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GORENSTEIN MODULES UNDER FROBENIUS EXTENSIONS

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Abstract. Let $R \subset S$ be a Frobenius extension of rings and M a left S-module and let $\mathcal{X}$ be a class of left R-modules and $\mathcal{Y}$ a class of left S-modules. Under some conditions it is proven that M is a $\mathcal{Y}$-Gorenstein left S-module if and only if M is an $\mathcal{X}$-Gorenstein left R-module if and only if $S \otimes_R M$ and $\mathrm{Hom}_R(S, M)$ are $\mathcal{Y}$-Gorenstein left S-modules. This statement extends a known corresponding result. In addition, the situations of Ding modules, Gorenstein AC modules and projectively coresolved Gorenstein flat modules are considered under Frobenius extensions.

1. Introduction

Auslander and Buchsbaum [2] showed that a commutative noetherian local ring R with residue field k is regular if and only if k has finite projective dimension if and only if every R-module has finite projective dimension in 1956. It is a crucial motivation for the study of homological dimensions of modules. In 1969, Auslander and Bridger [1] introduced a new invariant, called Gorenstein dimension (G-dimension), for finitely generated modules over a commutative noetherian ring and proved that R is Gorenstein if and only if k has finite G-dimension if and only if every finitely generated R-module has finite G-dimension. Over any associative rings, Enochs, Jenda, and Torrecillas [7] introduced the notion of Gorenstein flat modules in 1993 and Enochs and Jenda [5] introduced the concept of Gorenstein projective (injective) modules in 1995. For finitely generated modules over commutative noetherian rings Gorenstein projective dimension coincides with the Auslander and Bridger's Gorenstein dimension. In particular, a finitely generated module M is Gorenstein projective if and only if G-dimension of M is zero. The study of Gorenstein homological algebra takes cues from the classical homological algebra.

As a generalization of Frobenius algebras, the theory of Frobenius extensions was studied by Kasch [15] and was developed by Nakayama and Tsuzuku [18,19]

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and Morita [17]. For instance, for a finite group G the integral group ring extension $\mathbb{Z} \subset \mathbb{Z}G$, and the ring extension of dual numbers of an algebra $R \subset R[x]/(x^2)$ are Frobenius extensions. Also notice that an excellent extension is a Frobenius extension; see Huang and Sun [13, Lemma 4.7].

Ren [20, 21] discussed Gorenstein projective (injective) modules and Gorenstein projective (injective) dimensions over Frobenius extensions. Zhao [25] studied Gorenstein homological invariant properties under Frobenius extensions. Let $R \subset S$ be a Frobenius extension of rings and M a left S-module. Ren and Zhao proved that M is a Gorenstein projective left S-module if and only if M is a Gorenstein projective left R-module. In this paper, we will further study Gorenstein modules along Frobenius extensions and extend some main results in [20, 21, 25]. Let $\mathcal{X}$ be a class of left R-modules and $\mathcal{Y}$ a class of left S-modules. Under some conditions it is shown that M is a $\mathcal{Y}$-Gorenstein left S-module if and only if M is an $\mathcal{X}$-Gorenstein left R-module if and only if $S \otimes_R M$ and $\mathrm{Hom}_R(S, M)$ are $\mathcal{Y}$-Gorenstein left S-modules. As corollaries, one could obtain that M is a Gorenstein projective (injective) left S-module if and only if M is a Gorenstein projective (injective) left R-module if and only if $S \otimes_R M$ and $\mathrm{Hom}_R(S, M)$ are Gorenstein projective (injective) left S-modules; if $\mathrm{Gpd}_S M$ is finite, then $\mathrm{Gpd}_S M = \mathrm{Gpd}_R M$; if $\mathrm{Gid}_S M$ is finite, then $\mathrm{Gid}_S M = \mathrm{Gid}_R M$; $\mathrm{Gpd}_R M = \mathrm{Gpd}_S(S \otimes_R M) = \mathrm{Gpd}_R(S \otimes_R M)$ and $\mathrm{Gid}_R M = \mathrm{Gid}_S(S \otimes_R M) = \mathrm{Gid}_R(S \otimes_R M)$; see [20, Theorems 2.2 and 2.3], [21, Theorem 2.5], [25, Theorem 3.2] and [20, Propositions 3.1 and 3.2]. Moreover, the situations of Ding modules, Gorenstein AC modules and projectively coresolved Gorenstein flat modules are discussed under Frobenius extensions.

2. Preliminaries

Throughout this paper, all rings are associative with a unit. Let X be a complex of R-modules (R-complex for short). With homological grading, X has the form

$$\cdots \longrightarrow X_{i+1} \xrightarrow{\partial_{i+1}^X} X_i \xrightarrow{\partial_i^X} X_{i-1} \longrightarrow \cdots .$$

We use the notations $\mathrm{Z}_i(X)$ for the kernel of differential ∂_i^X and $\mathrm{C}_i(X)$ for the cokernel of the differential ∂_{i+1}^X. An R-complex X is called *acyclic* if the homology complex $\mathrm{H}(X)$ is the zero complex. An acyclic complex $\mathbf{P}$ of projective R-modules is *totally acyclic*, if the complex $\mathrm{Hom}_R(\mathbf{P}, Q)$ is acyclic for every projective R-module Q. A left R-module M is called *Gorenstein projective* if there exists a totally acyclic complex $\mathbf{P}$ of projective left R-modules such that $\mathrm{C}_0(\mathbf{P}) \cong M$. An acyclic complex $\mathbf{F}$ of flat right R-modules is *F-totally acyclic*, if the complex $\mathbf{F} \otimes_R I$ is acyclic for every injective left R-module I. A right R-module N is called *Gorenstein flat* if there exists an F-totally acyclic complex $\mathbf{F}$ such that $\mathrm{C}_0(\mathbf{F}) \cong N$. An acyclic complex $\mathbf{U}$ of injective R-modules is called a *totally acyclic* if the complex $\mathrm{Hom}_R(J, \mathbf{U})$ is acyclic for every injective R-module J. A left R-module E is called *Gorenstein injective* if there exists a totally acyclic complex $\mathbf{U}$ of injective left R-modules such that $\mathrm{Z}_0(\mathbf{U}) \cong E$; see Enochs and Jenda [6].

