

CLEANNES OF SKEW GENERALIZED POWER SERIES RINGS

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ABSTRACT. A skew generalized power series ring $R[[S, \omega]]$ consists of all functions from a strictly ordered monoid S to a ring R whose support contains neither infinite descending chains nor infinite antichains, with pointwise addition, and with multiplication given by convolution twisted by an action ω of the monoid S on the ring R . Special cases of the skew generalized power series ring construction are skew polynomial rings, skew Laurent polynomial rings, skew power series rings, skew Laurent series rings, skew monoid rings, skew group rings, skew Mal'cev-Neumann series rings, the “untwisted” versions of all of these, and generalized power series rings. In this paper we obtain some necessary conditions on R , S and ω such that the skew generalized power series ring $R[[S, \omega]]$ is (uniquely) clean. As particular cases of our general results we obtain new theorems on skew Mal'cev-Neumann series rings, skew Laurent series rings, and generalized power series rings.

1. Introduction

Given a ring R , a strictly ordered monoid (S, \leq) and a monoid homomorphism $\omega : S \rightarrow \text{End}(R)$, one can construct the skew generalized power series ring $R[[S, \omega]]$ (see Section 2 for details). Skew generalized power series rings are a common generalization of skew polynomial rings, skew power series rings, skew Laurent polynomial rings, skew Laurent series rings, skew monoid rings, skew group rings, skew Mal'cev-Neumann series rings, and of course the “untwisted” versions of all of these. Hence any result on skew generalized power series rings has its counterpart for each of these particular ring extensions, and these counterparts follow immediately from a single proof. This property makes skew generalized power series rings a useful tool for unifying results on the ring extensions listed above; such an approach was applied, e.g., in [18], [19], [20], [22], [25], [31], [32], [33], [34], [36], [39] and [42].

An element a of a ring R is called (*uniquely*) *clean* if it can be expressed (uniquely) as the sum of an idempotent and a unit in R . The ring R is called a

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(*uniquely*) *clean ring* if every element of R is (uniquely) clean. It follows that every local ring is clean. More generally, Camillo and Yu [1, Theorem 9] showed that a ring is semiperfect if and only if it is clean and contains no infinite family of orthogonal idempotents. In addition, they showed that every unit-regular ring is clean [1, Theorem 5]. Clean rings were first studied by Nicholson [27] in connection with exchange rings and lifting of idempotents.

A ring R is *Boolean* in case every element in R is idempotent. Clearly, the class of uniquely clean rings is a natural generalization of that of Boolean rings. In fact, a ring R is uniquely clean if and only if $R/J(R)$ is Boolean and idempotents lift uniquely modulo $J(R)$ (cf. [29, Theorem 20]), where $J(R)$ is the Jacobson radical of R . Studies of (uniquely) clean of some ring extensions was considered in many papers (see [1], [5], [9], [27], [28], [29], [38], and [46]).

Because of the importance of (uniquely) clean rings in general theory of rings, it is natural to ask under what conditions on a ring R , a strictly ordered monoid (S, \leq) and a monoid homomorphism $\omega : S \rightarrow \text{End}(R)$, the skew generalized power series ring $R[[S, \omega]]$ is (uniquely) clean. In this paper we obtain some necessary conditions on R , S and ω such that the skew generalized power series ring $R[[S, \omega]]$ is (uniquely) clean.

