

## ON SUBCLASSES OF FUNCTIONS WITH BOUNDARY AND RADIUS ROTATIONS ASSOCIATED WITH CRESCENT DOMAINS

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**ABSTRACT.** The present work is aimed at presenting some characteristic properties of functions that map open unit disk onto a lune in the right half plane. Furthermore, we introduce subclasses of functions with boundary and radius rotations which are related to crescent regions. Some useful results, which include coefficient inequalities and some subordination properties associated with these subclasses are derived. Consequently, related problems concerning these classes are also studied.

### 1. Introduction

Let  $H(E)$  be a linear space of all analytic functions defined in the open unit disc  $E = \{z: |z| < 1\}$ . Given  $a \in \mathbb{C}, n \in \mathbb{N}$ , let

$$(1) \quad H(a, n) = \{f \in H(E): f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots\},$$

and denote by  $A$  and  $P$  respectively, the special classes  $H(0, 1)$  and  $H(1, 1)$  whose members are of the form

$$(2) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

and

$$(3) \quad p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$$

with  $p(0) = 1$  and  $\operatorname{Re} p(z) > 0$ . We say a function  $f \in \mathcal{A}$  is subordinate to  $g \in \mathcal{A}$  (written as  $f(z) \prec g(z)$ ) if there exists a function  $w(z)$  with  $|w(z)| < 1$  and  $w(0) = 0$  such that  $f(z) = g(w(z))$ . In addition, if  $g(z)$  is univalent in  $E$ , then  $f(0) = g(0)$  and  $f(E) \subset g(E)$  [2].

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Very recently, K. Rainaa and J. Sokół [7] introduced the class  $P_c$  of functions  $p \in H(1, 1)$  satisfying the inequality

$$(4) \quad |p^2(z) - 1| < 2|p(z)| \quad (z \in E).$$

This class of functions maps  $E$  on to a crescent-like domain (see Figure 1) and are also subordinate to  $z + \sqrt{1 + z^2}$ .

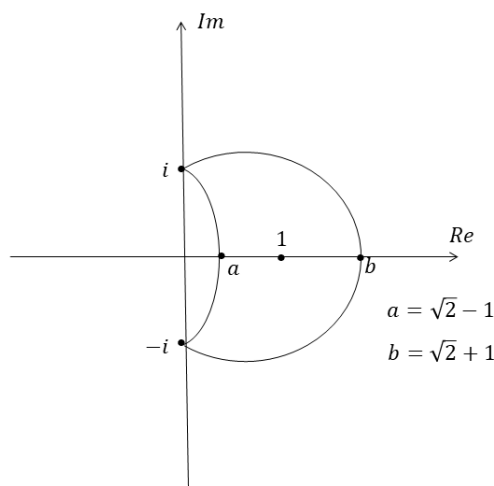


FIGURE 1.

Associated with  $P_c$ , are the classes  $\Delta_s$  and  $\Delta_c$  which were intensively investigated in [1, 7, 8, 10] and defined as follows:

$$(5) \quad \Delta_s = \left\{ f \in H(0, 1) : \frac{zf'(z)}{f(z)} \prec z + \sqrt{1 + z^2} \quad (z \in E) \right\}$$

and

$$(6) \quad \Delta_c = \left\{ f \in H(0, 1) : \frac{(zf'(z))'}{f'(z)} \prec z + \sqrt{1 + z^2} \quad (z \in E) \right\}.$$

Functions in these classes have the integral representations

$$(7) \quad f_s(z) = z \exp \int_0^z \frac{p(t) - 1}{t} dt \quad \text{and} \quad f_c(z) = \int_0^z \left( \exp \int_0^u \frac{p(t) - 1}{t} dt \right) du,$$

where  $p \in P_c$ . But for  $p(z) = p_0(z) = z + \sqrt{1 + z^2}$ ,

$$(8) \quad f_{s_0}(z) = \frac{2z \exp(\sqrt{1 + z^2} + z - 1)}{\sqrt{1 + z^2} + 1} \quad \text{and} \quad f_{c_0}(z) = \int_0^z \frac{2 \exp(\sqrt{1 + t^2} + t - 1)}{\sqrt{1 + t^2} + 1} dt$$

are the extremal functions for these classes, respectively.

