*
2. *$I = I_1 \times \cdots \times I_n$ and either for some $k \in \{1, ..., n\}$ such that I_k is a 2-absorbing $\delta_\times$-semiprimary ideal of R_k and $\delta_j(I_j) = R_j$ for all $j \in \{1, ..., n\} \backslash \{k\}$ or I_k and I_t are $\delta_{k,t}$-semiprimary ideals of R_k and R_t, respectively for some $k, t \in \{1, 2, ..., n\}$ and $\delta_j(I_j) = R_j$ for all $j \in \{1, ..., n\} \backslash \{k, t\}$*

Proof. It can be obtained from Theorem 2.30. by using mathematical induction on n. □

Theorem 2.32. *Let R_1 and R_2 be commutative rings with identity, $R = R_1 \times R_2$, and let δ_1, δ_2 be expansion functions of $\mathcal{I}(\mathcal{R}_1)$ and $\mathcal{I}(\mathcal{R}_2)$, respectively. Then the following statements are equivalent:*

1. *$I = I_1 \times R_2$ is a weakly 2-absorbing $\delta_\times$-semiprimary ideal of R.*
2. *$I = I_1 \times R_2$ is a 2-absorbing $\delta_\times$-semiprimary ideal of R.*
3. *I_1 is a 2-absorbing δ_1-semiprimary ideal of R_1*

Proof.

(1)⇒(2): Suppose that $I = I_1 \times R_2$ is a weakly 2-absorbing $\delta_\times$-semiprimary ideal of R. Since $I^3 \neq 0$, $I = I_1 \times R_2$ is a 2-absorbing $\delta_\times$-semiprimary ideal of R by Theorem 2.17.

(2)⇒(3)⇒(1) is clear from Theorem 2.30. □

Definition 2.33. Let R be a ring and δ an expansion function of $\mathcal{I}(\mathcal{R})$. We say δ has (★) property if the following condition is satisfied for all ideals J of R:

$$(\bigstar)\ \delta(J) = R \text{ if and only if } J = R.$$

Theorem 2.34. *Let $R_1, R_2, ..., R_n$ be commutative rings with identity and $R = R_1 \times \cdots \times R_n$ where $n \geq 3$. Let δ_i be an expansion function of $\mathcal{I}(\mathcal{R}_i)$ which has (★) property for each $i = 1, ..., n$. For a nonzero ideal I of R, the following statements are equivalent:*

1. *I is a weakly 2-absorbing $\delta_\times$-semiprimary ideal of R.*
2. *I is a 2-absorbing $\delta_\times$-semiprimary ideal of R.*
3. *$I = I_1 \times \cdots \times I_n$ and either for some $k \in \{1, ..., n\}$ such that I_k is a 2-absorbing $\delta_\times$-semiprimary ideal of R_k and $I_j = R_j$ for all $j \in \{1, ..., n\} \backslash \{k\}$ or I_k and I_t are $\delta_{k,t}$-semiprimary ideals of R_k and R_t, respectively for some $k, t \in \{1, 2, ..., n\}$ and $I_j = R_j$ for all $j \in \{1, ..., n\} \backslash \{k, t\}$*

Proof.

(1)⇔(2): Suppose that $I = I_1 \times \cdots \times I_n$ is a weakly 2-absorbing $\delta_\times$-semiprimary ideal of R. Since I is nonzero, there exists an element $0 \neq (x_1, x_2, ..., x_n) \in I$. Hence $0 \neq (x_1, 1, ..., 1)(1, x_2, ..., 1) \cdots (1, 1, ..., x_n) \in I$ implies that $1 \in \delta(I_k)$ for some $k \in \{1, ..., n\}$. Thus $I_k = R_k$ for some $k \in \{1, ..., n\}$; so I^3 can not be 0. Therefore, I is a 2-absorbing $\delta_\times$-semiprimary ideal of R by Theorem 2.17. The converse is obvious.

(2)⇔(3): From Theorem 2.31., the claim is clear. □

Let R be a commutative ring and M an R-module. The idealization $R(+)M = \{(r, m) : r \in R, m \in M\}$ is a commutative ring with addition and multiplication, respectively: $(r, m)(s, m') = (r + s, m + m')$ and $(r, m)(s, m') = (rs, rm' + sm)$ for each $r, s \in R$; $m, m' \in M$. Additionally, $I(+)N$ is an ideal of $R(+)M$ where I is an ideal of R and N is a submodule of M if and only if $IM \subseteq N$ ([2] and [9]). In this circumstances, $I(+)N$ is called a homogeneous ideal of $R(+)M$. Let δ be an expansion function of R. Clear that $\delta_{(+)}$ is defined as $\delta_{(+)}(I(+)N) = \delta(I)(+)M$ for all ideal $I(+)N$ of $R(+)M$ is an expansion function of $R(+)M$.