The paper is organized as follows. In Section 2, we recall the skew generalized power series ring construction and show that (skew) polynomial rings, (skew) Laurent polynomial rings, (skew) power series rings, (skew) Laurent series rings, (skew) monoid rings and the Mal'cev-Neumann construction are special cases of the construction. In Section 3, we study when the skew generalized power series ring $R[[S, \omega]]$ is (uniquely) clean. In particular, it is proved that, under suitable conditions, for a 2-primal ring R , the skew generalized power series ring $R[[S, \omega]]$ is clean if and only if R is semiregular with $J(R)$ nil, where (S, \leq) is a totally ordered group or (S, \leq) is an abelian torsion-free group such that \leq is subtotal and $\omega : S \rightarrow \text{Aut}(R)$ a group homomorphism (see Theorem 3.7). As a consequence of the main result of this paper (Corollary 3.9), we obtain some characterizations a skew Laurent series ring $R[[x, x^{-1}; \alpha]]$ to be clean. In particular, we will show that, $R[[x, x^{-1}; \alpha]]$ is clean if and only if R is semiregular with $J(R)$ nil, where R is an (α, δ) -compatible 2-primal ring and it is either right Goldie, or has right Krull dimension, or is a ring with ACC on both right and left annihilators. The results were motivated by [46, Theorem 2.5] of Zhou and Ziembowski. Finally, we prove that $R[[S, \omega]]$ is uniquely clean if and only if R is uniquely clean and ω_s is idempotent-stabilizing for all $s \in S$, where (S, \leq) is a positively strictly ordered monoid, and $\omega : S \rightarrow \text{End}(R)$ a monoid homomorphism (Theorem 3.12). As an application, we provide (apparently) new examples of (uniquely) clean rings.

Throughout this paper all monoids and rings are with identity element that is inherited by submonoids and subrings and preserved under homomorphisms, but neither monoids nor rings are assumed to be commutative. We will denote by $\text{End}(R)$ the monoid of ring endomorphisms of R , and by $\text{Aut}(R)$ the group of ring automorphisms of R . If S is a monoid or a ring, then the group of

invertible elements of S is denoted by $U(S)$. When we consider an ordering relation \leq on a set S , then the word “order” means a partial ordering unless otherwise stated. The order \leq is *total* (respectively *trivial*) if any two different elements of S are comparable (respectively incomparable) with respect to \leq . We will use the symbol 1 to denote the identity elements of the monoid S , the ring R , and the ring $R[[S, \omega]]$, as well as the trivial monoid homomorphism $1 : S \rightarrow \text{End}(R)$ that sends every element of S to the identity endomorphism. Also we use \mathbb{Z} , \mathbb{N} , \mathbb{Q} and \mathbb{R} for the integers, positive integers, rational numbers and the field of real numbers, respectively. For a ring R , we denote by $J(R)$ the Jacobson radical of R . The prime radical of a ring R and the set of all nilpotent elements in R are denoted by $\mathbf{P}(R)$ and $\text{nil}(R)$, respectively.

2. Preliminaries

A partially ordered set (S, \leq) is called *artinian* if every strictly decreasing sequence of elements of S is finite, and (S, \leq) is called *narrow* if every subset of pairwise order-incomparable elements of S is finite. Thus, (S, \leq) is artinian and narrow if and only if every nonempty subset of S has at least one but only a finite number of minimal elements. An *ordered monoid* is a pair (S, \leq) consisting of a monoid S and an order \leq on S such that for all $a, b, c \in S$, $a \leq b$ implies $ca \leq cb$ and $ac \leq bc$. An ordered monoid (S, \leq) is said to be *strictly ordered* if for all $a, b, c \in S$, $a < b$ implies $ca < cb$ and $ac < bc$.

For a strictly ordered monoid S and a ring R , Ribenboim [42] defined the ring of generalized power series $R[[S]]$ consisting of all maps from S to R whose support is artinian and narrow with the pointwise addition and the convolution multiplication. This construction provided interesting examples of rings (e.g., Elliott and Ribenboim, [4]; Ribenboim, [40], [41]) and it was extensively studied by many authors.

In [23], Mazurek and Ziemkowski, introduced a “twisted” version of the Ribenboim construction and studied when it produces a von Neumann regular ring. Now we recall the construction of the skew generalized power series ring introduced in [23]. Let R be a ring, (S, \leq) a strictly ordered monoid, and $\omega : S \rightarrow \text{End}(R)$ a monoid homomorphism. For $s \in S$, let ω_s denote the image of s under ω , that is $\omega_s = \omega(s)$. Let A be the set of all functions $f : S \rightarrow R$ such that the support $\text{supp}(f) = \{s \in S : f(s) \neq 0\}$ is artinian and narrow. Then for any $s \in S$ and $f, g \in A$ the set

$$X_s(f, g) = \{(x, y) \in \text{supp}(f) \times \text{supp}(g) : s = xy\}$$

is finite. Thus one can define the product $fg : S \rightarrow R$ of $f, g \in A$ as follows:

$$fg(s) = \sum_{(u, v) \in X_s(f, g)} f(u)\omega_u(g(v)),$$

(by convention, a sum over the empty set is 0). With pointwise addition and multiplication as defined above, A becomes a ring, called *the ring of skew generalized power series* with coefficients in R and exponents in S (one can

think of a map $f : S \rightarrow R$ as a formal series $\sum_{s \in S} r_s s$, where $r_s = f(s) \in R$ and denoted either by $R[[S^{\leq}, \omega]]$ or by $R[[S, \omega]]$ (see [18] and [23]).