The concept of the class  $V_k$  of functions  $f(z)$  for which  $f(E)$  has a boundary rotation of at most  $k\pi$  ( $k \geq 2$ ) was first introduced by Loewner [3] and later extensively studied by Paatero [5]. Pinchuk later defined the class  $P_k$  (see [6] for formal definition) and used it to give the characterization of the classes  $V_k$  and  $U_k$  (class of functions with bounded radius rotation bounded by  $k\pi$ ).

It is worth mentioning other function whose geometric property is similar to that of  $q(z) = z + \sqrt{1+z^2}$ . For instance,  $q_1(z) = \sqrt{1+z}$  ( $z \in E$ ) maps  $E$  on to the interior of the right half of the lemniscate of Bernoulli [11, 12].

Motivated by earlier work, we introduce classes  $P_{c_k}$ ,  $\Delta_{s_k}$  and  $\Delta_{c_k}$  ( $k \geq 2$ ) as follows:

Let  $p \in H(1, 1)$ . Then  $p(z)$  is said to be in the class  $P_{c_k}$  if there exist  $p_1, p_2 \in P_c$  such that

$$(9) \quad p(z) = \frac{1}{2} \left( \frac{k}{2} + 1 \right) p_1(z) - \frac{1}{2} \left( \frac{k}{2} - 1 \right) p_2(z) \quad (k \geq 2),$$

$$(10) \quad \Delta_{s_k} = \left\{ f \in H(0, 1) : \frac{zf'(z)}{f(z)} \in P_{c_k} \right\},$$

and

$$(11) \quad \Delta_{c_k} = \left\{ f \in H(0, 1) : zf'(z) \in \Delta_{s_k} \right\}.$$

We note that

$$P_{c_k} \subset P_k, \Delta_{s_k} \subset U_k \text{ and } \Delta_{c_k} \subset V_k.$$

The objective of this work is to study the classes  $P_c, P_{c_k}, \Delta_{s_k}$  and  $\Delta_{c_k}$  for their respective properties. Related details and definitions of these classes are mentioned in Section 1. In Section 2, we give some auxiliary lemmas which are used to obtain our results. In Section 3, results relating to the properties of the classes  $P_c$  and  $P_{c_k}$  are given. Some lemmas in [4] are used to obtain certain subordination results in Section 3. Finally, in Section 4, we give a result relating to coefficient estimate for functions belonging to the classes  $\Delta_{s_k}$  and  $\Delta_{c_k}$ , and other related results are also obtained. From the idea presented herein, a few known results are deduced as corollaries. We state some of them for the convenience of the reader(s).

### 2. Auxiliary lemmas

**Lemma 2.1** ([2]). *Let  $h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, G(z) = 1 + \sum_{n=1}^{\infty} d_n z^n \in H(1, 1)$ . If  $h(z) \prec G(z)$  and  $G(E)$  is convex, then  $|c_n| \leq |d_1|$  for all  $n \geq 1$ .*

**Lemma 2.2** ([9]). *The function  $q(z) = z + \sqrt{1+z^2}$  is convex in the disc  $|z| < \frac{\sqrt{2}}{2}$ .*

**Lemma 2.3** ([4]). *Let  $q(z)$  be univalent in  $E$  and let  $\theta$  and  $\varphi$  be analytic in a domain  $D$  containing  $q(E)$  with  $\varphi(w) \neq 0$ , when  $w \in q(E)$ . Set  $Q(z) = zq'(z) \cdot \varphi(q(z))$ ,  $h(z) = \theta(q(z)) + Q(z)$ , and suppose that either*

