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CORRIGENDUM TO "TRANSLATION SURFACES OF TYPE 2 IN THE THREE DIMENSIONAL SIMPLY ISOTROPIC SPACE $\mathbb{I}_3^1$", [BULL. KOREAN MATH. SOC. 54 (2017), NO. 3, 953–965]

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Abstract. In [1], there are some mistakes in calculations and solutions of differential equations and theorems that appeared in the paper. We here provide correct solutions and theorems.

4. Translation surfaces satisfying $\Delta^{\mathbf{I}}\mathbf{x}_i = \lambda_i \mathbf{x}_i$

In this section, we classify the translation surfaces $\mathbf{M}$ of Type 2 in $\mathbb{I}_3^1$, that is the surface

$$\mathbf{x} = (u,\ f(u) + g(v),\ v) \tag{4.1}$$

satisfying the equation

$$\Delta^{\mathbf{I}}\mathbf{x}_i = \lambda_i \mathbf{x}_i, \tag{4.2}$$

where $\lambda_i \in \mathbb{R}$, $i = 1, 2, 3$. The coefficients of the first and second fundamental forms are

$$E = 1 + f'^2,\ F = f'g',\ G = g'^2, \tag{4.3}$$

$$L = -\frac{f''}{g'},\ M = 0,\ N = -\frac{g''}{g'}, \tag{4.4}$$

respectively. The Gaussian curvature $\mathbf{K}$ and the mean curvature $\mathbf{H}$ are

$$\mathbf{K} = \frac{f''(u)g''(v)}{g'^4(v)},\quad \mathbf{H} = -\frac{g'^2(v)f''(u) + \left(1 + f'^2(u)\right)g''(v)}{2g'^3(v)}, \tag{4.5}$$

respectively.

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By straightforward computations, the Laplacian operator on $\mathbf{M}$ with the help of (4.2) and (4.3) turns out to be

$$\Delta^{\mathbf{I}}\mathbf{x} = \left(\Delta^{\mathbf{I}}\mathbf{x}_1, \Delta^{\mathbf{I}}\mathbf{x}_2, \Delta^{\mathbf{I}}\mathbf{x}_3\right) = \left(0, 0, \frac{g'^2 f'' + \left(1 + f'^2\right) g''}{g'^3}\right). \tag{4.6}$$

Suppose that $\mathbf{M}$ satisfies (4.2). Then from (4.6), we have

$$\frac{g'^2 f'' + \left(1 + f'^2\right) g''}{g'^3} = \lambda v, \tag{4.7}$$

where $\lambda \in \mathbb{R}$. First of all, we assume that $\mathbf{M}$ satisfies the condition $\Delta^{\mathbf{I}}\mathbf{x} = 0$. We call a surface satisfying that condition a harmonic surface or isotropic minimal. In this case, from (4.7), we get

$$g'^2 f'' + \left(1 + f'^2\right) g'' = 0. \tag{4.8}$$

The above equation can be written in the form:

$$\frac{f''}{1 + f'^2} = p = -\frac{g''}{g'^2}, \; p \in \mathbb{R}.$$

If $p \in \mathbb{R}\backslash\{0\}$, then we get

$$\begin{aligned} f(u) = &\ c_1 - \frac{\ln|\cos(pu - c_2)|}{p}, \\ g(v) = &\ c_3 + \frac{\ln|pv - c_4|}{p}, \end{aligned} \tag{4.9}$$

where $c_i \in \mathbb{R}$. In this case, $\mathbf{M}$ is parametrized by

$$\mathbf{x}(u, v) = \left(u, \left(c_1 - \frac{\ln|\cos(pu - c_2)|}{p}\right) + \left(c_3 + \frac{\ln|pv - c_4|}{p}\right), v\right). \tag{4.10}$$

Therefore upto a translation and dilation of $\mathbb{I}_3^1$, (4.10) can be written as

$$\mathbf{x}(u, v) = \left(u, \ln\left|\frac{v}{\cos u}\right|, v\right).$$

The above is an isotropic Schrek's surface given by

$$\mathbf{x}(u, v) = \left(u, \ln\frac{v}{\cos u}, v\right). \tag{4.11}$$

The above equation is transformed into $z = e^y \cos x$ or in the parametric form

$$\mathbf{x}(u, v) = (u, v, e^v \cos u). \tag{4.12}$$

The graph of (4.11) is contained in the image of (4.12), but they are not the same. The image of (4.11) is not connected while the graph of (4.12) is connected.