Theorem 2.35. *Let δ be an expansion function of R and $I(+)N$ be a homogeneous ideal of $R(+)M$. Then, I is a 2-absorbing δ-semiprimary ideal of R if and only if $I(+)N$ is a 2-absorbing $\delta_{(+)}$-semiprimary ideal of $R(+)M$.*

Proof. Let $(r_1, m_1)(r_2, m_2)(r_3, m_3) = (r_1r_2r_3, r_2r_3m_1 + r_1r_3m_2 + r_1r_2m_3) \in I(+)N$. Then $r_1r_2r_3 \in I$. Since I is 2-absorbing δ-semiprimary, we have $r_1r_2 \in \delta(I)$ or $r_2r_3 \in \delta(I)$ or $r_1r_3 \in \delta(I)$ Since $\delta_{(+)}(I(+)N) = \delta(I)(+)M$, we conclude that $(r_1, m_1)(r_2, m_2) \in \delta_{(+)}(I(+)N)$ or $(r_2, m_2)(r_3, m_3) \in \delta_{(+)}(I(+)N)$ or $(r_1, m_1)(r_3, m_3) \in \delta_{(+)}(I(+)N)$. Conversely, suppose that $r_1r_2r_3 \in I$ for some $r_1, r_2, r_3 \in R$. Then $(r_1, 0), (r_2, 0)(r_3, 0) = (r_1r_2r_3, 0) \in I(+)N$. The remain of the proof is clear. □

3. Strongly (weakly) 2-absorbing δ-semiprimary ideal

First, we state the following theorem which gives a characterization for 2-absorbing δ-semiprimary ideals in terms of ideals of R.

Theorem 3.1. *Let δ be an expansion function of $\mathfrak{I}(\mathfrak{R})$ and I a proper ideal of R. Then the following are equivalent:*

1. *I is a 2-absorbing δ-semiprimary ideal of R.*
2. *For every elements $a, b \in R$ with $ab \notin \delta(I)$, $(I : ab) \subseteq (\delta(I) : a) \cup (\delta(I) : b)$.*
3. *For every elements $a, b \in R$ with $ab \notin \delta(I)$, $(I : ab) \subseteq (\delta(I) : a)$ or $(I : ab) \subseteq (\delta(I) : b)$.*
4. *For every elements $a, b \in R$ with $abJ \subseteq I$ and $ab \notin \delta(I)$ implies either $aJ \subseteq \delta(I)$ or $bJ \subseteq \delta(I)$.*
5. *For any ideals J, K and L of R with $JKL \subseteq I$ implies $JK \subseteq \delta(I)$ or $JL \subseteq \delta(I)$ or $KL \subseteq \delta(I)$.*

Proof. (1)⇒(2): Let $c \in (I : ab)$. Since $abc \in I$, $ab \notin \delta(I)$ and I is 2-absorbing δ-semiprimary, we have $ac \in \delta(I)$ or $bc \in \delta(I)$. Hence $c \in (\delta(I) : a) \cup (\delta(I) : b)$.

(2)⇒(3): It is straightforward.

(3)⇒(4): Suppose that (3) holds and $abJ \subseteq I$ and $ab \notin \delta(I)$. Hence, we have $J \subseteq (I : ab) \subseteq (\delta(I) : a)$ or $J \subseteq (I : ab) \subseteq (\delta(I) : b)$ by our assumption. Thus, $aJ \subseteq \delta(I)$ or $bJ \subseteq \delta(I)$.