- (ii)  $h(z)$  is convex, or  $Q(z)$  is starlike,
- (iii)

$$\operatorname{Re} \frac{zh'(z)}{Q(z)} = \operatorname{Re} \left( \frac{\theta'(q(z))}{\varphi(q(z))} + \frac{zQ'(z)}{Q(z)} \right) > 0.$$

If  $p(z)$  is analytic in  $E$  with  $p(0) = q(0)$ ,  $p(E) \subset D$  and

$$(12) \quad \theta(p(z)) + zp'(z)\varphi(p(z)) \prec \theta(q(z)) + zq'(z)\varphi(q(z)) = h(z),$$

then  $p(z) \prec q(z)$ , and  $q(z)$  is the best dominant in the sense that  $p \prec s \Rightarrow q \prec s$  for all  $s$ .

**Lemma 2.4** ([4]). Assume that  $Q$  is the set of analytic functions  $f(z)$  that are injective on  $\overline{E} \setminus U(f)$ , where

$$U(f) := \left\{ \xi: \xi \in \partial E \text{ and } \lim_{z \rightarrow \xi} f(z) = \infty \right\},$$

and such that  $f'(\xi) \neq 0$  ( $\xi \in \partial E \setminus U(f)$ ). Let  $\psi \in Q$  with  $\psi(0) = 1$  and let  $\phi(z) = 1 + c_1z + c_2z^2 + \dots$  be analytic in  $E$ , with  $\phi(z) \neq 1$ . If  $\phi(z)$  is not subordinate to  $\psi(z)$ , then there exist points  $z_0 \in E$ ,  $\xi_0 \in \partial E \setminus U(\psi)$ ,  $m > 1$  for which

- (i)  $\phi(|z| < |z_0|) \subset \psi(E)$ ,
- (ii)  $\phi(z_0) = \psi(\xi_0)$ ,
- (iii)  $z_0\phi'(z_0) = m\xi_0\psi'(\xi_0)$ .

### 3. Properties of the classes $P_c$ and $P_{c_k}$

**Theorem 3.1.** Let  $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \in P_c$ . Then for  $r < 1$ ,

- (i)  $|p(z)| \leq r + \sqrt{1+r^2}$ ,

- (ii)  $\frac{1}{|p(z)|} \leq -r + \sqrt{1+r^2}$ ,

- (iii)  $|zp'(z)| \leq \frac{r(r + \sqrt{1+r^2})}{\sqrt{1+r^2}}$ ,

- (iv)  $\left| \frac{zp'(z)}{p(z)} \right| \leq \frac{r}{\sqrt{1+r^2}}$ ,

and for  $r < \frac{\sqrt{2}}{2}$ ,

- (v)  $|p_n| \leq 1, n \geq 1$ ,

- (vi)  $\frac{1}{2\pi} \int_0^{2\pi} |p(z)|^2 d\theta \leq \frac{1}{1-r^2}, (z = re^{i\theta})$ .

The inequalities (i)-(iv) are sharp for the function  $q(z) = z + \sqrt{1 + z^2}$ .

*Proof.* Using subordination properties (see [2], Vol. I, p. 82, Theorem 5), the proofs of (i)-(iv) are direct. We next establish the proofs of (v) and (vi).

Since  $p(z) \in P_c$ , then  $1 + \sum_{n=1}^{\infty} p_n z^n \prec 1 + z + \frac{1}{2}z^2 + \dots$ . But  $z + \sqrt{1 + z^2}$  is univalent and by Lemma 2.2, it is convex in the disk  $|z| < \frac{\sqrt{2}}{2}$ . Then (v) follows directly from Lemma 2.1.

