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ZERO DISTRIBUTION OF SOME DELAY-DIFFERENTIAL POLYNOMIALS

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Abstract. Let f be a meromorphic function of finite order ρ with few poles in the sense $S_\lambda(r,f) := O(r^{\lambda+\varepsilon}) + S(r,f)$, where $\lambda < \rho$ and $\varepsilon \in (0, \rho-\lambda)$, and let $g(f) := \sum_{j=1}^{k} b_j(z) f^{(k_j)}(z+c_j)$ be a linear delay-differential polynomial of f with small meromorphic coefficients b_j in the sense $S_\lambda(r,f)$. The zero distribution of $f^n(g(f))^s - b_0$ is considered in this paper, where b_0 is a small function in the sense $S_\lambda(r,f)$.

1. Introduction

In this paper, we use key notions of the Nevanlinna theory and related results, as to those, we refer the reader to [6, 7, 9]. A meromorphic function α is said to be a λ-small function of a meromorphic function f of finite order ρ, if there exists $\lambda < \rho$, such that for any $\varepsilon \in (0, \rho - \lambda)$,

$$T(r,\alpha) = O(r^{\lambda+\varepsilon}) + S(r,f), \tag{1}$$

outside a possible exceptional set F of finite logarithmic measure. Here, $S(r,f)$ is any quantity that satisfies $S(r,f) = o(T(r,f))$ as $r \to \infty$ outside a set F. For the sake of simplicity, the right hand side in (1) will be denoted by $S_\lambda(r,f)$. In addition, we say that f has few poles in the sense of (1), if $N(r,f) = S_\lambda(r,f)$.

The first author studied in [8] the zero distribution of $f^n(h(f))^s - b_0$, where n, s are positive integers, b_0 is a λ-small function of f, and $h(f)$ is a shift polynomial given by

$$h(f)(z) := \sum_{j=1}^{k} b_j(z) f(z+c_j),$$

where b_j are λ-small functions of f and c_j are complex numbers. A similar problem had been considered in [13] for $fg(f) - b_0$, where b_0 is a non-zero

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polynomial, and $g(f)$ is a delay-differential polynomial given by

$$g(f) := \sum_{j=1}^{k} b_j(z) f^{(k_j)}(z + c_j), \tag{2}$$

where b_j are small functions of f in the sense $T(r, b_j) = S(r, f)$, c_j are complex numbers and k_j are non-negative integers.

Our purpose is to improve and extend the results in [8, 13] for meromorphic function f with $N(r, f) = S_\lambda(r, f)$ by considering the zero distribution of $f^n(g(f))^s - b_0$, where b_0 is a λ-small function of f, and $g(f)$ is a delay-differential polynomial given in (2) with coefficients b_j being λ-small functions of f. In particular, we generalize some other results in [1, 4, 10, 11] and [3, Chapter 4].

The rest of the paper is organized as follows. Section 2 contains the results concerning the zero distribution of $f^n(g(f))^s - b_0$ in case $b_0 \neq 0$, while the results related to the case $b_0 = 0$ are given in Section 3. The lemmas needed for proving the main results are presented in Section 4, and proofs for the main results are given in Sections 5 and 6.

2. The case $b_0 \neq 0$

Our starting point is the following two examples that show the incompleteness of [8, Theorem 4.4] and [13, Theorem 1.1]. The first example shows that some exceptional cases may occur in [8, Theorem 4.4].

Example 2.1. Let $g_1(f) \equiv 1$ and $g_2(f) = f(z + \pi i)$, with $f(z) = e^z + 1$. Then

$$f g_1(f) - 1 = e^z \quad \text{and} \quad f g_2(f) - 1 = -e^{2z}$$

have no zeros.

Regarding [13, Theorem 1.1], we find that an exceptional case may occur as shown by the following example.

Example 2.2. Let $f(z) = e^{z/2} + e^{-z/2}$ and let the delay-differential polynomials

$$g_1(f) := \frac{1}{2} f(z + 4\pi i) + f'(z), \quad g_2(f) := \frac{1}{2} f(z + 4\pi i) - f'(z).$$

Then,

$$f g_1(f) - 1 = e^z \quad \text{and} \quad f g_2(f) - 1 = e^{-z}$$

have no zeros.

Due to the above examples, we tried to complete [8, Theorem 4.4] and [13, Theorem 1.1]. In fact, we proved the following theorem which extends and completes these results.

Theorem 2.3. *Let f be a transcendental meromorphic function of finite order ρ with $N(r, f) = S_\lambda(r, f)$, b_0 be a non-vanishing λ-small function of f and $g_1(f)$, $g_2(f)$ be non-vanishing linear delay-differential polynomials as in* (2) *with λ-small coefficients of f such that $g_1(f) \not\equiv g_2(f)$. Then for the two functions $F_1 := fg_1(f) - b_0$ and $F_2 := fg_2(f) - b_0$, we have* $\max\{\lambda(F_1), \lambda(F_2)\} = \rho$, *except when one of the of the following cases holds*:

(i) *$g_1(f) = L_1(z)f + M(z)f'$ and $g_2(f) = L_2(z)f - M(z)f'$, where L_1, L_2 and M are non vanishing λ-small functions of f, and $L_1 + L_2 \not\equiv 0$.*

(ii) *Only one of $g_i(f)$, $i = 1, 2$, is a λ-small function of f.*

If g_1 and g_2 are shift polynomials, then [13, Theorem 1.1] is correct. Moreover, if both of g_1 and g_2 are not small functions of f, then [8, Theorem 4.4] is correct.

The condition $g_1(f) \not\equiv g_2(f)$ cannot be dropped out of Theorem 2.3. For example, if $f(z) = e^z + z$ and $g_k(f)(z) = 2f^{(k+1)}(z + \pi i) + f(z)$ for $k = 1, 2$, then $f(z)g_k(f)(z) - z^2 = -e^{2z}$ has no zeros.

Three recent papers should be mentioned here related to Theorem 2.3: The paper [12] is considering zeros of expressions of type $ff^{(k)} - b$. The paper [10] is involving the shifts $f(z+c_1)$ and $f(z+c_2)$ instead of $g_1(f)$ and $g_2(f)$. In the paper [2], iterated differences replace $g_1(f)$ and $g_2(f)$. Moreover, b_0 is taken to be a non-zero polynomial in [2] and [10].

The next result extends [8, Theorem 2.1]. The proof is a simple modification of the corresponding proof of [8, Theorem 2.1].

Theorem 2.4. *Let f be a transcendental meromorphic function of finite order ρ with $N(r, f) = S_\lambda(r, f)$, b_0 be a non-vanishing λ-small function of f, $g(f)$ be a non-vanishing delay-differential polynomial as in* (2) *with λ-small coefficients of f, $n \geq 2$ and $s \geq 1$. Then $F := f^n g(f)^s - b_0$ has sufficiently many zeros to satisfy $\lambda(F) = \rho$.*

The condition $N(r, f) = S_\lambda(r, f)$ is necessary in Theorem 2.4. For example, the function $f(z) = \tan z$ is of order 1 and $N(r, f) = O(r)$. If $g(f)(z) = f(z + \pi/2) = -\cot z$, then $F(z) := f^2(z)g(f)^2(z) - 2 = -1$ has no zeros.

During preparing this paper, Z. Huang offered us the following example, which shows that [8, Theorem 3.1] does not hold always.

Example 2.5. Take $f(z) = e^z + z$ and define

$$g(f)(z) := 2f(z) - f(z + \log 2) = z - \log 2.$$

Then, for every integer $s \geq 1$, the delay polynomial

$$F(z) := f(z)g(f)^s(z) - z(z - \log 2)^s = (z - \log 2)^s e^z$$

has finitely many zeros only.

We give the following extension and a complete version of [8, Theorem 3.1].

Theorem 2.6. *Let f be a transcendental meromorphic function of finite order ρ with $N(r, f) = S_\lambda(r, f)$, b_0 be a non-vanishing λ-small function of f and $g(f)$ be a non-vanishing delay-differential polynomial as in* (2) *with λ-small coefficients of f. If $s \geq 2$, then $F := fg(f)^s - b_0$ satisfies*

$$\max\{\lambda(F), \lambda(f)\} = \rho.$$

In particular, if $\rho \notin \mathbb{N}$, then $\lambda(F) = \lambda(f) = \rho$.

Example 2.5 illustrates Theorem 2.6 in the case when $\lambda(F) < \rho$ and $\lambda(f) = \rho \in \mathbb{N}$.

If $\rho(f) \notin \mathbb{N}$, we see that [8, Theorem 3.1] is correct. This leads to ask, in case $\rho(f) \in \mathbb{N}$, what are the conditions on $g(f)$ that ensure $\lambda(F) = \rho$ in Theorem 2.6? To give a partial answer, we consider a particular form of the delay-differential polynomial $g(f)$, which is given by

$$\tilde{g}(f) := \sum_{i=0}^{n} \sum_{j=0}^{m} b_{i,j}(z) f^{(j)}(z + c_i), \tag{3}$$

where $b_{i,j}$ are λ-small functions of f, and $b_{im} \equiv 1$ for every $0 \leq i \leq n$. We prove the following result, which may be seen as another variant of [8, Theorem 3.1], besides Theorem 2.6.