Let M be a left R-module. The *Gorenstein projective dimension* of M, denoted by $\mathrm{Gpd}_R\, M$, is defined as $\inf\{n \geq 0 \,|\, \text{there exists an exact sequence } 0 \to G_n \to \cdots \to G_1 \to G_0 \to M \to 0 \text{ with } G_i \text{ Gorenstein projective modules}\}$. We set $\mathrm{Gpd}_R\, M$ infinity if no such integer exists. The *Gorenstein injective dimension* of M, denoted by $\mathrm{Gid}_R\, M$, can be defined dually; see Holm [11, Definition 2.8].

Recall the following definition of Frobenius extension from Kadison [14, Definition 1.1 and Theorem 1.2].

Definition 2.1. A ring extension $R \subset S$ is called a *Frobenius extension* if one of the following equivalent conditions holds.

(1) The functors $S \otimes_R -$ and $\mathrm{Hom}_R(S, -)$ are naturally equivalent.

(2) ${}_RS$ is a finitely generated projective module and ${}_SS_R \cong \mathrm{Hom}_R({}_RS_S, R)$.

(3) S_R is a finitely generated projective module and ${}_RS_S \cong \mathrm{Hom}_R({}_SS_R, R)$.

(4) There is an R-homomorphism $\tau : S \to R$ and elements $x_i, y_i \in S$ such that for any $s \in S$ one has $\sum\limits_i x_i\tau(y_is) = s$ and $\sum\limits_i \tau(sx_i)y_i = s$.

3. $\mathcal{X}$-Gorenstein modules

In this section, $\mathcal{X}$ denotes a class of left R-modules and $\mathcal{Y}$ denotes a class of left S-modules. We will study $\mathcal{X}$-Gorenstein modules under Frobenius extensions.

Definition 3.1. A class $\mathcal{X}$ of left R-modules is said to be *self-orthogonal* provided that it satisfies the condition: $\mathrm{Ext}_R^{i\geq 1}(X, X') = 0$ for all $X, X' \in \mathcal{X}$.

An $\mathcal{X}$ *resolution* of a left R-module M is an exact sequence $\mathbf{X} = \cdots \to X_1 \to X_0 \to M \to 0$ with $X_i \in \mathcal{X}$ for all $i \geq 0$; moreover, if the sequence $\mathrm{Hom}_R(\mathcal{X}, \mathbf{X})$ is exact, then $\mathbf{X}$ is said to be *proper*. Dually one has the definition of a (co-proper) $\mathcal{X}$ coresolution. Recall the following definition from Geng and Ding [9, Definition 2.2].

Definition 3.2. A left R-module M is called $\mathcal{X}$-*Gorenstein* if there exists an acyclic complex

$$\mathbf{X} = \cdots \to X_1 \to X_0 \to X_{-1} \to X_{-2} \to \cdots$$

of modules in $\mathcal{X}$ such that $M = \mathrm{Im}(X_0 \to X_{-1})$ and $\mathbf{X}$ is $\mathrm{Hom}_R(\mathcal{X}, -)$ and $\mathrm{Hom}_R(-, \mathcal{X})$ exact.

The next theorem is the central result in this note.

Theorem 3.1. *Let $R \subset S$ be a Frobenius extension of rings and M an S-module. Assume that the following conditions hold*:

(a) *$\mathcal{X}$ and $\mathcal{Y}$ are self-orthogonal.*

(b) $S \otimes_R \mathcal{X} \subseteq \mathcal{Y}$.

(c) $Y \in \mathcal{X}$ *as an* R*-module for any* $Y \in \mathcal{Y}$.

(d) Y *is a direct summand of* $S \otimes_R Y$ *as* S*-modules for any* $Y \in \mathcal{Y}$.

Then the following conditions are equivalent.

(1) M *is a* $\mathcal{Y}$*-Gorenstein left* S*-module.*

(2) M *is an* $\mathcal{X}$*-Gorenstein left* R*-module.*

(3) $S \otimes_R M$ *and* $\mathrm{Hom}_R(S, M)$ *are* $\mathcal{Y}$*-Gorenstein left* S*-modules.*

Proof. (1) $\Rightarrow$ (2): Let M be a $\mathcal{Y}$-Gorenstein left S-module. Then there exists an acyclic S-complex $\mathbf{Y} = \cdots \to Y_1 \to Y_0 \to Y_{-1} \to Y_{-2} \to \cdots$ of modules in $\mathcal{Y}$ with $M = \mathrm{Im}(Y_0 \to Y_{-1})$ and such that $\mathrm{Hom}_R(\mathcal{Y}, \mathbf{Y})$ and $\mathrm{Hom}_R(\mathbf{Y}, \mathcal{Y})$ are exact. It is clear that $\mathbf{Y}$ is an acyclic R-complex with $Y_i \in \mathcal{X}$. By Hom-tensor adjunction there are isomorphisms $\mathrm{Hom}_R(\mathcal{X}, \mathbf{Y}) \cong \mathrm{Hom}_R(\mathcal{X}, \mathrm{Hom}_S(S, \mathbf{Y})) \cong \mathrm{Hom}_S(S \otimes_R \mathcal{X}, \mathbf{Y})$ and

$$\mathrm{Hom}_R(\mathbf{Y}, \mathcal{X}) \cong \mathrm{Hom}_R(S \otimes_S \mathbf{Y}, \mathcal{X}) \cong \mathrm{Hom}_S(\mathbf{Y}, \mathrm{Hom}_R(S, \mathcal{X})).$$

Notice that $\mathrm{Hom}_R(S, X) \cong S \otimes_R X$. Hence $\mathrm{Hom}_R(\mathcal{X}, \mathbf{Y})$ and $\mathrm{Hom}_R(\mathbf{Y}, \mathcal{X})$ are acyclic by assumption. Thus M is an $\mathcal{X}$-Gorenstein left R-module.