Next, by Parseval's theorem and (v), (vi) follows easily. □

**Theorem 3.2.** Let  $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \in P_{c_k}$ . Then for  $r < \frac{\sqrt{2}}{2}$ ,

(i)

$$|p_n| \leq \frac{k}{2}, \quad n \geq 1,$$

(ii)

$$(13) \quad \frac{1}{2\pi} \int_0^{2\pi} |p(z)|^2 d\theta \leq \frac{1 + \left(\frac{k^2}{4} - 1\right)r^2}{1 - r^2}, \quad (z = re^{i\theta}) \text{ and}$$

(iii)  $P_{c_k} \subset P_c$  for all  $z$  in the disk

$$(14) \quad |z| \leq \min \left\{ \frac{\sqrt{2}}{2}, \frac{2}{\sqrt{k^2 - 4}} \right\}, \quad k \geq 3.$$

The function

$$(15) \quad p(z) = \frac{k + 2}{4} \left( -z + \sqrt{1 + z^2} \right) - \frac{k - 2}{4} \left( z + \sqrt{1 + z^2} \right)$$

shows that (14) cannot be improved.

*Proof.* Since  $p(z) \in P_{c_k}$ , there exist  $h_1(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ ,  $h_2(z) = 1 + \sum_{n=1}^{\infty} d_n z^n \in P_c$  such that

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n = \frac{k + 2}{4} \left( 1 + \sum_{n=1}^{\infty} c_n z^n \right) - \frac{k - 2}{4} \left( 1 + \sum_{n=1}^{\infty} d_n z^n \right).$$

On comparing the coefficients of the above equation and applying Theorem 3.1(v), we obtain (i). For (ii), we apply Parseval's identity and (i).

Applying a result (see [9]) for the bound of functions in  $P_c$  and using Theorem 3.1(i), we have that

$$(16) \quad \operatorname{Re} p(z) \geq \frac{1}{2} \left( 2\sqrt{1 + r^2} - kr \right).$$

The right side of (16) is positive provided

$$2\sqrt{1 + r^2} - kr > 0 \Rightarrow 4 - (k^2 - 4)r^2 > 0.$$

Let  $T(r) = 4 - (k^2 - 4)r^2$ . Then  $T(0) = 4 > 0$  and  $T(1) = 8 - k^2 < 0$ , since  $k \geq 3$ . Thus,  $T(r)$  has a solution in  $[0, 1]$ .

The result is sharp for the function given by (15) in the sense that

$$p(z) = \frac{1}{2} \left( 2\sqrt{1+z^2} - kz \right),$$

which is 0 for  $z = \frac{2}{\sqrt{k^2-4}}$ .  $\square$

#### 4. Subordination results

**Theorem 4.1.** *If  $p(z) \in H(1,1)$  satisfies*

$$(17) \quad \operatorname{Re} \left[ p(z) + \left( \frac{zp'(z)}{p(z)} \right)^2 \right] \leq \frac{1}{2} \quad (z \in E),$$

then  $p(z) \prec z + \sqrt{1+z^2} := q(z)$ ,  $z \in E$ .

*Proof.* We need to show that  $p(z) \prec q(z)$ . Suppose on the contrary that  $p(z) \not\prec q(z)$ . Then in view of Lemma 2.4, there exist points  $z_0$  and  $\xi_0$  for which

$$p(z_0) = q(\xi_0), \quad p(|z| < |z_0|) \subset q(E), \quad z_0 p'(z_0) = m \xi_0 q'(\xi_0) \quad \text{with } m \geq 1.$$

Thus

$$\operatorname{Re} \left[ p(z_0) + \left( \frac{z_0 p'(z_0)}{p(z_0)} \right)^2 \right] > \operatorname{Re} \left( \frac{m \xi_0 q'(\xi_0)}{q(\xi_0)} \right)^2,$$

since  $\operatorname{Re} q(\xi_0) > 0$  (see [9]). But

$$\begin{aligned} \left( \frac{z_0 q'(\xi_0)}{q(\xi_0)} \right)^2 &= \frac{\xi_0^2}{1 + \xi_0^2} \\ &= \frac{1}{1 + e^{-i2t_0}} \\ &= \frac{1}{2} + i \frac{\tan(\arg \xi_0)}{2}. \end{aligned}$$

Therefore,

$$\operatorname{Re} \left[ p(z_0) + \left( \frac{z_0 p'(z_0)}{p(z_0)} \right)^2 \right] > \frac{m}{2} > \frac{1}{2},$$

which contradicts (17). Hence,  $p(z) \prec q(z)$ .  $\square$

If we take  $p(z) = \frac{zf'(z)}{f(z)}$  and  $p(z) = \frac{(zf'(z))'}{f'(z)}$  in Theorem 4.1, we have the following results.