Theorem 2.7. *Let f be a transcendental meromorphic function of finite order ρ with $N(r, f) = S_\lambda(r, f)$, b_0 be a non-vanishing λ-small function of f and $\tilde{g}(f)$ is given in* (3) *satisfying*

$$T(r, \tilde{g}(f)) \neq S_\lambda(r, f), \tag{4}$$

and

$$T(r, w) = S_\lambda(r, f) \tag{5}$$

for every meromorphic solution w of $\tilde{g}(w) = 0$. If $s \geq 2$, then $F := f(\tilde{g}(f))^s - b_0$ satisfies $\lambda(F) = \rho$.

Example 2.5 above shows that Theorem 2.7 could fail without the condition (4). Meanwhile, the next example shows that Theorem 2.7 could also fail without the condition (5).

Example 2.8 ([1])**.** Suppose that $f(z) = e^z - \frac{2}{3}e^{-z/2}$ and let

$$\tilde{g}(f)(z) := f'(z + 4\pi i) - f(z) = e^{-z/2}.$$

Clearly $T(r, \tilde{g}(f)) \neq S_\lambda(r, f)$ and the function $w = e^z$ is a solution of $\tilde{g}(w) = 0$ without satisfying the condition (5). Finally, we can see that

$$F(z) := f(z)\tilde{g}(f)^2(z) - 1 = \left(e^z - \frac{2}{3}e^{-z/2}\right) e^{-z} - 1 = -\frac{2}{3}e^{-3z/2}$$

has no zeros.

3. The case $b_0 = 0$

In this section, we generalize some results from [3, Chapter 4] and [11]. In [3], the difference operator Δf has been used instead of $g(f)$, while in [11], $g(f)$ was considered to be a shift polynomial with constant coefficients.

Theorem 3.1. *Let f be a transcendental meromorphic function of finite order ρ such that $N(r,f) = S_\lambda(r,f)$. Let $g(f)$ be a non-vanishing linear delay-differential polynomial as in* (2) *with at least two terms and λ-small coefficients of f. Suppose $n \geq 1$ and define $F := f^n g(f)$. Then*

(1) *If $\lambda(f) = \rho$, then $\lambda(F) = \rho$ as well.*
(2) *If $\lambda(f) < \rho$, then $\lambda(F) < \rho$. Furthermore*
  (i) *If $\lambda(f) \leq \rho - 1$, $\lambda < \rho - 1$ and $\rho \neq 1$, then $\lambda(F) = \rho - 1$.*
  (ii) *If $\rho - 1 < \lambda(f) = \lambda^* < \rho$ and $\lambda < \lambda^*$, then $\lambda(F) = \lambda^*$.*
  (iii) *If $\lambda(f) = \lambda = 0$ and $\rho = 1$, then $\lambda(F) = 0$.*

Remark 3.2. The case (1) in Theorem 3.1 holds for $F := f^n g(f)^s$, $s \geq 1$.

The following example illustrates the case (2) in Theorem 3.1.

Example 3.3. (1) The function e^{z^2} is of order 2 and has no zeros. Define

$$g(f)(z) := f(z) + f(z+1) = e^{z^2}(e^{2z+1} + 1).$$

Then, for any integer n, $\lambda(F) = 1 = \rho(f) - 1$. This illustrates the case (2)-(i) in Theorem 3.1.

(2) The function $f(z) = e^z \cosh\sqrt{z}$ is an entire function of order 1 and $\lambda(f) = 1/2$. Let

$$g(f)(z) := f''(z) + \left(\frac{1}{2z} - 2\right) f'(z) + \left(2 - \frac{3}{4z}\right) f(z) = e^z \cosh\sqrt{z}.$$

Then, for every integer n, $\lambda(F) = \lambda(f) = 1/2$. This illustrates the case (2)-(ii) in Theorem 3.1.

The condition $\lambda < \rho - 1$ for $S_\lambda(r,f)$ is necessary for the case (2)-(i) in Theorem 3.1. For example, the function $f(z) = (e^z + 1)e^{z^2}$ is of order $\rho(f) = 2$ and $\lambda(f) = 1$. Let

$$g(f)(z) := \frac{1}{e^z + 1} f'(z) - \frac{2z e^{4\pi^2 - 4\pi i z}}{e^z + 1} f(z + 2\pi i) = \frac{e^z}{e^z + 1} e^{z^2}.$$

Clearly, the coefficients of $g(f)$ are of growth at most $S_\lambda(r,f)$, where $\lambda = 1 = \rho(f) - 1$. Then $F(z) := f(z)g(f)(z) = e^{2z^2 + z}$ such that $\lambda(F) = 0 \neq \rho(f) - 1$.

Theorem 3.4. *Let f be a transcendental meromorphic function of finite order ρ with a finite Borel exceptional value d and $N(r,f) = S_\lambda(r,f)$. Let $g(f)$ be a non-constant delay-differential polynomial as in* (2) *with λ-small coefficients of f. Defining $F := fg(f)$, the following statements hold*:

(i) *If $d \neq 0$ and*

$$\sum_{j=1, k_j=0}^{k} b_j(z) \not\equiv 0,$$

then $F(z)$ has at most one finite Borel exceptional value $d^* \neq 0$, which satisfies

$$\frac{d^* - F(z)}{(d - f(z))^2} = \frac{d^*}{d^2} = \sum_{j=1, k_j=0}^{k} b_j(z).$$

(ii) *If $d \neq 0$ and*

$$\sum_{j=1, k_j=0}^{k} b_j(z) \equiv 0,$$

then $F(z)$ has no finite Borel exceptional values.

(iii) *If $d = 0$, then 0 is a Borel exceptional value of $F(z)$ as well.*

The case (i) of Theorem 3.4 may occur. For example, the function $f(z) = e^z + 1$ has a Borel exceptional value 1. If $g(f) = f(z + \pi i)$, then $F(z) = 1 - e^{2z}$ has 1 as a Borel exceptional value as well, and $\frac{1-F(z)}{(1-f(z))^2} = 1$.

Remark 3.5. The case (iii) of Theorem 3.4 is in fact a special case of the case (2) of Theorem 3.1.

The following consequence of Theorem 3.4 generalizes [4, Theorem 1.2].

Corollary 3.6. *Under the hypotheses of Theorem 3.4, $F(z)$ has no Borel exceptional value b such that*

$$b - d^2 \sum_{j=1, k_j=0}^{k} b_j(z) \not\equiv 0.$$

The following example illustrates Corollary 3.6.

Example 3.7. (1) The function $f(z) = e^z + 1$ has the Borel exceptional value 1. If

$$g(f)(z) := f(z + \pi i) - 2f(z) + 4f'(z) = e^z - 1,$$

then $F(z) := f(z)g(f)(z) = e^{2z} - 1$, and for every $b \neq 1(1-2) = -1$, we have $\lambda(F - b) = 1$.

(2) The function $f(z) = e^z$ has 0 as Borel exceptional value. If

$$g(f)(z) := 2f(z + \pi i) + 2f'(z) + f(z) = e^z,$$

then $F(z) := f(z)g(f)(z) = e^{2z}$, and for every $b \neq 0(2+1) = 0$, we have $\lambda(F - b) = 1$.

Before we state the final result, we recall the definition of the multi-order exponent of convergence of zeros of f by

$$\lambda_{(2}(f) := \limsup_{r \to \infty} \frac{\log^+ N_{(2}\left(r, \frac{1}{f}\right)}{\log r},$$

where $N_{(2}\left(r, \frac{1}{f}\right)$ denotes the counting function of zeros of f whose multiplicities are not less than 2.

Theorem 3.8. *Let f be a transcendental meromorphic function of finite order ρ such that $\lambda_{(2}(f) = \rho$ and that $N(r, f) = S_\lambda(r, f)$, $g(f)$ be a non-constant linear delay-differential polynomial as in* (2) *with λ-small coefficients of f. If $n \geq 1$, then $F := f^n g(f)$ takes every value $a \in \mathbb{C}$ infinitely often and such that $\lambda(F - a) = \rho$.*

4. Auxiliary results

In this section, we collect the results that are needed for proving the main results.

Using the same reasoning as in the proof of [7, Lemma 2.4.2], we easily get the following lemma.

Lemma 4.1. *Let f be a transcendental meromorphic solution of finite order ρ of a differential-difference equation*

$$f^n P(z, f) = Q(z, f),$$

where $P(z, f)$ and $Q(z, f)$ are delay-differential polynomials in f with λ-small coefficients of f. If the total degree of $Q(z, f)$ is $\leq n$, then for each $\varepsilon > 0$

$$m(r, P(z, f)) = O(r^{\rho-1+\varepsilon}) + S_\lambda(r, f).$$

Lemmas 4.2 and 4.3 below are, respectively, slight modifications of [5, Theorem 3] and [8, Lemma 2.5].