Now, if $p = 0$, we have from (4.10)

$$f(u) = c_1 u + c_2, \quad g(v) = c_3 v + c_4, \tag{4.13}$$

where $c_i \in \mathbb{R}$. Again by the translation and dilation of $\mathbb{I}_3^1$, we can write the parametrization of $\mathbf{M}$ as

$$\mathbf{x}(u,v) = (u, u+v, v)\,. \tag{4.14}$$

Theorem 4.1. *Let* $\mathbf{M}$ *be a translation surface of Type* 2 *given by* (4.1) *in* $\mathbb{I}_3^1$*. If* $\mathbf{M}$ *is harmonic, then it is congruent to an open part of the surfaces in* (4.11), (4.12) *or an isotropic plane.*

If $\lambda \neq 0$, from (4.7), we have

$$g'^2 f'' + \left(1 + f'^2\right) g'' - \lambda v g'^3 = 0. \tag{4.15}$$

According to the choices of f and g, we discuss the following three cases:

Case 1: An obvious solution for (4.15) is g being constant for an arbitrary choice of f, but this contradicts to (4.4).

Now, supposing $f = c$, $c \in \mathbb{R} \setminus \{0\}$.

Subcase 1.1: For $\lambda > 0$, from (4.15), we obtain

$$g(v) = \pm \frac{1}{\sqrt{\lambda}} \tan^{-1}\left(\frac{\sqrt{\lambda} v}{\sqrt{-\lambda v^2 - 2c_1}}\right) + c_2,$$

where $c_1, c_2 \in \mathbb{R}$. Suppose $|v| < \sqrt{-2c/\lambda}$, the surface can be parameterized as

$$\mathbf{x}(u,v) = \left(u, \left(c \pm \frac{1}{\sqrt{\lambda}} \tan^{-1}\left(\frac{\sqrt{\lambda} v}{\sqrt{-\lambda v^2 - 2c_1}}\right) + c_2\right), v\right), \ |v| < \sqrt{-2c/\lambda}.$$

Subcase 1.2: For $\lambda < 0$, i.e., $\lambda = -\tilde{\lambda}$, where $\tilde{\lambda} > 0$. From (4.15), we obtain

$$g(v) = c_2 \pm \frac{1}{\sqrt{\tilde{\lambda}}} \log\left[\tilde{\lambda} v + \sqrt{\tilde{\lambda}}\sqrt{\tilde{\lambda} v^2 - 2c_1}\right],$$

where $c_1, c_2 \in \mathbb{R}$

Subcase 1.2.1: Suppose $c_1 < 0$, then $\mathbf{M}$ can be parameterized as

$$\mathbf{x}(u,v) = \left(u, \left(c + c_2 \pm \frac{1}{\sqrt{\tilde{\lambda}}} \log\left[\tilde{\lambda} v + \sqrt{\tilde{\lambda}}\sqrt{\tilde{\lambda} v^2 - 2c_1}\right]\right), v\right), \ c_1 < 0.$$

Subcase 1.2.1: Suppose $c_1 < 0$, then $\mathbf{M}$ can be parameterized as

$$\mathbf{x}(u,v) = \left(u, \left(c + c_2 \pm \frac{1}{\sqrt{\tilde{\lambda}}} \log\left[\tilde{\lambda} v + \sqrt{\tilde{\lambda}}\sqrt{\tilde{\lambda} v^2 - 2c_1}\right]\right), v\right), \ |v| > \sqrt{2c_1/\tilde{\lambda}_1}.$$

Case 2: Suppose $g(v)$ is linear, i.e., $g(v) = c_1 v + c_2$, where $c_1 \in \mathbb{R} \setminus \{0\}$, $c_2 \in \mathbb{R}$. From (4.15), we obtain

$$f'' = v \lambda c_1.$$

The above differential equation has a solution only for $\lambda = 0$, which is a contradiction to our assumption $\lambda \neq 0$.