**Corollary 4.2.** *If  $f \in A$  and*

$$\operatorname{Re} \left[ \frac{zf'(z)}{f(z)} + \left( \frac{(zf'(z))'}{f'(z)} - \frac{zf'(z)}{f(z)} \right)^2 \right] < \frac{1}{2}, \quad z \in E,$$

then  $f \in \Delta_s$ .

**Corollary 4.3.** *If  $f \in A$  and*

$$\operatorname{Re} \left[ \frac{(zf'(z))'}{f'(z)} + \left( 1 + \frac{z(zf'(z))''}{(zf'(z))'} - \frac{(zf'(z))'}{f'(z)} \right)^2 \right] < \frac{1}{2}, \quad z \in E,$$

*then  $f \in \Delta_c$ .*

**Theorem 4.4.** *If  $p(z) \in H(1, 1)$  satisfies*

$$(18) \quad p(z) + z \left( \frac{p'(z)}{p(z)} \right)^2 \prec z + \sqrt{1+z^2} + \frac{z}{1+z^2} \quad (z \in E),$$

*then  $p(z) \prec z + \sqrt{1+z^2}$ ,  $z \in E$ .*

*Proof.* Let  $q(z) = z + \sqrt{1+z^2}$ , and  $\theta(w) = w$ ,  $\varphi(w) = \frac{w'(z)}{w^2(z)}$  be analytic functions in the domain  $D$  such that  $q(E) \subset D$ . Then

$$Q(z) = \varphi(q(z)) \cdot zq'(z) = \frac{zq'(z)}{q^2(z)} \cdot zq'(z) = \frac{z^2}{1+z^2}$$

and

$$\frac{zQ'(z)}{Q(z)} = 1 + \frac{2z^2}{1+z^2},$$

so that

$$\operatorname{Re} \frac{zQ'(z)}{Q(z)} = 2 > 0.$$

Furthermore,

$$h(z) = z + \sqrt{1+z^2} + \frac{z}{1+z^2}$$

and

$$\begin{aligned} \operatorname{Re} \frac{\theta'(q(z))}{\varphi(q(z))} &= \operatorname{Re} q^2(z) \\ &= 2\operatorname{Re}(z^2 + \sqrt{1+z^2}) + 1 \\ &> 1. \end{aligned}$$

Thus, the conditions (i) and (ii) of Lemma 2.3 are satisfied. By Lemma 2.3, the subordination relation (18) implies the subordination (12).  $\square$

### 5. Some results on classes $\Delta_{s_k}$ and $\Delta_{c_k}$

**Theorem 5.1.** *Let  $f \in A$ . Then  $f \in \Delta_{s_k}$  if and only if there exist  $f_1, f_2 \in \Delta_s$  such that*

$$(19) \quad f(z) = \frac{(f_1(z))^{\frac{k+2}{4}}}{(f_2(z))^{\frac{k-2}{4}}}, \quad k \geq 2.$$

*Proof.* The proof is straightforward from the definition of  $\Delta_{s_k}$ .  $\square$

**Theorem 5.2.** *If  $f \in \Delta_{s_k}$ , then*

$$(20) \quad \sum_{n=2}^{\infty} \left(4n^2 - k^2(3 + 2\sqrt{2})\right) |a_n|^2 \leq k^2(3 + 2\sqrt{2}) - 4.$$