Lemma 4.2. *Let f be a transcendental meromorphic function of finite order ρ, and let g be a λ-small function of f. Then for all z such that $|z| \notin E \cup [0, 1]$, where E is of finite logarithmic measure, and for all $k > j$,*

$$\left| \frac{g^{(k)}(z)}{g^{(j)}(z)} \right| \leq |z|^{(k-j)(\lambda-1+\varepsilon)}.$$

Lemma 4.3. *Let f be a transcendental meromorphic function of finite order ρ with $N(r, f) = S_\lambda(r, f)$, and let $g(f)$ be a non-vanishing linear delay-differential polynomial with λ-small coefficients of f. If $n \geq 1$ and $s \geq 1$, then we have $\rho(f^n g(f)^s) = \rho$.*

The following lemma is the complete version of [8, Proposition 4.1].

Lemma 4.4. *Let f be a transcendental meromorphic function of finite order ρ with $N(r, f) = S_\lambda(r, f)$, b_0 be a non-vanishing λ-small function of f and $g_1(f)$, $g_2(f)$ be non-vanishing linear delay-differential polynomials as in* (2) *with λ-small coefficients of f such that $g_1(f) \not\equiv g_2(f)$. Suppose that*

$$f g_j(f) = b_0 + \beta_j e^{h_j}, \quad j = 1, 2, \tag{6}$$

where β_1, β_2 are λ-small functions of f, and h_1, h_2 are polynomials. Then,

$$\deg h_1 = \deg h_2 = \rho.$$

Furthermore, if $\deg(h_1+h_2)<\rho$, *then the delay-differential polynomials* $g_1(f)$ *and* $g_2(f)$ *reduce to*

$$g_1(f)=L_1(z)f+M(z)f' \quad \text{and} \quad g_2(f)=L_2(z)f-M(z)f',$$

where L_1, L_2, M *are non-vanishing* λ*-small functions of* f, *and* $L_1+L_2\not\equiv 0$.

Proof. As to the claims $\deg h_1=\deg h_2=\rho$, the proof in [8] may be repeated, verbatim.

Now, suppose that $\deg(h_1+h_2)<\rho$. Differentiating (6) and eliminating exponentials, we obtain

$$\frac{f'}{f}+\frac{g_j(f)'}{g_j(f)}-\frac{\beta_j'}{\beta}-h_j'=A_j\frac{1}{fg_j(f)}, \tag{7}$$

where

$$A_j=b_0\left(\frac{b_0'}{b_0}-\frac{\beta_j'}{\beta_j}-h_j'\right).$$

Moreover, A_1 and A_2 are not vanishing identically by the reasoning used in the proof of [8, Theorem 2.1]. From (7), we obtain

$$m\left(r,\frac{1}{fg_j(f)}\right)=S_\lambda(r,f), \quad j=1,2$$

and

$$N_{(2}\left(r,\frac{1}{fg_j(f)}\right)=S_\lambda(r,f), \quad j=1,2,$$

where $N_{(2}(r,\cdot)$ stands for the non-simple zeros. Therefore

$$T(r,fg_j(f))=N_{1)}\left(r,\frac{1}{fg_j(f)}\right)+S_\lambda(r,f), \quad j=1,2,$$

where $N_{1)}(r,.)$ counts the simple zeros only.

Clearly, we also have $N_{(2}(r,1/f)=S_\lambda(r,f)$, hence

$$N(r,1/f)=N_{1)}(r,1/f)+S_\lambda(r,f).$$

Making use of the identity $\frac{1}{f^2}=\frac{g_j(f)}{f}\frac{1}{fg_j(f)}$, we obtain $m(r,1/f)=S_\lambda(r,f)$. Assuming, as we may, that $\rho-1\le\lambda<\rho$, we obtain

$$T(r,f) \;=\; N_{1)}(r,1/f)+\; S_\lambda(r,f).$$

Writing now (7) in the form

$$f'g_j(f)+f(g_j(f))'-\left(\frac{\beta_j'}{\beta_j}+h_j'\right)fg_j(f)=A_j, \quad j=1,2, \tag{8}$$

we observe that $f'(z_0)g_j(f)(z_0)-A_j(z_0)=0$ as soon z_0 is a simple zero of f, outside of all possible zeros and poles of A_j,β_j. Since (8) holds for both of $j=1,2$, it is easy to see that all possible poles of

$$H=\frac{A_1g_2(f)-A_2g_1(f)}{f} \tag{9}$$

are multiple except perhaps at the shift values $f(z_0 + c_j)$ and the poles of A_j, hence $N(r, H) = S_\lambda(r, f)$. Moreover, $m(r, H) = S_\lambda(r, f)$, hence, $T(r, H) = S_\lambda(r, f)$.

Since $\deg(h_1 + h_2) \le \rho - 1 \le \lambda$, we have $T(r, e^{h_1+h_2}) = O(r^{\lambda+\varepsilon})$. Moreover, for $\varphi := \beta_1\beta_2 e^{h_1+h_2}$, we have $T(r, \varphi) = S_\lambda(r, f)$.

Consider now a simple zero, say z_0, of f. At the same time, we may assume that b_0, β_1, β_2, φ as well as all coefficients of $g_1(f)$, $g_2(f)$ are non-zero and finite at z_0. Write now (6) in the form

$$fg_1(f) = b_0 + \beta_1 e^{h_1}, \quad fg_2(f) = b_0 + \frac{\varphi}{\beta_1} e^{-h_1}.$$

Thus, we obtain

$$e^{h_1} = -\frac{b_0}{\beta_1} = -\frac{\varphi}{b_0\beta_1}$$

and so $b_0^2 = \varphi$ at z_0. If $b_0^2 - \varphi$ is not vanishing identically, we conclude that

$$N_{1)}(r, 1/f) \le N\left(r, \frac{1}{b_0^2 - \varphi}\right) + S_\lambda(r, f) = S_\lambda(r, f).$$

Hence, $T(r, f) = S_\lambda(r, f)$, resulting in a contradiction. It remains to consider the case that $b_0^2 \equiv \varphi$. We have

$$b_0^2 = \beta_1\beta_2 e^{h_1+h_2} = (fg_1(f) - b_0)(fg_2(f) - b_0),$$

resulting in

$$fg_1(f)g_2(f) = b_0(g_1(f) + g_2(f)). \tag{10}$$

But now, from

$$g_1(f)g_2(f) = b_0\frac{g_1(f) + g_2(f)}{f},$$

we see that $m(r, g_1(f)g_2(f)) = S_\lambda(r, f)$, hence, $T(r, g_1(f)g_2(f)) = S_\lambda(r, f)$, as well. Denote now $\psi := g_1(f)g_2(f)$. Making use of (10), we get

$$b_0 + \beta_1 e^{h_1} = fg_1(f) = \frac{b_0}{\psi} g_1(f)(g_1(f) + g_2(f)) = \frac{b_0}{\psi} g_1(f)^2 + b_0.$$

Thus we obtain

$$g_1(f)^2 = \frac{\beta_1\psi}{b_0} e^{h_1}. \tag{11}$$

Similarly,

$$g_2(f)^2 = \frac{\beta_2\psi}{b_0} e^{h_2}, \tag{12}$$

and, further

$$\left(\frac{g_1(f)}{g_2(f)}\right)^2 = \frac{\beta_1}{\beta_2} e^{h_1-h_2}.$$

Recalling the identity (9), we now proceed to considering

$$A_1 g_2(f) - A_2 g_1(f) = Hf. \tag{13}$$

If H vanishes identically, we see that

$$\left(\frac{A_1}{A_2}\right)^2 = \left(\frac{g_1(f)}{g_2(f)}\right)^2 = \frac{\beta_1}{\beta_2}e^{h_1-h_2},$$

hence $T(r, e^{h_1-h_2}) = S_\lambda(r,f)$. Therefore, $\deg(h_1-h_2) \leq \rho - 1$. Combining with $\deg(h_1+h_2) \leq \rho - 1$, we obtain $\deg h_1 < \rho$, contradicting $\deg h_1 = \rho$.