Now supposing $f(u)$ is linear, i.e., $f(u) = c_1 u + c_2$, where $c_1 \in \mathbb{R} \setminus \{0\}$, $c_2 \in \mathbb{R}$.

Subcase 2.1: For $\lambda > 0$, from (4.15), we obtain

$$g(v) = c_5 \pm \frac{1}{\sqrt{\lambda}} \sqrt{-1 - c_1^2} \log \left[v\lambda + \sqrt{\lambda}\sqrt{v^2\lambda + 2c_3 + 2c_1^2 c_4} \right],$$

where $c_i \in \mathbb{R}$, $i \in \{3, 4, 5\}$. The above found $g(v)$ is again a complex-valued function giving rise to a contradiction.

Subcase 2.2: For $\lambda < 0$, i.e., $\lambda = -\tilde{\lambda}$, where $\tilde{\lambda} > 0$. From (4.15), we obtain

$$g(v) = c_5 \pm \frac{1}{\sqrt{\tilde{\lambda}}} \sqrt{1 + c_1^2} \log \left[v\tilde{\lambda} + \sqrt{\tilde{\lambda}}\sqrt{v^2\tilde{\lambda} - 2c_3 - 2c_1^2 c_4} \right], \ v^2\tilde{\lambda} - 2c_3 - 2c_1^2 c_4 \geq 0,$$

where $c_i \in \mathbb{R}$, $i \in \{3, 4, 5\}$. In this case, $\mathbf{M}$ is parametrized by

$$\mathbf{x}(u, v) = \left(u, \left((c_1 u + c_2) + \left(c_5 \pm \frac{1}{\sqrt{\tilde{\lambda}}} \sqrt{1 + c_1^2} \log \left[v\tilde{\lambda} + \sqrt{\tilde{\lambda}}\sqrt{v^2\tilde{\lambda} - 2c_3 - 2c_1^2 c_4} \right] \right) \right), v \right),$$

where $v^2 > (2c_3 + 2c_1^2 c_4)/\tilde{\lambda}$.

Case 3: f and g are both non-linear and $f(u)$ is at least of C^3 class. Differentiating (4.15) with respect to u we get

$$2f'f''g'' + g'^2 f''' = 0 \tag{4.16}$$

which can be written as

$$\frac{2f'f''}{f'''} = \frac{-g'^2}{g''} = p, \tag{4.17}$$

where $p \in \mathbb{R} \setminus \{0\}$. From (4.17), we obtain

$$\begin{cases} f(u) = c_1 - p \ln \left(\cos \left(\sqrt{\frac{c_2}{p}} \, (u + c_3) \right) \right), \\ g(v) = c_4 + p \ln (v - pc_5), \end{cases} \tag{4.18}$$

where $c_i \in \mathbb{R}$. Substituting (4.18) into (4.15), we get

$$pv(1 - p^2 + p^2\lambda^2) + p(-pc_5 + p^2 c_5) = 0.$$

Since $\{1, v\}$ are linearly independent, we can write

$$p(1 - p^2 + p^2\lambda^2) = 0, \text{ and } p(-pc_5 + p^2 c_5) = 0.$$

This implies

$$p = \left\{ 0, \frac{1 \pm \sqrt{1 - 4\lambda}}{2\lambda} \right\} \text{ and } p = \{0, 1\}.$$

The only common solution from above relations is $p = 0$, which eventually gives rise to contradiction from (4.18) to our assumption that f and g are both non-linear.