(2) $\Rightarrow$ (3): Let M be an $\mathcal{X}$-Gorenstein left R-module. Then there exists an acyclic R-complex $\mathbf{X} = \cdots \to X_1 \to X_0 \to X_{-1} \to X_{-2} \to \cdots$ of modules in $\mathcal{X}$ with $M = \mathrm{Im}(X_0 \to X_{-1})$ and such that $\mathrm{Hom}_R(\mathcal{X}, \mathbf{X})$ and $\mathrm{Hom}_R(\mathbf{X}, \mathcal{X})$ are exact. Applying the functor $S \otimes_R -$ to $\mathbf{X}$ yields an acyclic S-complex $S \otimes_R \mathbf{X}$ with $S \otimes_R X_i \in \mathcal{Y}$ and $S \otimes_R M = \mathrm{Im}(S \otimes_R X_0 \to S \otimes_R X_{-1})$. There are isomorphisms

$$\mathrm{Hom}_S(\mathcal{Y}, S \otimes_R \mathbf{X}) \cong \mathrm{Hom}_S(\mathcal{Y}, \mathrm{Hom}_R(S, \mathbf{X})) \cong \mathrm{Hom}_R(\mathcal{Y}, \mathbf{X})$$

and $\mathrm{Hom}_S(S \otimes_R \mathbf{X}, \mathcal{Y}) \cong \mathrm{Hom}_R(\mathbf{X}, \mathcal{Y})$. So the complexes $\mathrm{Hom}_S(\mathcal{Y}, S \otimes_R \mathbf{X})$ and $\mathrm{Hom}_S(S \otimes_R \mathbf{X}, \mathcal{Y})$ are acyclic by assumption. Thus $S \otimes_R M$ is a $\mathcal{Y}$-Gorenstein left S-module. Similarly, one gets $\mathrm{Hom}_R(S, M)$ is a $\mathcal{Y}$-Gorenstein left S-module.

(3) $\Rightarrow$ (2): By (1) $\Rightarrow$ (2) one has that $S \otimes_R M$ is an $\mathcal{X}$-Gorenstein left R-module. Since M is a direct summand of $S \otimes_R M$ as R-modules, it follows from Huang [12, Theorem 1.4] that M is an $\mathcal{X}$-Gorenstein left R-module.

(3) $\Rightarrow$ (1): Let $Y \in \mathcal{Y}$ and let $S \otimes_R M$ and $\mathrm{Hom}_R(S, M)$ be $\mathcal{Y}$-Gorenstein left S-modules. Then M is an $\mathcal{X}$-Gorenstein left R-module by (3) $\Rightarrow$ (2). Note that $0 = \mathrm{Ext}_R^{i \geq 1}(M, Y) \cong \mathrm{Ext}_S^{i \geq 1}(M, \mathrm{Hom}_R(S, Y)) \cong \mathrm{Ext}_S^{i \geq 1}(M, S \otimes_R Y)$ and $0 = \mathrm{Ext}_R^{i \geq 1}(Y, M) = \mathrm{Ext}_R^{i \geq 1}(Y, \mathrm{Hom}_S(S, M)) = \mathrm{Ext}_S^{i \geq 1}(S \otimes_R Y, M)$. As Y is a direct summand of $S \otimes_R Y$ as S-modules by assumption, one has $\mathrm{Ext}_S^{i \geq 1}(M, Y) = 0$ and $\mathrm{Ext}_S^{i \geq 1}(Y, M) = 0$.

Next we show that M has a proper $\mathcal{Y}$-resolution. As $S \otimes_R M$ is a $\mathcal{Y}$-Gorenstein left S-module, there exists an exact sequence

$$0 \longrightarrow K \longrightarrow Y_0 \xrightarrow{u} S \otimes_R M \longrightarrow 0$$