*Proof.* From the representation (19), we have for  $f_1, f_2 \in \Delta_s$ ,

$$(21) \quad \frac{zf'(z)}{f(z)} = \frac{k+2}{4} \cdot \frac{zf'_1(z)}{f_1(z)} - \frac{k-2}{4} \cdot \frac{zf'_2(z)}{f_2(z)}.$$

But

$$f_i(z) \in \Delta_{s_k} \Rightarrow \frac{zf'_i(z)}{f_i(z)} \prec z + \sqrt{1+z^2}, \quad z \in E, \quad i = 1, 2.$$

Therefore, there exist Schwartz functions  $w_i(z)$ ,  $i = 1, 2$  such that

$$\frac{zf'_i(z)}{f_i(z)} = w_i + \sqrt{1+w_i^2}.$$

Thus, (21) can be written as

$$(22) \quad f(z) = \frac{4zf'(z)}{(k+2)(w_1(z) + \sqrt{1+w_1^2(z)}) - (k-2)(w_2(z) + \sqrt{1+w_2^2(z)})}.$$

By Parseval's identity, it follows that

$$\begin{aligned} & 2\pi \sum_{n=1}^{\infty} |a_n|^2 r^{2n} \\ &= \int_0^{2\pi} |f(z)|^2 d\theta, \quad z = re^{i\theta} \\ &= 16 \int_0^{2\pi} \left| \frac{zf'(z)}{(k+2)(w_1(z) + \sqrt{1+w_1^2(z)}) - (k-2)(w_2(z) + \sqrt{1+w_2^2(z)})} \right|^2 d\theta \\ &\geq 16 \int_0^{2\pi} \frac{|zf'(z)|^2}{\left[ (k+2)(1+\sqrt{2}) + (k-2)(1+\sqrt{2}) \right]^2} d\theta \\ &= 4 \int_0^{2\pi} \frac{|zf'(z)|^2 d\theta}{k^2(3+2\sqrt{2})} \\ &= \frac{8}{k^2(3+2\sqrt{2})} \pi \sum_{n=1}^{\infty} n^2 |a_n|^2 r^{2n}. \end{aligned}$$

This implies that

$$\sum_{n=1}^{\infty} \left(4n^2 - k^2(3 + 2\sqrt{2})\right) |a_n|^2 r^{2n} \leq 0, \quad a_1 = 1.$$

As  $r \rightarrow 1^-$ , we obtain the required result.  $\square$



**Corollary 5.3.** *If  $f \in \Delta_{c_k}$ , then*

$$(23) \quad \sum_{n=2}^{\infty} n^2 (4n^2 - k^2(3 + 2\sqrt{2})) |a_n|^2 \leq k^2(3 + 2\sqrt{2}) - 4.$$

**Corollary 5.4.** *If  $f \in \Delta_{s_k}$ , then*

$$|a_n| \leq \sqrt{\frac{k^2(3 + 2\sqrt{2}) - 4}{4n^2 - k^2(3 + 2\sqrt{2})}}, \quad n > k.$$

**Corollary 5.5.** *If  $f \in \Delta_{c_k}$ , then*

$$|a_n| \leq \frac{1}{n} \sqrt{\frac{k^2(3 + 2\sqrt{2}) - 4}{4n^2 - k^2(3 + 2\sqrt{2})}}, \quad n > k.$$

For  $k = 2$ , we obtain the results of P. Sharma et al. (see [10, Sec. 3]).