(4)⇒(5): Suppose that $JKL \subseteq I$ and $KL \nsubseteq \delta(I)$. Then $ab \notin \delta(I)$ for some $a \in K$ and $b \in L$. Hence $aJ \subseteq \delta(I)$ or $bJ \subseteq \delta(I)$. Assume that $aJ \subseteq \delta(I)$ and $bJ \nsubseteq \delta(I)$. We show that $JK \subseteq \delta(I)$. If $kJ \nsubseteq \delta(I)$ for some $k \in K$, then $(a+k)bJ \subseteq I$. Since $bJ \nsubseteq \delta(I)$, we have $(a+k)J \subseteq \delta(I)$; and so, we get $kJ \subseteq \delta(I)$, a contradiction. Assume that $aJ \nsubseteq \delta(I)$ and $bJ \subseteq \delta(I)$. Similar to the previous argument, we conclude that $JL \subseteq \delta(I)$. Now, suppose that $aJ \subseteq \delta(I)$ and $bJ \subseteq \delta(I)$. Assume that neither $JK \subseteq \delta(I)$ nor $JL \subseteq \delta(I)$. Then there exist $k \in K$ and $l \in L$ such that $kJ \nsubseteq \delta(I)$ and $lJ \nsubseteq \delta(I)$. Since $klJ \subseteq I$, we conclude $kl \in \delta(I)$. Since $(a+k)lJ \subseteq I$, $lJ \nsubseteq \delta(I)$ and $(a+k)J = aJ + kJ \nsubseteq \delta(I)$, we have $(a+k)l \in \delta(I)$. Since $kl \in \delta(I)$, we have $al \in \delta(I)$. Since $(b+l)kJ \subseteq I$, $kJ \nsubseteq \delta(I)$ and $(b+l)J = bJ + lJ \nsubseteq \delta(I)$, we have $(b+l)k \in \delta(I)$. Since $kl \in \delta(I)$, we have $bk \in \delta(I)$. Since $(a+k)(b+l)J \subseteq I$, $(a+k)J \nsubseteq \delta(I)$ and $(b+l)J \nsubseteq \delta(I)$, we have $(a+k)(b+l) \in \delta(I)$. Since $al, bk, kl \in \delta(I)$, we conclude $ab \in \delta(I)$, a contradiction. Thus $K \subseteq \delta(I)$ or $JL \subseteq \delta(I)$.

(5)⇒(1): Suppose that $abc \in I$ for some $a, b, c \in R$. Put $J = (a)$, $K = (b)$ and $L = (c)$ in (5). Hence, the result is clear. □

Now, we define strongly (weakly) 2-absorbing δ-semiprimary ideals as follows:

Definition 3.2. Let δ be an expansion function of $\mathcal{I}(\mathcal{R})$. We call a proper ideal I of R a strongly (weakly) 2-absorbing δ-semiprimary ideal if whenever $J, K, L \in \mathcal{I}(\mathcal{R})$ with $(0 \neq JKL \subseteq I)$ $JKL \subseteq I$ implies $JK \subseteq \delta(I)$ or $JL \subseteq \delta(I)$ or $KL \subseteq \delta(I)$.

As a result of Theorem 3.1., I is a 2-absorbing δ-semiprimary ideal of R if and only if I is a strongly 2-absorbing δ-semiprimary ideal of R. Motivated from this result, we search the answer for the following question:

Question 3.3. If I is a weakly 2-absorbing δ-semiprimary ideal of R, then does I need to be a strongly weakly 2-absorbing δ-semiprimary ideal of R ?

Theorem 3.4. *Let I be a weakly 2-absorbing δ-semiprimary ideal of R and suppose that $0 \neq JKL \subseteq I$ for some ideals J, K and L of R. If I is a free δ-triple-zero with respect to JKL, then $JK \subseteq \delta(I)$ or $KL \subseteq \delta(I)$ or $JL \subseteq \delta(I)$.*

To prove the theorem above, we need the following definition and lemmas.

Definition 3.5. Let I be a weakly 2-absorbing δ-semiprimary ideal of R and suppose that $JKL \subseteq I$ for some ideals J, K and L of R. We call I a free δ-triple-zero with respect to JKL if (a, b, c) is not a δ-triple-zero of I for every $a \in J$, $b \in K$ and $c \in L$. (Equivalently, if $a \in J$, $b \in K$, $c \in L$, then $ab \in \delta(I)$ or $bc \in \delta(I)$ or $ac \in \delta(I)$.)

Lemma 3.6. *Let I be a weakly 2-absorbing δ-semiprimary ideal of R and let $abL \subseteq I$ for some $a, b \in R$ and an ideal L of R. If (a, b, l) is not a δ-triple-zero of I for all $l \in L$ and $ab \notin \delta(I)$, then $aL \subseteq \delta(I)$ or $bL \subseteq \delta(I)$.*