of S-modules such that $Y_0 \in \mathcal{Y}$ and $K = \operatorname{Ker} u$ is $\mathcal{Y}$-Gorenstein. There is an S-epimorphism $\theta : S \otimes_R M \to M$ given by $\theta(s \otimes m) = sm$. It is split as an R-homomorphism. Hence there exists an R-homomorphism $\theta' : M \to S \otimes_R M$ such that $\theta\theta' = 1_M$. Now one has an exact sequence $0 \longrightarrow K_0 \longrightarrow Y_0 \xrightarrow{\theta u} M \longrightarrow 0$ of S-modules, where $K_0 = \operatorname{Ker}(\theta u)$. Let $X \in \mathcal{X}$ and $v : X \to M$ be any R-homomorphism. Since K is also an $\mathcal{X}$-Gorenstein R-module, there is an R-homomorphism $w : X \to Y_0$ such that $uw = \theta' v$ and so $v = \theta(\theta' v) = \theta u(w)$. As $\mathcal{X}$ is self-orthogonal, $\operatorname{Ext}^1_R(X, Y_0) = 0$ and so $\operatorname{Ext}^1_R(X, K_0) = 0$. It follows from Sather-Wagstaff, Sharif and White [23, Corollary 4.5] and [9, Corollary 2.6] that K_0 is an $\mathcal{X}$-Gorenstein R-module. Let $Y \in \mathcal{Y}$. By assumption Y is a direct summand of $S \otimes_R Y$ as S-modules and so there exist S-homomorphisms $\psi : S \otimes_R Y \to Y$ and $\psi' : Y \to S \otimes_R Y$ such that $\psi\psi' = 1_Y$. As $\operatorname{Ext}^1_S(S \otimes_R Y, K_0) \cong \operatorname{Ext}^1_R(Y, K_0) = 0$, one has the exact sequence $0 \longrightarrow K_0 \longrightarrow Y_0 \xrightarrow{\theta u} M \longrightarrow 0$ is $\operatorname{Hom}_S(S \otimes_R Y, -)$ exact. For any S-homomorphism $\alpha : Y \to M$ and $\alpha\psi : S \otimes_R Y \to M$, there exists an S-homomorphism $\beta : S \otimes_R Y \to Y_0$ such that $\alpha\psi = \theta u(\beta)$ and so $\alpha = \alpha(\psi\psi') = (\theta u)(\beta\psi')$. It follows that the exact sequence $0 \longrightarrow K_0 \longrightarrow Y_0 \xrightarrow{\theta u} M \longrightarrow 0$ is $\operatorname{Hom}_S(Y, -)$ exact. Thus K_0 is an $\mathcal{X}$-Gorenstein left R-module and so $S \otimes_R K_0$ is a $\mathcal{Y}$-Gorenstein left S-module. Now proceeding in this manner, one could get the desired proper $\mathcal{Y}$-resolution of M.

Next we show that M has a co-proper $\mathcal{Y}$-coresolution. As $\operatorname{Hom}_R(S, M)$ is a $\mathcal{Y}$-Gorenstein left S-module, there is an exact sequence $0 \longrightarrow \operatorname{Hom}_R(S, M) \xrightarrow{f} Y_{-1} \longrightarrow C \longrightarrow 0$ of S-modules such that $Y_{-1} \in \mathcal{Y}$ and $C = \operatorname{Coker} f$ is $\mathcal{Y}$-Gorenstein. There is an S-monomorphism $\varphi : M \to \operatorname{Hom}_R(S, M)$ given by $\varphi(m)(s) = sm$. It is split as an R-homomorphism. Hence there exists an R-homomorphism $\varphi' : \operatorname{Hom}_R(S, M) \to M$ such that $\varphi'\varphi = 1_M$. Now one has an exact sequence $0 \longrightarrow M \xrightarrow{f\varphi} Y_{-1} \longrightarrow C_{-1} \longrightarrow 0$ of S-modules, where $C_{-1} = \operatorname{Coker}(f\varphi)$. Let $X \in \mathcal{X}$ and $g : M \to X$ be any R-homomorphism. Since C is also an $\mathcal{X}$-Gorenstein R-module, there is an R-homomorphism $h : Y_{-1} \to X$ such that $g\varphi' = hf$ and so $g = (g\varphi')\varphi = h(f\varphi)$. As $\mathcal{X}$ is self-orthogonal, $\operatorname{Ext}^1_R(Y_{-1}, X) = 0$ and so $\operatorname{Ext}^1_R(C_{-1}, X) = 0$. It follows from [23, Corollary 4.5] and [9, Corollary 2.6] that C_{-1} is an $\mathcal{X}$-Gorenstein R-module. Let $Y \in \mathcal{Y}$. By assumption Y is a direct summand of $S \otimes_R Y$ as S-modules and so there exist S-homomorphisms $\psi : S \otimes_R Y \to Y$ and $\psi' : Y \to S \otimes_R Y$ such that $\psi\psi' = 1_Y$. As $\operatorname{Ext}^1_S(C_{-1}, S \otimes_R Y) \cong \operatorname{Ext}^1_R(C_{-1}, Y) = 0$, one has the exact sequence $0 \longrightarrow M \xrightarrow{f\varphi} Y_{-1} \longrightarrow C_{-1} \longrightarrow 0$ is $\operatorname{Hom}_S(-, S \otimes_R Y)$ exact. For any S-homomorphism $\mu : M \to Y$ and $\psi'\mu : M \to S \otimes_R Y$, there exists an S-homomorphism $\nu : Y_{-1} \to S \otimes_R Y$ such that $\psi'\mu = \nu(f\varphi)$ and so $\mu = (\psi\psi')\mu = (\psi\nu)(f\varphi)$. It follows that the exact sequence $0 \longrightarrow M \xrightarrow{f\varphi} Y_{-1} \longrightarrow C_{-1} \longrightarrow 0$ is $\operatorname{Hom}_S(-, Y)$ exact. Thus C_{-1} is an $\mathcal{X}$-Gorenstein left R-module and so $\operatorname{Hom}_R(S, C_{-1})$ is a $\mathcal{Y}$-Gorenstein left S-module. Now proceeding in this manner, one could get the desired co-proper $\mathcal{Y}$-coresolution of M. Therefore M is a $\mathcal{Y}$-Gorenstein left S-module by [9, Proposition 2.4]. □

Now we have the following two results by Theorem 3.1.