**Theorem 5.6.** *If  $f \in \Delta_{s_k}$ , then for  $k \geq 2$ ,*

$$(24) \quad \frac{2r \exp\left(\sqrt{1+r^2} - \frac{kr}{2} - 1\right)}{1 + \sqrt{1+r^2}} \leq |f(z)| \leq \frac{2r \exp\left(\sqrt{1+r^2} + \frac{kr}{2} - 1\right)}{1 + \sqrt{1+r^2}}.$$

*Proof.* The proof is direct from the representation (19) and the distortion theorem for the functions in the class  $\Delta_s$  given in [10]. □

**Corollary 5.7.** *If  $f \in \Delta_{c_k}$ , then*

$$\int_0^r \frac{2 \exp\left(\sqrt{1+t^2} - \frac{kt}{2} - 1\right)}{1 + \sqrt{1+t^2}} dt \leq |f(z)| \leq \int_0^r \frac{2 \exp\left(\sqrt{1+t^2} + \frac{kt}{2} - 1\right)}{1 + \sqrt{1+t^2}} dt.$$

**Corollary 5.8.** *If  $f \in \Delta_{s_k}$ , then for  $r \rightarrow 1^-$ ,*

$$\left| \frac{z}{f(z)} \right| \leq \frac{1 + \sqrt{2}}{2 \exp\left(\sqrt{2} - 1 - \frac{k}{2}\right)}.$$

**Theorem 5.9.** *If  $f \in \Delta_{s_k}$  and maps  $E$  onto a domain  $\mathcal{D}$  of finite area  $\mathcal{B}$ , then  $G(z) = \sqrt{f(z^2)}$  maps  $E$  onto a domain  $D'$  of area*

$$\mathcal{A} \leq \frac{1 + \sqrt{2}}{2 \exp\left(\sqrt{2} - 1 - \frac{k}{2}\right)} \mathcal{B}, \quad k \geq 2.$$

*Proof.* By definition, we have

$$\begin{aligned} \mathcal{A} &= \int_0^{2\pi} \int_0^1 |G'(t)|^2 R dR d\psi \quad (t = Re^{i\psi}) \\ &= \int_0^{2\pi} \int_0^1 \left| \frac{t f'(t^2)}{\sqrt{f(t^2)}} \right|^2 R dR d\psi. \end{aligned}$$

Letting  $z = t^2$  ( $z = \rho e^{i\theta}$ ), we obtain

$$\begin{aligned} \mathcal{A} &= \frac{1}{4} \int_0^{4\pi} \int_0^1 \left| \frac{z}{f(z)} \right| \left| f'(z) \right|^2 d\rho d\theta \\ &\leq \frac{1 + \sqrt{2}}{2 \exp\left(\sqrt{2} - 1 - \frac{k}{2}\right)} \int_0^{2\pi} \int_0^1 \left| f'(z) \right|^2 d\rho d\theta, \end{aligned}$$

where we have used Corollary 5.8. Therefore,

$$\begin{aligned} \mathcal{A} &\leq \frac{(1 + \sqrt{2})\pi}{2 \exp\left(\sqrt{2} - 1 - \frac{k}{2}\right)} \int_0^1 \sum_{n=1}^{\infty} n^2 |a_n|^2 \rho^{2n-2} d\rho \\ &= \frac{(1 + \sqrt{2})\pi}{2 \exp\left(\sqrt{2} - 1 - \frac{k}{2}\right)} \sum_{n=1}^{\infty} \frac{n^2}{2n-1} |a_n|^2 \\ &\leq \frac{(1 + \sqrt{2})\pi}{2 \exp\left(\sqrt{2} - 1 - \frac{k}{2}\right)} \sum_{n=1}^{\infty} n |a_n|^2 \\ &= \frac{(1 + \sqrt{2})}{2 \exp\left(\sqrt{2} - 1 - \frac{k}{2}\right)} \mathcal{B}. \end{aligned}$$

□

## 6. Conclusion

In this article, we established some properties of  $P_c$  and  $P_{c_k}$ , and a few subordination conditions concerning the class  $P_c$ . In addition, we proved the coefficient inequalities associated with  $\Delta_{s_k}$  and  $\Delta_{c_k}$ , and give the importance of our work by connecting them to the existing literature. We hope that our work will be source of motivation for researchers in this direction. It is worthy of note that the idea presented here can be improved when extended to Quantum Calculus and related problems.

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