Now, we need to consider (13), assuming that H does not vanish identically. Squaring (13), and making use of (11) and (12), we get

$$f^2 = \frac{A_1^2}{H^2}\frac{\beta_2\psi}{b_0}e^{h_2} + \frac{A_2^2}{H^2}\frac{\beta_1\psi}{b_0}e^{h_1} - \frac{2A_1A_2}{H^2}\psi.$$

Multiplying by $g_1(f)^2$, making use of (11) and (12) again, and recalling that $f^2g_1(f)^2 = (b_0+\beta_1e^{h_1})^2$, an elementary computation results in

$$\left(\beta_1^2 - \frac{A_2^2}{H^2}\left(\frac{\beta_1\psi}{b_0}\right)^2\right)e^{2h_1} + \left(2b_0\beta_1 + \frac{2A_1A_2\beta_1}{H^2b_0}\psi^2\right)e^{h_1} + \left(b_0^2 - \frac{A_1^2}{H^2}\psi^2\right) = 0. \tag{14}$$

Then, we have $T(r,e^{h_1}) = S_\lambda(r,f)$, resulting in a contradiction $\deg h_1 < \rho$, provided not all coefficients in (14) are vanishing identically. Suppose finally that all coefficients in (14) vanish. Then we immediately observe that $A_1 + A_2 \equiv 0$ and the equation (13) becomes

$$g_1(f) = \kappa(z)f - g_2(f), \tag{15}$$

where $\kappa = \frac{H}{A_1}$. Differentiating (15), we get

$$(g_1(f))' = \kappa'(z)f + \kappa(z)f' - (g_2(f))'. \tag{16}$$

Substituting (15) and (16) into (8) for $j=1$, we obtain

$$\left(\kappa' - \kappa\left(\frac{\beta_1'}{\beta_1} + h_1'\right)\right)f^2 + 2\kappa f'f + \left(\frac{\beta_1'}{\beta_1} + h_1'\right)fg_2(f) - (g_2(f))'f - g_2(f)f' = A_1. \tag{17}$$

By adding (17) to the equation (8) for $j=2$ and keeping in mind $A_1 + A_2 \equiv 0$, we get

$$\left(\frac{\beta_2'}{\beta_2} - \frac{\beta_1'}{\beta_1} + (h_2-h_1)'\right)g_2(f) = \left(\kappa' - \kappa\left(\frac{\beta_1'}{\beta_1} + h_1'\right)\right)f + 2\kappa f'. \tag{18}$$

It's easy to see that

$$\frac{\beta_2'}{\beta_2} - \frac{\beta_1'}{\beta_1} + (h_2-h_1)' \not\equiv 0 \quad \text{and} \quad \kappa' - \kappa\left(\frac{\beta_1'}{\beta_1} + h_1'\right) \not\equiv 0.$$

Otherwise, we get $\deg(h_1-h_2) < \rho$ and $\deg h_1 < \rho$, respectively, which is a contradiction. Therefore, (18) can be rewritten as

$$g_2(f) = L_2(z)f - M(z)f', \tag{19}$$

where

$$L_2 = -\frac{\kappa' - \kappa\left(\frac{\beta_1'}{\beta_1} + h_1'\right)}{\frac{\beta_1'}{\beta_1} - \frac{\beta_2'}{\beta_2} + (h_1 - h_2)'}, \quad M = \frac{2\kappa}{\frac{\beta_1'}{\beta_1} - \frac{\beta_2'}{\beta_2} + (h_1 - h_2)'}.$$

Similarly, we have

$$g_1(f) = L_1(z)f + M(z)f', \tag{20}$$

where

$$L_1 = \frac{\kappa' - \kappa\left(\frac{\beta_2'}{\beta_2} + h_2'\right)}{\frac{\beta_1'}{\beta_1} - \frac{\beta_2'}{\beta_2} + (h_1 - h_2)'}.$$

This completes the proof of Lemma 4.4. □

5. Proofs of theorems of Section 2

The proof of Theorem 2.3 follows, to large extent, the corresponding proof of [8, Theorem 4.4].

Proof of Theorem 2.3. It suffices to show that the only cases which may occur, when $\max\{\lambda(F_1), \lambda(F_2)\} < \rho$, are (i) and (ii).

Suppose that $\max\{\lambda(F_1), \lambda(F_2)\} < \lambda$ for some $\lambda < \rho$. Then

$$fg_j(f) - b_0 = \beta_j e^{h_j}, \quad j = 1, 2, \tag{21}$$

where β_j are non-vanishing λ-small functions of f and h_1, h_2 are polynomials. Therefore, from Lemma 4.4, we have $\deg h_1 = \deg h_2 = \rho$.

Now, if $\deg(h_1 + h_2) < \rho$, then, from Lemma 4.4, we obtain the exceptional case (i) in Theorem 2.3.

Next, we consider the case $\deg(h_1 + h_2) = \rho$. In this case, we show that the exceptional case (ii) in Theorem 2.3 is the only possible one. To this end, we proceed as follows.

(a) Suppose that $g_1(f)$ and $g_2(f)$ both are λ-small functions of f. Then from the second main theorem of Nevanlinna, we know that

$$\begin{aligned} T(r,f) &\le N(r,f) + N\left(r, \frac{1}{f - \frac{b_0}{g_1(f)}}\right) + N\left(r, \frac{1}{f - \frac{b_0}{g_2(f)}}\right) + S(r,f) \\ &= N\left(\frac{1}{\frac{\beta_1}{g_1(f)}}\right) + N\left(\frac{1}{\frac{\beta_2}{g_2(f)}}\right) + S_\lambda(r,f) = S_\lambda(r,f), \end{aligned}$$

which is impossible.

(b) Suppose that both of $g_1(f)$ and $g_2(f)$ are not λ-small functions of f. First, we claim that $\deg(h_1 - h_2) = \rho$. If this is not the case, that is, $\deg(h_1 - h_2) < \rho$, then from (21), we have

$$f\left(g_1(f) - \frac{\beta_1}{\beta_2}e^{h_1 - h_2}g_2(f)\right) = b_0\left(1 - \frac{\beta_1}{\beta_2}e^{h_1 - h_2}\right). \tag{22}$$

By Lemma 4.1, we obtain

$$m\left(r, g_1(f) - \frac{\beta_1}{\beta_2}e^{h_1-h_2}g_2(f)\right) = O(r^{\rho-1+\varepsilon}) + S_\lambda(r,f).$$

Without loss of generality, we may assume that $\rho - 1 < \lambda < \rho$. Then

$$T\left(r, g_1(f) - \frac{\beta_1}{\beta_2}e^{h_1-h_2}g_2(f)\right) = S_\lambda(r,f). \tag{23}$$

If $g_1(f) - \frac{\beta_1}{\beta_2}e^{h_1-h_2}g_2(f) \equiv 0$, then from (22), we get

$$f(g_2(f) - g_1(f)) = \beta_2 e^{h_2} - \beta_1 e^{h_1} = 0,$$

which contradicts the assumption $g_1(f) \not\equiv g_2(f)$. Thus $g_1(f) - \frac{\beta_1}{\beta_2}e^{h_1-h_2}g_2(f) \not\equiv 0$, and therefore, (22) and (23) yield

$$T(r,f) = T\left(r, \frac{b_0\left(1 - \frac{\beta_1}{\beta_2}e^{h_1-h_2}\right)}{g_1(f) - \frac{\beta_1}{\beta_2}e^{h_1-h_2}g_2(f)}\right) = S_\lambda(r,f),$$

which is a contradiction. Thus $\deg(h_1 - h_2) = \rho$.

Second, recall that the function H defined in (9), i.e.,

$$H := \frac{A_1(g_2(f)) - A_2(g_1(f))}{f} \tag{24}$$

is a λ-small function of f, where A_1 and A_2 are non-vanishing λ-small functions of f.

Suppose that H is vanishing identically. Then (21) implies that

$$-\beta_1 A_2 e^{h_1} + \beta_2 A_1 e^{h_2} + (A_1 - A_2)b_0 = 0.$$

Since $\deg(h_1 - h_2) = \rho$, we see from [14, Theorem 1.51] that all coefficients in this equation are vanishing identically, which gives a contradiction.

Suppose now that H is not vanishing identically. From here on, we follow the same reasoning of the proof of [8, Theorem 4.4] omitting most of the details. From (24), we have

$$g_1(f) = \frac{A_1}{A_2}g_2(f) - \frac{H}{A_2}f. \tag{25}$$

Differentiating (25), substituting $g_1(f)$ and $(g_1(f))'$ into (8), and then adding the result to (8), in the case $j = 2$, multiplied by $-A_1/A_2$, we obtain

$$\left(\frac{B_1 H}{A_2} - \left(\frac{H}{A_2}\right)'\right)f + \left(\frac{-2H}{A_2}\right)f' - Dg_2(f) = 0,$$

where

$$D := \frac{B_1 A_1}{A_2} - \left(\frac{A_1}{A_2}\right)' - \frac{A_1 B_2}{A_2},$$

and $B_j := \beta_j'/\beta_j + h_j'$, $j = 1, 2$. The coefficients here, denoted as $T_1 D$ for f and $T_2 D$ for f' (and D for $g_2(f)$) are not vanishing identically, see [8, p. 818]. Hence, we may write (25) in the form

$$g_2(f) = T_1 f + T_2 f' \tag{26}$$

with λ-small coefficients of f. Differentiating now (26), substituting this expression and (26) into (8) with $j = 2$ results in

$$(T_1' - B_2 T_1) f^2 + (2T_1 + T_2' - B_2 T_2) f f' + T_2 (f')^2 + T_2 f f'' = A_2. \tag{27}$$