Corollary 3.1 ([20, Theorem 2.2], [25, Theorem 3.2])**.** *Let $R \subset S$ be a Frobenius extension and M an S-module. Then the following conditions are equivalent.*

(1) *M is a Gorenstein projective left S-module.*

(2) *M is a Gorenstein projective left R-module.*

(3) *$S \otimes_R M$ and $\mathrm{Hom}_R(S, M)$ are Gorenstein projective left S-modules.*

Proof. In Theorem 3.1 $\mathcal{X}$ is taken to be the class of all projective left R-modules and $\mathcal{Y}$ the class of all projective left S-modules. □

Corollary 3.2 ([20, Theorem 2.3])**.** *Let $R \subset S$ be a Frobenius extension and M an S-module. Then the following conditions are equivalent.*

(1) *M is a Gorenstein injective left S-module.*

(2) *M is a Gorenstein injective left R-module.*

(3) *$S \otimes_R M$ and $\mathrm{Hom}_R(S, M)$ are Gorenstein injective left S-modules.*

Proof. In Theorem 3.1 $\mathcal{X}$ is taken to be the class of all injective left R-modules and $\mathcal{Y}$ the class of all injective left S-modules. □

We denote by $\mathrm{G}(\mathcal{X})$ the class of $\mathcal{X}$-Gorenstein left R-modules and $\mathrm{G}(\mathcal{Y})$ the class of $\mathcal{Y}$-Gorenstein left S-modules.

Definition 3.3. Let M be a left R-module. $\mathrm{G}(\mathcal{X})\text{-}\dim_R M$ is defined as $\inf\{n \geq 0 \,|\, \text{there exists an exact sequence } 0 \to X_n \to \cdots \to X_1 \to X_0 \to M \to 0 \text{ with } X_i \in G(\mathcal{X})\}$. We set $\mathrm{G}(\mathcal{X})\text{-}\dim_R M$ infinity if no such integer exists.

Proposition 3.1. *Let $R \subset S$ be a Frobenius extension. Assume that the following conditions hold*:

(a) *$\mathcal{X}$ and $\mathcal{Y}$ are self-orthogonal.*

(b) *$S \otimes_R \mathcal{X} \subseteq \mathcal{Y}$.*

(c) *$Y \in \mathcal{X}$ as an R-module for any $Y \in \mathcal{Y}$.*

(d) *Y is a direct summand of $S \otimes_R Y$ as S-modules for any $Y \in \mathcal{Y}$.*

For a left S-module M, if $\mathrm{G}(\mathcal{Y})\text{-}\dim_S M$ is finite, then

$$\mathrm{G}(\mathcal{X})\text{-}\dim_R M = \mathrm{G}(\mathcal{Y})\text{-}\dim_S M.$$

Proof. Since each $\mathcal{Y}$-Gorenstein left S-module is an $\mathcal{X}$-Gorenstein left R-module by Theorem 3.1, one has $\mathrm{G}(\mathcal{X})\text{-}\dim_R M \leq \mathrm{G}(\mathcal{Y})\text{-}\dim_S M$. Conversely, assume that $\mathrm{G}(\mathcal{X})\text{-}\dim_R M = n$ is finite. Let $Y \in \mathcal{Y}$. It follows from [12, Theorem 5.8] that $\mathrm{Ext}_R^{n+i}(M, Y) = 0$ for $i \geq 1$. As $\mathrm{Ext}_S^{n+i}(M, S \otimes_R Y) \cong \mathrm{Ext}_R^{n+i}(M, Y) = 0$ and Y is a direct summand of $S \otimes_R Y$ as S-modules by assumption, one has

$\operatorname{Ext}_S^{n+i}(M,Y)=0$ for $i\geq 1$. Hence $\mathrm{G}(\mathcal{Y})\text{-}\dim_S M\leq n$ by [12, Theorem 5.8]. Therefore, $\mathrm{G}(\mathcal{X})\text{-}\dim_R M=\mathrm{G}(\mathcal{Y})\text{-}\dim_S M$. □

Proposition 3.2. *Let $R\subset S$ be a Frobenius extension. Assume that the following conditions hold*:

(a) *$\mathcal{X}$ and $\mathcal{Y}$ are self-orthogonal.*

(b) *$S\otimes_R\mathcal{X}\subseteq\mathcal{Y}$.*

(c) *$Y\in\mathcal{X}$ as an R-module for any $Y\in\mathcal{Y}$.*

(d) *Y is a direct summand of $S\otimes_R Y$ as S-modules for any $Y\in\mathcal{Y}$.*

For a left S-module M,

$$\mathrm{G}(\mathcal{X})\text{-}\dim_R M=\mathrm{G}(\mathcal{Y})\text{-}\dim_S(S\otimes_R M)=\mathrm{G}(\mathcal{X})\text{-}\dim_R(S\otimes_R M).$$

Proof. By Theorem 3.1, one has

$$\mathrm{G}(\mathcal{X})\text{-}\dim_R(S\otimes_R M)\leq\mathrm{G}(\mathcal{Y})\text{-}\dim_S(S\otimes_R M)\leq\mathrm{G}(\mathcal{X})\text{-}\dim_R M.$$

As M is a direct summand of $S\otimes_R M$ as R-modules, one has $\mathrm{G}(\mathcal{X})\text{-}\dim_R M\leq\mathrm{G}(\mathcal{X})\text{-}\dim_R(S\otimes_R M)$ by [12, Theorem 5.8]. □

Now in Propositions 3.1 and 3.2 let $\mathcal{X}$ be the class of all projective (injective) left R-modules and $\mathcal{Y}$ the class of all projective (injective) left S-modules. Then one has the next two results.