In particular, if $p = 1$, we have

$$\frac{1 \pm \sqrt{1-4\lambda}}{2\lambda} = 1$$

implying $\lambda = 0$, which is again a contradiction to our assumption that $\lambda \neq 0$. So, there is no solution. Hence we can state the following:

Theorem 4.2. *Let* $\mathbf{M}$ *be a translation surface of Type* 2 *given by* (4.1) *in the three dimensional simply isotropic space* $\mathbb{I}_3^1$*. Then,* $\mathbf{M}$ *is congruent to an open part of the following*:

(1)

$$\mathbf{x}(u,v) = \left(u, \left(c \pm \frac{1}{\sqrt{\lambda}} \tan^{-1}\left(\frac{\sqrt{\lambda}v}{\sqrt{-\lambda v^2 - 2c_1}}\right) + c_2\right), v\right), \ |v| < \sqrt{-2c/\lambda}.$$

(2)

$$\mathbf{x}(u,v) = \left(u, \left(c + c_2 \pm \frac{1}{\sqrt{\tilde{\lambda}}} \log\left[\tilde{\lambda}v + \sqrt{\tilde{\lambda}}\sqrt{\tilde{\lambda}v^2 - 2c_1}\right]\right), v\right), \ c_1 < 0.$$

(3)

$$\mathbf{x}(u,v) = \left(u, \left(c + c_2 \pm \frac{1}{\sqrt{\tilde{\lambda}}} \log\left[\tilde{\lambda}v + \sqrt{\tilde{\lambda}}\sqrt{\tilde{\lambda}v^2 - 2c_1}\right]\right), v\right), \ |v| > \sqrt{2c_1/\tilde{\lambda}_1}.$$

(4)

$$\mathbf{x}(u,v) = \left(u, \left((c_1u + c_2) + \left(c_5 \pm \frac{1}{\sqrt{\tilde{\lambda}}}\sqrt{1+c_1^2}\log\left[v\tilde{\lambda} + \sqrt{\tilde{\lambda}}\sqrt{v^2\tilde{\lambda} - 2c_3 - 2c_1^2c_4}\right]\right)\right), v\right),$$

$v^2 > (2c_3 + 2c_1^2 c_4)/\tilde{\lambda}$.

5. Translation surfaces of Type 2 satisfying $\boldsymbol{\Delta}^{\mathbf{II}}\mathbf{x}_i = \boldsymbol{\lambda}_i\mathbf{x}_i$

In this section, we classify the translation surfaces of Type 2 with non-degenerate second fundamental form in $\mathbb{I}_3^1$, that is, the surfaces of the form (4.1) satisfy the equation

$$\boldsymbol{\Delta}^{\mathbf{II}}\mathbf{x}_i = \lambda_i \mathbf{x}_i, \tag{5.1}$$

where $\lambda_i \in \mathbb{R}$, $i = 1, 2, 3$. By straightforward computations, we see that the expression in (5.1) turns out to be

(5.2)
$$\boldsymbol{\Delta}^{\mathbf{II}}\mathbf{x} = \left(\boldsymbol{\Delta}^{\mathbf{II}}\mathbf{x}_1, \boldsymbol{\Delta}^{\mathbf{II}}\mathbf{x}_2, \boldsymbol{\Delta}^{\mathbf{II}}\mathbf{x}_3\right) = \left(-\frac{g'f'''}{2f''^2}, \frac{g'}{2}\left(4 - \frac{f'f'''}{f''^2} - \frac{g'g'''}{g''^2}\right), -\frac{g'g'''}{2g''^2}\right).$$

The equation (4.1) by means of (5.1) gives rise to the following system of ordinary differential equations

$$-\frac{g'f'''}{2f''^2} = \lambda_1 u, \tag{5.3}$$

$$\frac{g'}{2}\left(4-\frac{f'f'''}{f''^2}-\frac{g'g'''}{g''^2}\right)=\lambda_2\left(f(u)+g(v)\right), \tag{5.4}$$

$$-\frac{g'g'''}{2g''^2}=\lambda_3 v, \tag{5.5}$$

where $\lambda_i \in \mathbb{R}$. On combining equations (5.3), (5.4) and (5.5), we can write:

$$(2+\lambda_3 v)\,g'-\lambda_2 g=p=-\lambda_1 u f'+\lambda_2 f, \tag{5.6}$$

where $p \in \mathbb{R}\setminus\{0\}$. We discuss eight cases according to constants $\lambda_1, \lambda_2, \lambda_3$. We have summarized the solutions of ordinary differential equation in (5.6) in the following table.