Differentiate (27). By a careful analysis of simple zeros of f at this expression and at (27), we obtain

$$f'' = \frac{\widetilde{H}}{3A_2T_2} f - \frac{2A_2T_1 + 2A_2T_2 - A_2' - A_2B_2T_2}{3A_2T_2} f', \tag{28}$$

where

$$\widetilde{H} := \frac{(2A_2T_1 + 2A_2T_2 - A_2' - A_2B_2T_2) f' + 3A_2T_2 f''}{f} \tag{29}$$

is a λ-small function of f. To continue, substitute (28) into (27), implying

$$Q_1 f^2 + Q_2 f f' + T_2 (f')^2 = A_2, \tag{30}$$

where

$$Q_1 := T_1' - B_2 T_1 + 3\frac{\widetilde{H}}{3A_2} \quad \text{and} \quad Q_2 := \frac{1}{3}\left(4T_1 + T_2' - 2T_2B_2 + \frac{A_2'}{A_2} T_2\right).$$

Here, in particular, Q_2 is not vanishing identically, as one may easily see. Differentiation of (30) now results in

$$Q_1' f^2 + (2Q_1 + Q_2') f f' + (Q_2 + T_2')(f')^2 + Q_2 f f'' + 2T_2 f' f'' = A_2'. \tag{31}$$

Analyzing simple zeros of f at (27) and (31) we obtain

$$f'' = \frac{\widetilde{R}}{2A_2T_2} f + \frac{A_2' - A_2(Q_2 + T_2')}{2A_2T_2} f', \tag{32}$$

where

$$\widetilde{R} := \frac{(-A_2' + A_2(Q_2 + T_2')) f' + 2A_2T_2 f''}{f}$$

is a λ-small function of f. Substitute now (32) into (31) to obtain

$$\left(Q_1' + \frac{Q_2\widetilde{R}}{2A_2T_2}\right) f^2 + \left(2Q_1 + Q_2' - \frac{1}{2}\frac{Q_2}{T_2}(Q_2 + T_2') + \frac{1}{2}\frac{A_2'}{A_2} Q_2 + \frac{\widetilde{R}}{A_2}\right) f f' + \frac{A_2'}{A_2} T_2 (f')^2 = A_2'. \tag{33}$$

Adding now (30) multiplied by $-A_2'/A_2$ in (33) results in

(34)
$$\left(Q_1'+\frac{Q_2\widetilde{R}}{2A_2T_2}-\frac{A_2'}{A_2}Q_1\right)f+\left(2Q_1+Q_2'-\frac{1}{2}\frac{Q_2}{T_2}(Q_2+T_2')-\frac{1}{2}\frac{A_2'}{A_2}Q_2+\frac{\widetilde{R}}{A_2}\right)f'=0.$$

Looking at the simple zeros of f at (34) results in an immediate contradiction unless its coefficients satisfy

$$Q_1'+\frac{Q_2\widetilde{R}}{2A_2T_2}-\frac{A_2'}{A_2}Q_1\equiv 0 \tag{35}$$

and

$$2Q_1+Q_2'-\frac{1}{2}\frac{Q_2}{T_2}(Q_2+T_2')-\frac{1}{2}\frac{A_2'}{A_2}Q_2+\frac{\widetilde{R}}{A_2}\equiv 0. \tag{36}$$

Eliminate $\widetilde{R}/A_2$ from (35) and (36) to obtain

$$\begin{aligned} &T_2(4Q_1T_2-Q_2^2)\frac{A_2'}{A_2}+Q_2(4Q_1T_2-Q_2^2)\\ &\qquad -T_2(4Q_1T_2-Q_2^2)'+T_2'(4Q_1T_2-Q_2^2)=0. \end{aligned} \tag{37}$$

We are now approaching to the final reasoning for a contradiction. If $(4Q_1T_2-Q_2^2)$ does not vanish identically, it is not difficult to conclude that $e^{h_1+h_2}$ is of order less than ρ, a contradiction with the assumption $\deg(h_1+h_2)=\rho$. Therefore, we must have $4Q_1T_2=Q_2^2$. Denoting $h_1(z)=\alpha z^\rho+\cdots$ and $h_2(z)=\beta z^\rho+\cdots$, we may repeat the reasoning in [8, pp. 821–822], to see that

$$\lim_{|z|\to\infty}\frac{h_1'h_2'}{(h_1'+h_2')^2}=\frac{\alpha\beta}{(\alpha+\beta)^2}=\frac{2}{9}.$$

Solving the equation $\frac{\alpha\beta}{(\alpha+\beta)^2}=\frac{2}{9}$ results in either $\alpha=2\beta$ or $\alpha=\frac{1}{2}\beta$. We proceed to considering the case $\alpha=2\beta$. We may now write

$$e^{h_1(z)}=e^{2\beta z^\rho}e^{P_1(z)},\quad e^{h_2(z)}=e^{\beta z^\rho}e^{P_2(z)},$$

where $P_1(z)$ and $P_2(z)$ are two polynomials of degree $\rho-1$ at most. Recall that we have $g_2(f)=T_1f+T_2f'$ and

$$g_1(f)=\left(\frac{A_1}{A_2}T_1-\frac{H}{A_2}\right)f+\frac{A_1}{A_2}T_2f'.$$

Therefore,

$$b_0+\beta_2e^{\beta z^\rho}e^{P_2(z)}=fg_2(f)=T_1f^2+T_2f'f$$

and

$$b_0+\beta_1e^{2\beta z^\rho}e^{P_1(z)}=fg_1(f)=\left(\frac{A_1}{A_2}T_1-\frac{H}{A_2}\right)f^2+\frac{A_1}{A_2}T_2f'f.$$

By a simple computation,

$$g_2(f)=\frac{b_0}{f}+\beta_2e^{P_2-\frac{1}{2}P_1}\left(\frac{1}{\beta_1}\left(\frac{A_1}{A_2}T_1-\frac{H}{A_2}+\frac{A_1}{A_2}T_2\frac{f'}{f}-\frac{b_0}{f^2}\right)\right)^{1/2}.$$

We next show that $g_2(f)$ is a small function by computing $T(r, T_1 f + T_2 f')$. Indeed,

$$\begin{aligned}T(r, T_1 f + T_2 f') &= m(r, T_1 f + T_2 f') + S_\lambda(r, f)\\ &= \frac{1}{2\pi}\int_{E_1} \log^+ |T_1 f + T_2 f'| d\theta + \frac{1}{2\pi}\int_{E_2} \log^+ |T_1 f + T_2 f'| d\theta\\ &\quad + S_\lambda(r, f),\end{aligned}$$

where E_1, resp. E_2, means the part on the circle of radius r such that $|f| \leq 1$, resp. $|f| > 1$. Again, we may repeat the reasoning in [8, pp. 823–824], to get

$$\frac{1}{2\pi}\int_{E_1} \log^+ |T_1 f + T_2 f'| d\theta = S_\lambda(r, f) \text{ and } \frac{1}{2\pi}\int_{E_2} \log^+ |T_1 f + T_2 f'| d\theta = S_\lambda(r, f).$$

Thus, we obtain

$$T(r, g_2(f)) = T(r, T_1 f + T_2 f') = S_\lambda(r, f),$$

which is a contradiction. Similarly, for the case $\alpha/\beta = 1/2$, we obtain $T(r, g_1(f)) = S_\lambda(r, f)$, which is a contradiction too. This shows that the case, when $g_1(f)$ and $g_2(f)$ are not λ-small functions of f, is not possible. Thus the case (ii) in Theorem 2.3 is the only possible case.

This completes the proof of Theorem 2.3. □

Proof of Theorem 2.4. Suppose, contrary to the assertion, that $\lambda(F) = \lambda < \rho$. Since $N(r, f) = S_\lambda(r, f)$, we have $N(r, F) = S_\lambda(r, f)$ as well. By the standard Hadamard representation, we may write

$$f^n g(f)^s = b_0 + \beta e^h, \tag{38}$$

where β is a non-vanishing λ-small function of f and h is a polynomial of degree $\leq \rho$. Actually, $\deg h = \rho$. Indeed, if $\deg h \leq \mu < \rho$, then

$$T(r, f^n g(f)^s) = O(r^{\mu+\varepsilon}) + S_\lambda(r, f),$$

leading to $\rho(f) \leq \max\{\mu, \lambda\} < \rho$, a contradiction with Lemma 4.3. Differentiating now (38) and eliminating e^h, we obtain

$$n\frac{f'}{f} + s\frac{g(f)'}{g(f)} - \frac{\beta'}{\beta} - h' = \frac{A}{f^n (g(f))^s}, \tag{39}$$

where $A := b_0' - b_0\frac{\beta'}{\beta} - b_0 h'$ cannot vanish identically as shown in [8, p. 811].