Corollary 3.3 ([20, Proposition 3.1])**.** *Let $R\subset S$ be a Frobenius extension. For any left S-module M, if $\operatorname{Gpd}_S M$ is finite, then $\operatorname{Gpd}_S M=\operatorname{Gpd}_R M$. Dually, if $\operatorname{Gid}_S M$ is finite, then $\operatorname{Gid}_S M=\operatorname{Gid}_R M$.*

Corollary 3.4 ([20, Proposition 3.2])**.** *Let $R\subset S$ be a Frobenius extension. If M is a left S-module, then $\operatorname{Gpd}_R M=\operatorname{Gpd}_S(S\otimes_R M)=\operatorname{Gpd}_R(S\otimes_R M)$. Similarly, one has $\operatorname{Gid}_R M=\operatorname{Gid}_S(S\otimes_R M)=\operatorname{Gid}_R(S\otimes_R M)$.*

4. Gorenstein AC-modules

In this section, we will study Gorenstein AC modules under Frobenius extensions. Recall that a left R-module M is said to be of *type FP_∞* if M possesses a projective resolution by finitely generated projective modules. A left R-module M is called *absolutely clean* if $\operatorname{Ext}_R^1(N,M)=0$ for all modules N of type FP_∞. Similarly, M is called *level* if $\operatorname{Tor}_1^R(N,M)=0$ for all right R-modules N of type FP_∞. Notice that the modules of *type FP_∞* are also called super finitely presented modules, and the absolutely clean and level modules are also called weak injective and weak flat modules in Gao and Wang [8]. An R-module M is called *Gorenstein AC-projective* if there exists an acyclic complex $\mathbf{P}$ of projective modules such that $\operatorname{Hom}_R(\mathbf{P},\mathcal{L})$ is acyclic and $\mathrm{Z}_0(\mathbf{P})\cong M$, where $\mathcal{L}$ denotes the subcategory of level modules. Dually, an R-module N is called *Gorenstein AC-injective* if there exists an acyclic complex $\mathbf{I}$ of injective modules such that $\operatorname{Hom}_R(\mathcal{AC},\mathbf{I})$ is acyclic and $\mathrm{Z}_0(\mathbf{I})\cong N$, where $\mathcal{AC}$ denotes the subcategory of absolutely clean modules; see Bravo, Gillespie and Hovey [3].

Lemma 4.1. *Let $0 \to X \to Y \to Z \to 0$ be an exact sequence of left R-modules. If X and Y are Gorenstein AC-projective, then Z is Gorenstein AC-projective if and only if $\mathrm{Ext}_R^1(Z, L) = 0$ for all level modules L.*

Proof. By analogy with the proof of Yang, Liu and Liang [24, Theorem 2.10], one could obtain the result. □

One has a dual version of Lemma 4.1.

Lemma 4.2. *Let $0 \to X \to Y \to Z \to 0$ be an exact sequence of left R-modules. If Y and Z are Gorenstein AC-injective, then X is Gorenstein AC-injective if and only if $\mathrm{Ext}_R^1(A, X) = 0$ for all absolutely clean modules A.*

Theorem 4.1. *Let $R \subset S$ be a Frobenius extension and M a left S-module. Considering the following conditions*:

(1) *M is a Gorenstein AC projective left S-module.*

(2) *M is a Gorenstein AC projective left R-module.*

(3) *$S \otimes_R M$ and $\mathrm{Hom}_R(S, M)$ are Gorenstein AC projective left S-modules.*

Then one has (1) ⇒ (2) ⇔ (3)*. The converse holds if each level left S-module L is a direct summand of S-module $S \otimes_R L$.*

Proof. (1) ⇒ (2): Let M be a Gorenstein AC projective left S-module and L a level left R-module. Then there exists an acyclic complex of projective S-modules $\mathbf{P} = \cdots \to P_1 \to P_0 \to P_{-1} \to \cdots$ such that $M = \mathrm{Ker}(P_0 \to P_{-1})$. It is clear that $\mathbf{P}$ is also an acyclic complex of projective R-modules. Since $\mathrm{Tor}_1^R(X, L) = 0$ for all right R-modules X of type FP_∞, $\mathrm{Tor}_1^S(Y, S \otimes_R L) = 0$ for all right S-modules Y of type FP_∞. There are isomorphisms $\mathrm{Hom}_R(\mathbf{P}, L) \cong \mathrm{Hom}_S(\mathbf{P}, \mathrm{Hom}_R(S, L)) \cong \mathrm{Hom}_S(\mathbf{P}, S \otimes_R L)$. It follows that M is a Gorenstein AC projective left R-module.

(2) ⇒ (3): Let M be a Gorenstein AC projective left R-module and L a level left S-module. Then there exists an acyclic complex of projective R-modules $\mathbf{P} = \cdots \to P_1 \to P_0 \to P_{-1} \to \cdots$ such that $M = \mathrm{Ker}(P_0 \to P_{-1})$. It is easy to see that $S \otimes_R M = \mathrm{Ker}(S \otimes_R P_0 \to S \otimes_R P_{-1})$ and $S \otimes_R \mathbf{P}$ is also an acyclic complex of projective S-modules. As $\mathrm{Tor}_1^S(X, L) = 0$ for all S-modules X of type FP_∞, one has $\mathrm{Tor}_1^R(Y, L) = 0$ for all R-modules Y of type FP_∞. Notice that $\mathrm{Hom}_S(S \otimes_R \mathbf{P}, L) \cong \mathrm{Hom}_R(\mathbf{P}, L)$. Thus $S \otimes_R M$ and $\mathrm{Hom}_R(S, M)$ are Gorenstein AC projective left S-modules.

(3) ⇒ (2): By (1) ⇒ (2) one has that $S \otimes_R M$ is a Gorenstein AC projective left R-module. Since M is a direct summand of $S \otimes_R M$ as R-modules, it follows from [3, Lemma 8.3] that M is a Gorenstein AC projective left R-module.