Cases	$(\lambda_1, \lambda_2, \lambda_3)$	$f(u)$	$g(v)$
1	$(\lambda_1=0, \lambda_2=0, \lambda_3=0)$	$f(u)$	c_2
2	$(\lambda_1=0, \lambda_2\neq 0, \lambda_3=0)$	$\frac{p}{\lambda_2}$	$c_2 e^{\frac{\lambda_2}{2}v}-\frac{p}{\lambda_2}$
3	$(\lambda_1=0, \lambda_2=0, \lambda_3\neq 0)$	$f(u)$	c_2
4	$(\lambda_1=0, \lambda_2\neq 0, \lambda_3\neq 0)$	$\frac{p}{\lambda_2}$	$c_2\left\|2+\lambda_3 v\right\|^{\frac{\lambda_2}{\lambda_3}}-\frac{p}{\lambda_2}$
5	$(\lambda_1\neq 0, \lambda_2=0, \lambda_3=0)$	$c_1-\frac{p\ln(\|u\|)}{\lambda_1}$	$c_2+\frac{p}{2}v$
6	$(\lambda_1\neq 0, \lambda_2\neq 0, \lambda_3=0)$	$\frac{p}{\lambda_2}+c_1\left\|u\right\|^{\frac{\lambda_2}{\lambda_1}}$	$c_2 e^{\frac{\lambda_2}{2}v}-\frac{p}{\lambda_2}$
7	$(\lambda_1\neq 0, \lambda_2=0, \lambda_3\neq 0)$	$c_1-\frac{p\ln(\|u\|)}{\lambda_1}$	$c_2+\frac{p\ln\|2+\lambda_3 v\|}{\lambda_3}$
8	$(\lambda_1\neq 0, \lambda_2\neq 0, \lambda_3\neq 0)$	$c_1\left\|u\right\|^{\frac{\lambda_2}{\lambda_1}}+\frac{p}{\lambda_2}$	$c_2\left\|2+\lambda_3 v\right\|^{\frac{\lambda_2}{\lambda_3}}-\frac{p}{\lambda_2}$

In the cases 1 and 3, $f(u)$ can be any differentiable function. In the cases 1, 2, 3, 4 and 5 we have $L=0$ or $N=0$. So the second fundamental form in these cases is degenerate, that contradicts the assumption. In the cases 6, 7 and 8 substituting the f and g into (5.3), (5.4) and (5.5), respectively, we can easily see that they do not satisfy these equations, where $p, c_i \in \mathbb{R}$.

Definition. A surface in the three dimensional simply isotropic space is said to be **II**-harmonic if it satisfies the condition $\Delta^{\mathbf{II}}\mathbf{x}=\mathbf{0}$.

Theorem 5.1. *There is no translation surface of Type 2 in* $\mathbb{I}_3^1$ *satisfying* $\Delta^{\mathbf{II}}\mathbf{x}_i=\lambda_i\mathbf{x}_i$ *for any real number* $\lambda_i, (i\in\{1,2,3\})$.

6. Translation surfaces of Type 2 satisfying $\Delta^{\mathbf{III}}\mathbf{x}_i=\lambda_i\mathbf{x}_i$

In the original paper [1], the Laplacian operator $\Delta^{\mathbf{III}}$ of the third fundamental form is not correct. The third fundamental form and the Laplacian operator $\Delta^{\mathbf{III}}$ with respect to the non-degenerate third fundamental form **III** on **M** in simply isotropic 3-space are defined by