Proof. Assume on the contrary that $abL \subseteq I$ but neither $ab \in \delta(I)$ nor $aL \subseteq \delta(I)$ nor $bL \subseteq \delta(I)$. Hence there exist $l_1, l_2 \in L$ such that $al_1 \notin \delta(I)$ and $bl_2 \notin \delta(I)$. Since $abl_1 \in I$ but neither $ab \in \delta(I)$ nor $al_1 \in \delta(I)$, we have $bl_1 \in \delta(I)$ by our hypothesis that (a, b, l_1) is not a δ-triple-zero of I. Similarly, since $abl_2 \in I$ but neither $ab \in \delta(I)$ nor $bl_2 \in \delta(I)$, we conclude that $al_2 \in \delta(I)$. Now $ab(l_1 + l_2) \in I$ and since $ab \notin \delta(I)$, we have either $a(l_1 + l_2) \in \delta(I)$ or $b(l_1 + l_2) \in \delta(I)$. Thus, we conclude $al_2 \in \delta(I)$ or $bl_2 \in \delta(I)$, a contradiction. Thus, $aL \subseteq \delta(I)$ or $bL \subseteq \delta(I)$. □

Lemma 3.7. *Let I be a weakly 2-absorbing δ-semiprimary ideal of R and let $aKL \subseteq I$ for some $a \in R$ and for an ideal J of R. If (a, k, l) is not a δ-triple-zero of I for all $k \in K$, $l \in L$, then $aK \subseteq \delta(I)$ or $aL \subseteq \delta(I)$ or $KL \subseteq \delta(I)$.*

Proof. Assume that neither $aK \subseteq \delta(I)$ nor $aL \subseteq \delta(I)$ nor $KL \subseteq \delta(I)$. Thus there exist $k, k_1 \in K$ such that $ak \notin \delta(I)$ and $k_1L \nsubseteq \delta(I)$. Since $akL \subseteq I$, $ak \notin \delta(I)$ and $aL \nsubseteq \delta(I)$, we have $kL \subseteq \delta(I)$ by Lemma 3.6. Since $ak_1L \subseteq I$, $aL \nsubseteq \delta(I)$ and $k_1L \nsubseteq \delta(I)$, we have by $ak_1 \in \delta(I)$ Lemma 3.6. Now, since $a(k + k_1)L \subseteq I$ and $aL \nsubseteq \delta(I)$, from Lemma 3.6. we conclude that either $a(k + k_1) \in \delta(I)$ or $(k + k_1)L \subseteq I$. Hence $ak \in \delta(I)$ or $k_1L \subseteq I$, a contradiction. Thus, $aK \subseteq \delta(I)$ or $aL \subseteq \delta(I)$ or $KL \subseteq \delta(I)$. □

Proof of Theorem 3.4. Assume on the contrary that neither $JK \subseteq \delta(I)$ nor $KL \subseteq \delta(I)$ nor $JL \subseteq \delta(I)$. Hence there exists $a, a_1 \in J$ such that $aK \nsubseteq \delta(I)$ and $a_1L \nsubseteq \delta(I)$. Since $aKL \subseteq I$, $KL \nsubseteq \delta(I)$ and $aK \nsubseteq \delta(I)$, we have $aL \subseteq \delta(I)$ by Lemma 3.7. Since $a_1KL \subseteq I$, $KL \nsubseteq \delta(I)$ and $a_1L \nsubseteq \delta(I)$, we have $a_1K \subseteq \delta(I)$ by Lemma 3.7. Now $(a + a_1)KL \subseteq I$ and since $KL \nsubseteq \delta(I)$, we conclude either $(a + a_1)K \subseteq \delta(I)$ or $(a + a_1)L \subseteq \delta(I)$. Hence, we have $aK \subseteq \delta(I)$ or $a_1L \subseteq \delta(I)$, a contradiction. Therefore, $JK \subseteq \delta(I)$ or $KL \subseteq \delta(I)$ or $JL \subseteq \delta(I)$. □

More general than 2-absorbing δ-semiprimary ideal of a commutative ring, the concept of n-absorbing δ-semiprimary ideal where n is a positive integer can be defined. We shall give just the definition of this concept which may be inspired the other work:

Definition 3.8. Let R be a commutative ring with nonzero identity, $\delta : \mathcal{I}(\mathcal{R}) \to \mathcal{I}(\mathcal{R})$ an expansion of ideals of R and n a positive integer. We call a proper ideal I of R a (weakly) n-absorbing δ-semiprimary ideal if whenever ($0 \neq x_1 \cdots x_{n+1} \in I$) $x_1 \cdots x_{n+1} \in I$ for some $x_1, ..., x_{n+1} \in R$, then there exists $1 \leq k \leq n$ such that $x_1 \cdots x_{k-1}x_{k+1} \cdots x_{n+1} \in \delta(I)$. In particular, for $n = 1, 2$, it is δ-semiprimary and 2-absorbing δ-semiprimary ideal, respectively.

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