Since $n \geq 2$ and $N(r, f) = S_\lambda(r, f)$, the equation (39) gives $N(r, 1/f) = S_\lambda(r, f)$. If $s \geq 2$, we similarly observe that $N(r, 1/g(f)) = S_\lambda(r, f)$. By the second main theorem,

$$\begin{aligned}T(r, f^n g(f)^s) &\leq N(r, f^n g(f)^s) + N(r, 1/f^n g(f)^s) + N(r, 1/F) + S(r, f)\\ &= S_\lambda(r, f),\end{aligned}$$

contradicting Lemma 4.3.

It remains to consider the case $s = 1$. Since f is a meromorphic function of finite order ρ such that $\max\{N(r,f), N(r,1/f)\} = S_\lambda(r,f)$, f may be represented as $f(z) = \gamma(z)e^{Q(z)}$, where $T(r,\gamma) = S_\lambda(r,f)$ and Q is a polynomial of degree $\deg Q = \rho$. Write now $g(f)(z) = G(f)(z)e^{Q(z)}$, where

$$T(r, G(f)) = O(r^{\rho-1+\varepsilon}) + S_\lambda(r,f).$$

Hence

$$N\left(r, \frac{1}{g(f)}\right) = N\left(r, \frac{1}{G(f)}\right) = O(r^{\rho-1+\varepsilon}) + S_\lambda(r,f).$$

Recalling $N(r,f) = S_\lambda(r,f)$, the second main theorem may be applied again to obtain

$$\begin{aligned} T(r, f^n g(f)) &\le nN\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{g(f)}\right) + N\left(r, \frac{1}{F}\right) + N(r, f^n g(f)) + S(r,f) \\ &= O(r^{\rho-1+\varepsilon}) + S_\lambda(r,f). \end{aligned}$$

Thus, $\rho(f^n g(f)) \le \max\{\rho - 1, \lambda\}$, contradicting Lemma 4.3. □

Proof of Theorem 2.6. Suppose, contrary to the assertion that

$$\max\{\lambda(F), \lambda(f)\} < \rho.$$

We may write

$$f(g(f))^s = b_0 + \beta e^h, \tag{40}$$

where β is a non-vanishing λ-small function of f and h is a polynomial of degree ρ. Differentiating (40) and eliminating e^h, we obtain

$$\frac{f'}{f} + s\frac{g(f)'}{g(f)} - \frac{\beta'}{\beta} - h' = \frac{b_0\left(\frac{b_0'}{b_0} - \frac{\beta'}{\beta} - h'\right)}{f(g(f))^s}. \tag{41}$$

Since $s \ge 2$, we conclude from (41) that

$$N\left(r, \frac{1}{g(f)}\right) = S_\lambda(r,f). \tag{42}$$

By this and (41), we conclude again that

$$N\left(r, \frac{1}{f(g(f))^s}\right) \le \overline{N}\left(r, \frac{1}{f}\right) + S_\lambda(r,f). \tag{43}$$

By using the second main theorem of Nevanlinna, we get

$$\begin{aligned} T(r, f(g(f))^s) &\le N(r, f(g(f))^s) + N\left(r, \frac{1}{f(g(f))^s}\right) + N\left(r, \frac{1}{F}\right) + S(r,f) \\ &\le \overline{N}\left(r, \frac{1}{f}\right) + S_\lambda(r,f). \end{aligned} \tag{44}$$

From (44) and Lemma 4.3, we obtain $\rho = \rho(f(g(f))^s) \le \lambda(f) < \rho$, a contradiction.

Suppose now that $\rho \notin \mathbb{N}$. Then, clearly, $\lambda(f) = \rho$. On the other hand, by Lemma 4.3 again, we have $\rho(F) = \rho \notin \mathbb{N}$. Hence $\lambda(F) = \rho$. $\square$

Proof of Theorem 2.7. As one may clearly see, our reasoning here is to some part similar to the reasoning applied by Alotaibi in [1].

Suppose, contrary to the assertion that $\lambda(F) < \rho$. On the other hand, since b_0 is a non-vanishing λ-small function of f, we have $F = b_0 F^*$, where

$$F^* := \frac{1}{b_0} f \tilde{g}(f)^s - 1.$$

Clearly, $\lambda(F) = \lambda(F^*)$ and $\rho(F) = \rho(F^*) = \rho$. So, in the following, we consider only F^*.

Put $b_{i,-1} = b_{i,m+1} = 0$ and since $b_{i,m}(z) \equiv 1$ for every $0 \leq i \leq n$, we have

$$\tilde{g}(f) = \sum_{i=0}^{n} \sum_{j=0}^{m} b_{i,j} f^{(j)}(z + c_i) = \sum_{i=0}^{n} \sum_{j=-1}^{m+1} b_{i,j} f^{(j)}(z + c_i).$$

Using the fact that $b'_{i,m+1} = 0$, we have

$$\begin{aligned} \tilde{g}(f)' &= \sum_{i=0}^{n} \sum_{j=0}^{m} b'_{i,j} f^{(j)}(z + c_i) + \sum_{i=0}^{n} \sum_{j=0}^{m} b_{i,j} f^{(j+1)}(z + c_i) \\ &= \sum_{i=0}^{n} \sum_{j=0}^{m+1} (b'_{i,j} + b_{i,j-1}) f^{(j)}(z + c_i). \end{aligned} \tag{45}$$

Since $\tilde{g}(f) = \sum_{i=0}^{n} \sum_{j=0}^{m+1} b_{i,j} f^{(j)}(z + c_i)$, we find that $w = f$ solves the delay-differential equation

$$\sum_{i=0}^{n} \sum_{j=0}^{m+1} d_{i,j} w^{(j)}(z + c_i) = 0, \tag{46}$$

where

$$d_{i,j} = b'_{i,j} + b_{i,j-1} - \frac{\tilde{g}(f)'}{\tilde{g}(f)} b_{i,j}, \quad \text{and} \quad d_{0,m+1} = 1$$

for every $0 \leq i \leq n$, $0 \leq j \leq m + 1$.

Let $w = uv$, where $v = b_0/\tilde{g}(f)^s$. By using Leibniz' rule in (46) and the convention $C_j^k = 0$ for $k > j$, we get

$$\begin{aligned} \sum_{i=0}^{n} \sum_{j=0}^{m+1} d_{i,j} (uv)^{(j)}(z + c_i) &= \sum_{i=0}^{n} \sum_{j=0}^{m+1} d_{i,j} \sum_{k=0}^{j} C_j^k u^{(k)}(z + c_i) v^{(j-k)}(z + c_i) \\ &= \sum_{i=0}^{n} \sum_{j=0}^{m+1} d_{i,j} \sum_{k=0}^{m+1} C_j^k u^{(k)}(z + c_i) v^{(j-k)}(z + c_i) = 0. \end{aligned}$$

Dividing the right hand side by $v(z)$ we get

$$0=\sum_{i=0}^{n}\sum_{k=0}^{m+1}u^{(k)}(z+c_i)\sum_{j=0}^{m+1}C_j^k d_{i,j}\frac{v^{(j-k)}(z+c_i)}{v(z)}=\sum_{i=0}^{n}\sum_{k=0}^{m+1}A_{i,k}(z)u^{(k)}(z+c_i),$$

where, again since $C_j^k=0$ for $k>j$,

$$A_{i,k}(z)=\sum_{j=k}^{m+1}C_j^k d_{i,j}\frac{v^{(j-k)}(z+c_i)}{v(z)}. \tag{47}$$

In particular, this gives

$$\begin{aligned}\sum_{i=0}^{n}A_{i,0} &= \sum_{i=0}^{n}\sum_{j=0}^{m+1}d_{i,j}\frac{v^{(j)}(z+c_i)}{v(z)}\\ &= \sum_{i=0}^{n}\sum_{j=0}^{m+1}\left(b'_{i,j}+b_{i,j-1}-\frac{\tilde{g}(f)'}{\tilde{g}(f)}b_{i,j}\right)\frac{v^{(j)}(z+c_i)}{v(z)}\\ &= \frac{\tilde{g}(v)'-\frac{\tilde{g}(f)'}{\tilde{g}(f)}g(v)}{v(z)}. \qquad (48)\end{aligned}$$

We claim now that

$$\sum_{i=0}^{n}A_{i,0}\not\equiv 0. \tag{49}$$

To prove this, we suppose the contrary. By using (48), we get

$$\tilde{g}(v)'=\frac{\tilde{g}(f)'}{\tilde{g}(f)}\tilde{g}(v).$$

We consider two cases:

Case 1: If $\tilde{g}(v)\not\equiv 0$, then by simple integration of the above equation, we get

$$\tilde{g}(v)=c\tilde{g}(f),$$

where c is a non-zero constant. Defining $H:=v-cf$, linearity of $\tilde{g}$ implies that $\tilde{g}(H)=0$. By assumption, $T(r,H)=S_\lambda(r,f)$. Further defining $G:=f+\frac{1}{c}H$, we see that

$$v=cG \text{ and } \tilde{g}(f)=\tilde{g}\left(f+\frac{1}{c}H\right)=\tilde{g}(G).$$

Therefore, $T(r,G)=T(r,f)+S_\lambda(r,f)$. On the other hand, since $v=b_0/\tilde{g}(f)^s$, we get

$$1=\frac{c}{b_0}G\tilde{g}(f)^s=\frac{c}{b_0}G\tilde{g}(G)^s.$$

This leads to $T(r,G\tilde{g}(G)^s)=S_\lambda(r,f)$, which is a contradiction with Lemma 4.3.