$$\mathbf{III}=Xdu^2+2Ydudv+Zdv^2 \tag{6.1}$$

and

$$\Delta^{\mathbf{III}}\mathbf{x} = -\frac{1}{\sqrt{|XZ-Y^2|}}\left(\partial_u\left(\frac{Z\mathbf{x}_u - Y\mathbf{x}_v}{\sqrt{|XZ-Y^2|}}\right) - \partial_v\left(\frac{Y\mathbf{x}_u - X\mathbf{x}_v}{\sqrt{|XZ-Y^2|}}\right)\right)$$
$$= -\frac{1}{(LN-M^2)\sqrt{EG-F^2}}\left(\begin{array}{c} \partial_u\left(\frac{Z\mathbf{x}_u - Y\mathbf{x}_v}{(LN-M^2)\sqrt{EG-F^2}}\right) \\ -\partial_v\left(\frac{Y\mathbf{x}_u - X\mathbf{x}_v}{(LN-M^2)\sqrt{EG-F^2}}\right)\end{array}\right),$$

where

$$X = EM^2 - 2FLM + GL^2,$$
$$Y = EMN - FLN + GLM - FM^2,$$
$$Z = GM^2 - 2FNM + EN^2,$$

respectively. In fact, the third fundamental form **III** is expressed in terms of the first fundamental form **I** and the second fundamental form **II** in simply isotropic 3-space, that is,

$$\mathbf{III} - 2\mathbf{H}\ \mathbf{II} + \mathbf{K}\ \mathbf{I} = 0,$$

where **K** and **H** are the Gaussian curvature and the mean curvature, respectively. Now following the similar type of steps as in Section 4 and Section 5, we can easily find out:

$$\Delta^{\mathbf{III}}\mathbf{x} = \left(\Delta^{\mathbf{III}}\mathbf{x}_1, \Delta^{\mathbf{III}}\mathbf{x}_2, \Delta^{\mathbf{III}}\mathbf{x}_3\right)$$

or

$$\Delta^{\mathbf{III}}\mathbf{x} = \left(\begin{array}{c} -\frac{2f'}{f''} + \frac{\left(1+f'^2\right)f'''}{f''^3}, \\ -\frac{1+3f'^2}{f''} + \frac{f'f'''\left(1+f'^2\right)}{f''^3} + \frac{g'^2\left(-3g''^2+g'g'''\right)}{g''^3}, \\ \frac{g'\left(-2g''^2+g'g'''\right)}{g''^3}\end{array}\right). \tag{6.2}$$

Hence the equation $\Delta^{III}\mathbf{x}_i = \lambda_i\mathbf{x}_i$ gives rise to the following system of differential equations

$$-\frac{2f'}{f''} + \frac{\left(1+f'^2\right)f'''}{f''^3} = \lambda_1 u, \tag{6.3}$$

$$-\frac{1+3f'^2}{f''} + \frac{f'f'''\left(1+f'^2\right)}{f''^3} + \frac{g'^2\left(-3g''^2+g'g'''\right)}{g''^3} = \lambda_2\left(f+g\right), \tag{6.4}$$

$$\frac{g'\left(-2g''^2+g'g'''\right)}{g''^3} = \lambda_3 v, \tag{6.5}$$

where $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$. We claim that if $\lambda_i \neq 0$ for some $i = 1, 2, 3$, then there do not exist $f = f(u)$ and $g = g(v)$ which satisfy the system (6.3), (6.4) and

(6.5). To see this, we first observe that by combining (6.3), (6.4) and (6.5), we get

$$\left(\lambda_2 f - \lambda_1 u f' + \frac{1+{f'}^2}{f''}\right) + \left(\frac{{g'}^2}{g''} - \lambda_3 v g' + \lambda_2 g\right) = 0. \tag{6.6}$$

Differentiating (6.6) with respect to u, we get

$$\lambda_2(f' + uf'') - (2+\lambda_2)f' + \frac{(1+{f'}^2)f'''}{{f''}^2} = 0. \tag{6.7}$$