(3) ⇒ (1): Let L be a level left S-module and let $S \otimes_R M$ and $\mathrm{Hom}_R(S, M)$ be Gorenstein AC projective left S-modules. Then M is a Gorenstein AC projective left R-module by (3) ⇒ (2). Note that there are isomorphisms $0 =$

$\mathrm{Ext}_R^{i\geq 1}(M,L) \cong \mathrm{Ext}_S^{i\geq 1}(M,\mathrm{Hom}_R(S,L)) \cong \mathrm{Ext}_S^{i\geq 1}(M,S\otimes_R L)$. As L is a direct summand of $S\otimes_R L$ as S-modules by assumption, one has $\mathrm{Ext}_S^{i\geq 1}(M,L) = 0$. Thus M has a left projective resolution with $\mathrm{Ext}_S^{i\geq 1}(M,L) = 0$.

Next we construct the right projective resolution of M. Since $\mathrm{Hom}_R(S,M)$ is a Gorenstein AC-projective left S-module, there exists an exact sequence $0 \longrightarrow \mathrm{Hom}_R(S,M) \xrightarrow{f} P_0 \longrightarrow C \longrightarrow 0$ of S-modules such that P_0 is projective and $C = \mathrm{Coker}\, f$ is Gorenstein AC-projective. There is an S-monomorphism $\varphi : M \to \mathrm{Hom}_R(S,M)$ given by $\varphi(m)(s) = sm$. It is split as an R-homomorphism. Hence there exists an R-homomorphism $\varphi' : \mathrm{Hom}_R(S,M) \to M$ such that $\varphi'\varphi = 1_M$. Now one has an exact sequence $0 \longrightarrow M \xrightarrow{f\varphi} P_0 \longrightarrow C_0 \longrightarrow 0$ of S-modules, where $C_0 = \mathrm{Coker}(f\varphi)$. Let U be a level R-module and $g : M \to U$ be any R-homomorphism. Since C is also a Gorenstein AC-projective R-module, there is an R-homomorphism $h : P_0 \to U$ such that $g\varphi' = hf$ and so $g = (g\varphi')\varphi = h(f\varphi)$. Hence $\mathrm{Ext}_R^1(C_0,U) = 0$. It follows from Lemma 4.1 that C_0 is a Gorenstein AC-projective R-module. Let V be a level S-module. By assumption, V is a direct summand of $S\otimes_R V$ as S-modules and so there exist S-homomorphisms $\psi : S\otimes_R V \to V$ and $\psi' : V \to S\otimes_R V$ such that $\psi\psi' = 1_V$. As $\mathrm{Ext}_S^1(C_0, S\otimes_R V) \cong \mathrm{Ext}_R^1(C_0,V) = 0$, one has the exact sequence $0 \longrightarrow M \xrightarrow{f\varphi} P_0 \longrightarrow C_0 \longrightarrow 0$ is $\mathrm{Hom}_S(-, S\otimes_R V)$ exact. For any S-homomorphism $\mu : M \to V$ and $\psi'\mu : M \to S\otimes_R V$, there exists an S-homomorphism $\nu : P_0 \to S\otimes_R V$ such that $\psi'\mu = \nu(f\varphi)$ and so $\mu = (\psi\psi')\mu = (\psi\nu)(f\varphi)$. It follows that the exact sequence $0 \longrightarrow M \xrightarrow{f\varphi} P_0 \longrightarrow C_0 \longrightarrow 0$ is $\mathrm{Hom}_S(-,V)$ exact. Now C_0 is a Gorenstein AC-projective left R-module and so $\mathrm{Hom}_R(S,C_0)$ is a Gorenstein AC-projective left S-module. Proceeding in this manner, one could obtain that M is a Gorenstein AC-projective left S-module. □

Dually, one has the following result.

Proposition 4.1. *Let $R \subset S$ be a Frobenius extension and M a left S-module. Considering the following conditions*:

(1) *M is a Gorenstein AC injective left S-module.*

(2) *M is a Gorenstein AC injective left R-module.*

(3) *$S\otimes_R M$ and $\mathrm{Hom}_R(S,M)$ are Gorenstein AC injective left S-modules.*

Then one has (1) ⇒ (2) ⇔ (3). *The converse holds if each absolutely clean left S-module L is a direct summand of S-module $S\otimes_R L$.*

5. Ding modules

In this section, we will study Ding modules under Frobenius extensions. Recall that an R-module M is called *Ding projective* (*strongly Gorenstein flat*) if there exists an acyclic complex P of projective modules such that $\mathrm{Hom}_R(P,\mathcal{F})$ is acyclic and $\mathrm{Z}_0(P) \cong M$, where $\mathcal{F}$ denotes the subcategory of flat modules;

see Ding, Li and Mao [4] and Gillespie [10]. Dually, an R-module N is called *Ding injective* (*Gorenstein FP-injective*) if there exists an acyclic complex I of injective modules such that $\mathrm{Hom}_R(\mathcal{FP}, I)$ is acyclic and $\mathrm{Z}_0(I) \cong N$, where $\mathcal{FP}$ denotes the subcategory of FP-injective modules; see Mao and Ding [16] and [10].

Theorem 5.1. *Let $R \subset S$ be a Frobenius extension of rings and M a left S-module. Considering the following conditions*:

(1) *M is a Ding projective left S-module.*

(2) *M is a Ding projective left R-module.*

(3) *$S \otimes_R M$ and $\mathrm{Hom}_R(S, M)$ are Ding projective left S-modules.*

Then one has (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3). *The converse holds if each flat left S-module F is a direct summand of S-module $S \otimes_R F$.*

Proof. Using [24, Theorem 2.10] and by analogy with the proof of Theorem 4.1, one could obtain the result. □

Dually, one has the following result.