Case 2: If $\tilde{g}(v)\equiv 0$, then

$$S_\lambda(r,f)=T(r,v)=sT(r,1/\tilde{g}(f))+S_\lambda(r,f)$$

$$= sT(r, \tilde{g}(f)) + S_\lambda(r, f),$$

a contradiction with the condition $T(r, \tilde{g}(f)) \neq S_\lambda(r, f)$.

Returning to our proof now, we have

$$u = \frac{w}{v} = \frac{1}{b_0} f\tilde{g}(f)^s = F^* + 1.$$

So $F^* + 1$ solves the linear delay-differential equation

$$\sum_{i=0}^{n} \sum_{k=0}^{m+1} A_{i,k}(z) u^{(k)}(z + c_i) = 0.$$

Hence

$$\sum_{i=0}^{n} \sum_{k=0}^{m+1} A_{i,k}(z) F^{*(k)}(z + c_i) = -\sum_{i=0}^{n} A_{i,0}(z). \tag{50}$$

From (47) and (42) we deduce that

$$T(r, A_{i,k}) = O(r^{\rho-1+\varepsilon}) + S_\lambda(r, f).$$

Since $\lambda(F^*) < \rho$, we may write

$$F^*(z) = \beta^*(z) e^{h(z)}, \tag{51}$$

where $T(r, \beta^*) = S_\lambda(r, f)$, and h is a polynomial of degree equal to ρ. Obviously

$$F^{*(k)}(z + c_i) = \psi_k(z + c_i) e^{h(z+c_i)}, \tag{52}$$

where ψ_k $(k = 0, \dots, m + 1)$ are differential polynomials in β and h. By substituting (51) and (52) into (50) and since $\sum_{i=0}^{n} A_{i,0} \not\equiv 0$, we get

$$\sum_{i=0}^{n} \sum_{k=0}^{m+1} \frac{A_{i,k}(z)\psi_k(z + c_i)}{\sum_{i=0}^{n} A_{i,0}(z)} e^{h(z+c_i)-h(z)} = -e^{-h(z)}.$$

Hence

$$T(r, e^h) = O(r^{\rho-1+\varepsilon}) + S_\lambda(r, f),$$

resulting in a contradiction $\deg h = \rho(f) \leq \max\{\rho - 1, \lambda\} < \rho$. This completes the proof of Theorem 2.7. □

6. Proofs of results from Section 3

Proof of Theorem 3.1. (1) Suppose that $\lambda(F) < \rho$. Writing $F = f^n g(f) = \beta e^h$, where $T(r, \beta) = S_\lambda(r, f)$ and h is a polynomial, we get

$$n\frac{f'}{f} + \frac{(g(f))'}{g(f)} - \frac{\beta'}{\beta} = h'.$$

If f vanishes at z_0, then $F(z_0) = 0$, unless $g(f)$ has a pole at z_0. This may happen at the coefficients of $g(f)$ and the poles of $f(z_0 + c_j)$ only, contributing at most by $S_\lambda(r, f)$. Therefore, $\lambda(f) < \rho$, and the claim follows.

(2) In this case, we may write $f(z) = \tau(z)e^{\alpha z^\rho}$, where α $(\alpha \neq 0)$ is a constant and τ is a λ-small function of f. Then, of course, $f(z+c_j) = \tau(z+c_j)\tau_j(z)e^{\alpha z^\rho}$,

where τ_j is a meromorphic function of order $\rho - 1$. Therefore, it is not difficult to see that $g(f) = T(z)e^{\alpha z^\rho}$, where $T(z)$ is a differential polynomial of τ, of its shifts and of its derivatives. Therefore $\rho(T) \le \lambda$, implying that $\lambda(F) < \rho$.

We divide the proof of Part (2)-(i) into three parts:

(a) If $\lambda(f) < \rho - 1$, then $\rho(F) = \rho$ by Lemma 4.3. Write now $f(z) = \tau(z)e^{h(z)}$, where h is a polynomial of degree $= \rho \ge 2$, and τ is a λ-small function of f where $\lambda < \rho - 1$. Recalling that

$$g(f) := \sum_{j=1}^{k} b_j(z) f^{(k_j)}(z + c_j)$$

with $k \ge 2$, we may write

$$g(f) = \sum_{j=1}^{k} \tau_j(z) f(z + c_j) := \sum_{j=1}^{k} b_j(z) \frac{f^{(k_j)}(z + c_j)}{f(z + c_j)} f(z + c_j).$$

Suppose first, contrary to the claim, that $\lambda(F) < \rho - 1$. Writing, as we may, $F(z) = \sigma(z)e^{Q(z)}$, where σ is a λ-small function of f where $\lambda < \rho - 1$ and Q is a polynomial of degree ρ. Therefore, we now have

$$\sigma(z) = \sum_{j=1}^{k} \alpha_j(z) e^{\beta_j(z)},$$

where $\alpha_j(z) = \tau^n(z)\tau_j(z)\tau(z + c_j)$ are λ-small functions of f with $\lambda < \rho - 1$ and $\beta_j(z) = nh(z) + h(z + c_j) - Q(z)$ for every $1 \le j \le k$. Set also,

$$h(z) = b_m z^m + b_{m-1} z^{m-1} + \cdots + b_0,\ b_m \ne 0,$$

where $b_m, b_{m-1}, \ldots, b_0$ are constants and $m = \rho \ge 2$. Hence, for every $i \ne j$

$$\beta_i(z) - \beta_j(z) = h(z + c_i) - h(z + c_j) = m b_m (c_i - c_j) z^{m-1} + \cdots .$$

If for all j, $\deg \beta_j(z) \ge \rho - 1$, then, by [14, Theorem 1.51], $\sigma(z) \equiv 0$ and $\alpha_j(z) \equiv 0$ for $j = 1, \ldots, k$. Hence F vanishes, a contradiction. If this is not the case, then $\deg \beta_j(z) < \rho - 1$ for some $1 \le j_0 \le k$, which is a contradiction. Therefore,

$$\sigma(z) - \alpha_{j_0}(z) e^{\beta_{j_0}(z)} = \sum_{j=1, j \ne j_0}^{k} \alpha_j(z) e^{\beta_j(z)}.$$

Since $\deg(\beta_i - \beta_j) = \rho - 1$ for every $1 \le i \ne j \le k$, by [14, Theorem 1.51] again, $\alpha_j(z) \equiv 0$ for all $j \ne j_0$, and $\sigma(z)e^{-\beta_{j_0}(z)}(z) \equiv \alpha_{j_0}(z)$. This implies that $\tau_j(z)\tau(z + c_j)$ vanishes for all $j \ne j_0$, hence $g(f)$ includes just one term, a contradiction.

We point out to the reader that it is impossible to have $\deg \beta_j < \rho - 1$ for all $1 < j < k$, since $\deg(\beta_i - \beta_j) = \rho - 1$ for every $1 \le i \ne j \le k$.

(b) Suppose next that $\lambda(F) > \rho - 1$. We may write $g(f)$ as

$$g(f) = \left(\sum_{j=1}^{k} \tau_j(z)\tau(z+c_j)e^{h(z+c_j)-h(z)} \right) e^{h(z)}. \tag{53}$$

Since $\deg(h(z+c_j) - h(z)) = \rho - 1$ and $\tau_j(z)$ $(1 \leq j \leq k)$, $\tau(z)$ are λ-small functions of f with $\lambda < \rho - 1$, then $\lambda(g(f)) \leq \rho - 1$. This inequality and $\lambda(f) < \rho - 1$ implies a contradiction

$$\rho - 1 < \lambda(F) = \lambda(f^n g(f)) \leq \rho - 1.$$

(c) If finally $\lambda(f) = \rho - 1$, then, from (53), $\lambda(F) \leq \rho - 1$. On the other hand, we have

$$nN\left(r, \frac{1}{f}\right) = N\left(r, \frac{g(f)}{F}\right) \leq N\left(r, \frac{1}{F}\right) + S_\lambda(r, f). \tag{54}$$

By (54) and since $\lambda < \rho - 1$ we obtain $\rho - 1 = \lambda(f) \leq \lambda(F)$. This completes the proof of Part (2)-(i).

To prove Part (2)-(ii), we may write $f(z) = \tau(z)e^{h(z)}$, where h is a polynomial of degree $\rho \geq 1$ and $\tau := \frac{\tau_1}{\tau_2}$ is a meromorphic function where τ_1, τ_2 are the canonical products of zeros and poles, respectively. Since $\lambda(f) = \lambda^*$ $(\rho - 1 < \lambda^* < \rho)$ and $\lambda < \lambda^*$, we have $\rho(\tau_1) = \lambda^*$ and $\rho(\tau_2) < \lambda^*$. This leads to $\rho(\tau) = \lambda^*$. By this and (53) we deduce that $\lambda(g(f)) \leq \lambda^*$, hence $\lambda(F) \leq \lambda^*$.