On the other hand, from (6.3) we obtain

$$f''' = \frac{{f''}^2(2f' + \lambda_1 u f'')}{1+{f'}^2}. \tag{6.8}$$

By combining (6.7) and (6.8), we obtain

$$\lambda_1(f' + 2uf'') - \lambda_2 f' = 0. \tag{6.9}$$

Suppose now that $\lambda_1 \neq 0$. Then the solution of this is

$$f = \begin{cases} c_1 + c_2 \frac{2\lambda_1 u^{\frac{1}{2}\frac{\lambda_1+\lambda_2}{\lambda_1}}}{\lambda_1+\lambda_2} & \text{if } \lambda_1 \neq 0 \text{ and } \lambda_1 + \lambda_2 \neq 0, \\ c_3 + c_4 \ln u & \text{if } \lambda_1 \neq 0 \text{ and } \lambda_1 + \lambda_2 = 0. \end{cases} \tag{6.10}$$

However, this f does not satisfy (6.3).

Suppose $\lambda_1 = 0$ and $\lambda_2 \neq 0$. Then (6.9) implies that $f' = 0$. But this contradicts the fact that $f'' = 0$.

Now suppose that $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 \neq 0$.

Differentiating (6.6) with respect to v, we get

$$2g' - \frac{{g'}^2 g'''}{{g''}^2} - \lambda_3(g' + vg'') = 0. \tag{6.11}$$

On the other hand, from (6.5) we see that

$$g''' = \frac{{g''}^2(2g' + \lambda_3 v g'')}{{g'}^2}. \tag{6.12}$$

By combining (6.11) and (6.12), we obtain

$$2vg'' + g' = 0,$$

whose solution is

$$g = c_5 + 2c_6\sqrt{v}.$$

If $\lambda_3 \neq -2$, this g does not satisfy (6.5). If $\lambda_2 = -2$, this g satisfies (6.5), but we can see there is no f which satisfies (6.3), (6.4) as follows. Equation (6.4) implies

$$\frac{(1+{f'}^2)f'''}{{f''}^3} = \frac{1+3{f'}^2}{f'f''}.$$

Plugging this into (6.3) gives

$$1 + f'^2 = 0$$

which has no solution.

Combining all the above cases, we can see that the claim holds.

Now suppose that $\mathbf{M}$ is $\mathbf{III}$-harmonic, i.e., $\lambda_1 = \lambda_2 = \lambda_3 = 0$. Then from (6.6), we get

$$-\frac{1}{f''} - \frac{f'^2}{f''} = \frac{g'^2}{g''}. \tag{6.13}$$

Therefore for some $p \in \mathbb{R}$, we obtain

$$-\left(\frac{1}{f''} + \frac{f'^2}{f''}\right) = \frac{g'^2}{g''} = p. \tag{6.14}$$

Note that if $p = 0$, then there is no f satisfying (6.14). So suppose $p \neq 0$, we have

$$f(u) = c_1 + p\ln\left|\cos\left(\frac{pc_2 - u}{p}\right)\right|,\ g(v) = c_3 - p\ln|v + pc_4|, \tag{6.15}$$

where $c_i \in \mathbb{R}$. The solution of (6.15) satisfies the equations (6.3), (6.4) and (6.5). In this case $\mathbf{M}$ is parametrized by

$$\mathbf{x}(u,v) = \left(u, \left(c_1 + p\ln\left|\cos\left(\frac{pc_2 - u}{p}\right)\right|\right) + (c_3 - p\ln|v + pc_4|), v\right). \tag{6.16}$$

Therefore for $v > 0$ and upto a translation and dilation of $\mathbb{I}_3^1$, (6.16) is a Scherk surfaces of the form

$$\mathbf{x}(u,v) = \left(u, \ln\frac{v}{\cos u}, v\right).$$

Thus on summarizing all the above cases, we can state the following result.

Theorem 6.1. *The only translation surface of Type* 2 *satisfying the condition* $\Delta^{\mathbf{III}}\mathbf{x}_i = \lambda_i\mathbf{x}_i$ *in* $\mathbb{I}_3^1$ *is the isotropic Scherk's surface parametrized by*

$$\mathbf{x}(u,v) = \left(u, \ln\frac{v}{\cos u}, v\right).$$

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