Proposition 5.1. *Let $R \subset S$ be a Frobenius extension and M a left S-module. Considering the following conditions*:

(1) *M is a Ding injective left S-module.*

(2) *M is a Ding injective left R-module.*

(3) *$S \otimes_R M$ and $\mathrm{Hom}_R(S, M)$ are Ding injective left S-modules.*

Then one has (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3). *The converse holds if each FP-injective left S-module E is a direct summand of S-module $S \otimes_R E$.*

6. Projectively coresolved Gorenstein flat modules

In this section, we will study projectively coresolved Gorenstein flat modules under Frobenius extensions. Recall from Šaroch and Šťovíček [22] that M is called a *projectively coresolved Gorenstein flat module*, or a *PGF-module* for short if there exists an acyclic complex $\mathbf{P}$ of projective left R-modules such that $I \otimes_R \mathbf{P}$ is acyclic for every injective right R-module I and $\mathrm{C}_0(\mathbf{P}) \cong M$.

Proposition 6.1. *Let $R \subset S$ be a Frobenius extension and M a left S-module. Considering the following conditions*:

(1) *M is a PGF-module as a left S-module.*

(2) *M is a PGF-module as a left R-module.*

(3) *$S \otimes_R M$ and $\mathrm{Hom}_R(S, M)$ are PGF-modules as left S-modules.*

Then one has (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3). *The converse holds if the following condition is satisfied*: *If $0 \to A \to B \to C \to 0$ is an exact sequence of left R-modules with A and B PGF-modules and $\mathrm{Tor}_1^R(I, C) = 0$ for any injective right R-modules I, then C is a PGF-module.*

Proof. (1) $\Rightarrow$ (2): Let M be a PGF left S-module and I an injective right R-module. Then there exists an acyclic complex of projective left S-modules $\mathbf{P} = \cdots \to P_1 \to P_0 \to P_{-1} \to \cdots$ such that $M = \mathrm{Ker}(P_0 \to P_{-1})$. It is clear that $\mathbf{P}$ is also an acyclic complex of projective R-modules. Notice that $\mathrm{Hom}_R(S, I) \cong I \otimes_R S$ is an injective S-module and $I \otimes_R \mathbf{P} \cong I \otimes_R S \otimes_S \mathbf{P}$. It follows that M is a PGF left R-module.

(2) $\Rightarrow$ (3): Let M be a PGF left R-module and I an injective right S-module. Then there exists an acyclic complex of projective left R-modules $\mathbf{P} = \cdots \to P_1 \to P_0 \to P_{-1} \to \cdots$ such that $M = \mathrm{Ker}(P_0 \to P_{-1})$. It is easy to see that $S \otimes_R M = \mathrm{Ker}(S \otimes_R P_0 \to S \otimes_R P_{-1})$ and $S \otimes_R \mathbf{P}$ is also an acyclic complex of projective S-modules. Note that I is also an injective right R-module and $I \otimes_R \mathbf{P} \cong I \otimes_S S \otimes_R \mathbf{P}$. Thus $S \otimes_R M$ and $\mathrm{Hom}_R(S, M)$ are PGF left S-modules.

(3) $\Rightarrow$ (2): By (1) $\Rightarrow$ (2) one has that $S \otimes_R M$ is a PGF R-module. Since M is a direct summand of $S \otimes_R M$ as R-modules, it follows from [22, Theorem 3.9] that M is a PGF left R-module.

(3) $\Rightarrow$ (1): Let I be an injective left S-module. Assume that $S \otimes_R M$ and $\mathrm{Hom}_R(S, M)$ are PGF left S-modules. Then M is a PGF left R-module by (3) $\Rightarrow$ (2). There are isomorphisms $0 = \mathrm{Tor}^R_{i\geq 1}(I, M) \cong \mathrm{Tor}^S_{i\geq 1}(I \otimes_R S, M) \cong \mathrm{Tor}^S_{i\geq 1}(\mathrm{Hom}_R(S, I), M)$. As I is a direct summand of $\mathrm{Hom}_R(S, I)$ as S-modules, one has $\mathrm{Tor}^S_{i\geq 1}(I, M) = 0$. Thus M has a left projective resolution with $\mathrm{Tor}^S_{i\geq 1}(I, M) = 0$.

Next we construct the right projective resolution of M. Since $\mathrm{Hom}_R(S, M)$ is a PGF left S-module, there exists an exact sequence $0 \longrightarrow \mathrm{Hom}_R(S, M) \xrightarrow{f} P_0 \longrightarrow C \longrightarrow 0$ of S-modules such that P_0 is projective and $C = \mathrm{Coker}\, f$ is a PGF-module. There is an S-monomorphism $\varphi : M \to \mathrm{Hom}_R(S, M)$ given by $\varphi(m)(s) = sm$. It is split as an R-homomorphism. Hence there exists an R-homomorphism $\varphi' : \mathrm{Hom}_R(S, M) \to M$ such that $\varphi'\varphi = 1_M$. Now one has an exact sequence $0 \longrightarrow M \xrightarrow{f\varphi} P_0 \longrightarrow C_0 \longrightarrow 0$ of S-modules, where $C_0 = \mathrm{Coker}(f\varphi)$. Let E be an injective right R-module. Since C is also a PGF R-module, there is an R-monomorphism $E \otimes_R f$. Note that $E \otimes_R \varphi$ is also an R-monomorphism and so $E \otimes_R f\varphi$ is an R-monomorphism. Hence $\mathrm{Tor}^R_1(E, C_0) = 0$. By assumption C_0 is a PGF R-module. Notice that $\mathrm{Tor}^S_1(I, C_0) = 0$ for any injective S-modules I. Thus the exact sequence $0 \longrightarrow M \xrightarrow{f\varphi} P_0 \longrightarrow C_0 \longrightarrow 0$ is $I \otimes_S -$ exact. Proceeding in this manner, one could obtain that M is a PGF left S-module. ☐

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