On the other hand, from (54) and since $\lambda < \lambda^*$, we deduce that $\lambda(F) \leq \lambda^*$.

As to Part (2)-(iii), we may use the same reasoning as in (2) above to obtain that $g(f) = T(z)e^{\alpha z}$ where α is a constant and T is a meromorphic function of order 0. Therefore, $\lambda(F) = 0$. □

Proof of Theorem 3.4. (i) Suppose that d $(\neq 0)$ is the Borel exceptional value of $f(z)$ and

$$\sum_{j=1, k_j=0}^{k} b_j(z) \not\equiv 0.$$

Clearly, the order ρ is an integer and $f(z)$ can be written in the form

$$f(z) = d + \pi(z)e^{\alpha z^\rho}, \tag{55}$$

where $\alpha \neq 0$ is a constant and $\pi(z)$ is a non-vanishing meromorphic function satisfying $\rho(\pi) < \rho$. Thus

$$f(z+c_j) = d + \pi(z+c_j)\pi_j(z)e^{\alpha z^\rho}, \tag{56}$$

where π_j is a meromorphic function of order $\rho - 1$. On the other hand, we may write $g(f)$ as

$$g(f) = \sum_{j \in I_1} b_j(z)f(z+c_j) + \sum_{j \in I_2} b_j(z)f^{(k_j)}(z+c_j), \tag{57}$$

where $I_1 = \{1 \leq j \leq k : k_j = 0\}$ and $I_2 = \{1 \leq j \leq k : k_j > 0\}$. Hence, for every $j \in I_2$

$$f^{(k_j)}(z + c_j) = Q_{k_j}(z)e^{\alpha z^\rho}, \tag{58}$$

where Q_{k_j} is a meromorphic function of order less than ρ. By substituting (56) and (58) into (57), we get

$$\begin{aligned} F(z) = d^2A(z) &+ d\,(B(z) + C(z) + \pi(z)A(z))\,e^{\alpha z^\rho} \\ &+ \pi(z)\,(B(z) + C(z))\,e^{2\alpha z^\rho}, \end{aligned} \tag{59}$$

where

$$A(z) := \sum_{j\in I_1} b_j(z)\,(\not\equiv 0),\ B(z) := \sum_{j\in I_1} \pi(z + c_j)\pi_j(z)b_j(z)$$

and

$$C(z) := \sum_{j\in I_2} b_j(z)Q_{k_j}(z).$$

By Lemma 4.3, we know that $\rho(F) = \rho$. If $F(z)$ has a Borel exceptional value d^*, then

$$F(z) = d^* + \pi^*(z)e^{\beta z^\rho}, \tag{60}$$

where β $(\neq 0)$ is a constant, and $\pi^*(z)$ $(\not\equiv 0)$ is a meromorphic function of order less than ρ. By (59) and (60), we have

(61)

$$d(B(z)+C(z)+\pi(z)A(z))e^{\alpha z^\rho}+\pi(z)(B(z)+C(z))e^{2\alpha z^\rho}-\pi^*(z)e^{\beta z^\rho}=d^*-d^2A(z).$$

Case 1. If $\beta \neq \alpha$ and $\beta \neq 2\alpha$, then by (61) and [14, Theorem 1.51], we get $\pi^*(z) \equiv 0$, which is a contradiction.

Case 2. If $\beta = \alpha$ and $\beta \neq 2\alpha$, then the equation (61) may written as

$$(d(B(z)+C(z)+\pi(z)A(z))-\pi^*(z))\,e^{\alpha z^\rho}+\pi(z)(B(z)+C(z))e^{2\alpha z^\rho}=d^*-d^2A(z).$$

By this and [14, Theorem 1.51], we get $\pi^*(z) = \frac{d^*}{d}\pi(z)$. Substituting this into (60) and combining the result with (55), we obtain

$$f(z)g(f)(z) = F(z) = d^* + \frac{d^*}{d}\pi(z)e^{\alpha z^\rho} = \frac{d^*}{d}f(z).$$

Thus, $g(f)(z) = \frac{d^*}{d}$, contradicting the assumption that $g(f)$ is non-constant.

Case 3. If $\beta \neq \alpha$ and $\beta = 2\alpha$, then the equation (61) may written as

$$d(B(z)+C(z)+\pi(z)A(z))e^{\alpha z^\rho}+(\pi(z)(B(z)+C(z)) - \pi^*(z))\,e^{2\alpha z^\rho}=d^*-d^2A(z).$$

By this and [14, Theorem 1.51], we get $\pi^*(z) = -\frac{d^*}{d^2}\pi^2(z)$. Substituting this into (60) and combining the result with (55) , we get

$$F(z) = d^* - \frac{d^*}{d^2}(f(z) - d)^2.$$

Clearly $d^* \neq 0$ as we would have $g(f) \equiv 0$ otherwise, which is a contradiction. Hence

$$\frac{d^* - F(z)}{(d - f(z))^2} = \frac{d^*}{d^2}. \tag{62}$$

We prove now the second identity, from (62) we get

$$\sum_{j \in I_1} b_j(z) \frac{f(z + c_j) - d}{f(z) - d} + \sum_{j \in I_2} b_j(z) \frac{f^{(k_j)}(z + c_j)}{f(z) - d} + \frac{d^*}{d^2} = \frac{\frac{d^*}{d} - d \sum_{j \in I_1} b_j(z)}{f(z) - d}.$$

Suppose contrary to the claim that $d^* - d^2 \sum_{j \in I_1} b_j(z) \not\equiv 0$, then from the above equation, we deduce

$$m\left(r, \frac{1}{f - d}\right) = O(r^{\rho - 1 + \varepsilon}) + S_\lambda(r, f).$$

By this and since d is a Borel exceptional value of f, we obtain

$$T\left(r, \frac{1}{f - d}\right) = O(r^{\rho - 1 + \varepsilon}) + S_\lambda(r, f),$$

which is a contradiction.

(ii) Suppose that $d \neq 0$ and

$$\sum_{j=1, k_j = 0}^{k} b_j(z) \equiv 0.$$

Suppose that F has a finite Borel exceptional value d^*. By the same proof as in (i), we obtain (61) as

$$d(B(z) + C(z))e^{\alpha z^\rho} + \pi(z)(B(z) + C(z))e^{2\alpha z^\rho} - \pi^*(z)e^{\beta z^\rho} = d^*.$$

If $\beta \neq \alpha$ and $\beta \neq 2\alpha$, $\beta = \alpha$ and $\beta \neq 2\alpha$ or $\beta \neq \alpha$ and $\beta = 2\alpha$, then by using [14, Theorem 1.51], we get $\pi^*(z) \equiv 0$ in all three cases, which is a contradiction.

(iii) Suppose that $d = 0$ is the Borel exceptional value of f. Using the same method as above, we obtain (59) with $d = 0$:

$$F(z) = \pi(z)(B(z) + C(z))e^{2\alpha z^\rho}.$$

Since $\rho(F) = \rho$, then $\pi(z)(B(z) + C(z)) \not\equiv 0$ and since $\rho(\pi(B + C)) < \rho$, we deduce that $d = 0$ is a Borel exceptional value of f. □

Proof of Theorem 3.8. We shall prove this theorem by contradiction. Suppose contrary to our assertion that $\lambda(F - a) < \rho$, then ρ is an integer ≥ 1.

If first $a = 0$, applying the principle of contraposition on the part (1) of Theorem 3.1, we get $\lambda_{(2}(f) \leq \lambda(f) < \rho$, which is a contradiction.

Suppose next that $a \neq 0$, then $F(z)$ can be written as the form

$$F(z) = f^n(z)g(f)(z) - a = \tau(z)e^{Q(z)}, \tag{63}$$

where $\tau(z)$ is a λ-small function of f and $Q(z)$ is a polynomial of degree $\rho \geq 1$. Differentiating (63) and eliminating $e^{Q(z)}$ yields

$$\frac{F'(z)}{F(z)} = \left(\frac{\tau'(z)}{\tau(z)} + Q'(z)\right)\left(1 - \frac{a}{F(z)}\right).$$

Clearly $\frac{\tau'(z)}{\tau(z)} + Q'(z) \not\equiv 0$. Indeed, if not, then $F'(z) \equiv 0$ which contradicts the fact $\rho(F) = \rho$. Since $\lambda_{(2}(f) = \rho \geq 1$, then there exists a multiple zero z_0 of f that is not a pole of the coefficients of $g(f)$ and such that $\frac{\tau'(z_0)}{\tau(z_0)} + Q'(z_0) \neq 0$. From the above equation, we observe that z_0 is a simple pole of $\frac{F'}{F}$ and a pole of multiplicity at least 2 of $\frac{a}{F}$, which is a contradiction. □

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