To each $r \in R$ and $s \in S$, we associate elements $\mathbf{c}_r, \mathbf{e}_s \in R[[S, \omega]]$ defined by

$$\mathbf{c}_r(x) = \begin{cases} r & x = 1 \\ 0 & x \in S \setminus \{1\}, \end{cases} \quad \mathbf{e}_s(x) = \begin{cases} 1 & x = s \\ 0 & x \in S \setminus \{s\}. \end{cases}$$

It is clear that $r \mapsto \mathbf{c}_r$ is a ring embedding of R into $R[[S, \omega]]$ and $s \mapsto \mathbf{e}_s$ is a monoid embedding of S into the multiplicative monoid of the ring $R[[S, \omega]]$, and $\mathbf{e}_s \mathbf{c}_r = \mathbf{c}_{\omega_s(r)} \mathbf{e}_s$.

Below we quote from [19], how the classical constructions mentioned in Section 1 can be viewed as special cases of the skew generalized power series ring construction.

Let R be a ring and σ an endomorphism of R . Then for the additive monoid $S = \mathbb{N} \cup \{0\}$ of nonnegative integers, the map $\omega : S \rightarrow \text{End}(R)$ given by

$$(2.1) \quad \omega(n) = \sigma^n \text{ for any } n \in S,$$

is a monoid homomorphism. If furthermore σ is an automorphism of R , then (2.1) defines also a monoid homomorphism $\omega : S \rightarrow \text{Aut}(R)$ for $S = \mathbb{Z}$, the additive monoid of integers. We can consider two different orders on each of the monoids $\mathbb{N} \cup \{0\}$ and \mathbb{Z} : the trivial order and the natural linear order. In both cases these monoids are strictly ordered, and thus in each of the cases we can construct the skew generalized power series ring $R[[S, \omega]]$. As a result, we obtain the following extensions of the ring R :

- (1) If $S = \mathbb{N} \cup \{0\}$ and \leq is the trivial order, then the ring $R[[S, \omega]]$ is isomorphic to the skew polynomial ring $R[x, \sigma]$.
- (2) If $S = \mathbb{N} \cup \{0\}$ and \leq is the natural linear order, then $R[[S, \omega]]$ is isomorphic to the skew power series ring $R[[x; \sigma]]$.
- (3) If $S = \mathbb{Z}$ and \leq is the trivial order, and σ is an automorphism of R , then $R[[S, \omega]]$ is isomorphic to the skew Laurent polynomial ring $R[x, x^{-1}; \sigma]$.
- (4) If $S = \mathbb{Z}$ and \leq is the natural linear order, and σ is an automorphism of R , then $R[[S, \omega]]$ is isomorphic to the skew Laurent series ring $R[[x, x^{-1}; \sigma]]$.

By applying the above points (1)-(4) to the case where σ is the identity map of R , we can see that also the following ring extensions are special cases of the skew generalized power series ring construction: the ring of polynomials $R[x]$, the ring of power series $R[[x]]$, the ring of Laurent polynomials $R[x, x^{-1}]$, and the ring of Laurent series $R[[x, x^{-1}]]$.

Furthermore, any monoid S is a strictly ordered monoid with respect to the trivial order on S . Hence if R is a ring, S is a monoid and $\omega : S \rightarrow \text{End}(R)$ is a monoid homomorphism, then we can impose the trivial order on S and construct the skew generalized power series ring $R[[S, \omega]]$, which in this case will be denoted by $R[S, \omega]$. It is clear that the ring $R[S, \omega]$ is isomorphic to the classical skew monoid ring built from R and S using the action ω of S on

R . If ω is trivial, we write $R[S]$ instead of $R[S, \omega]$. Obviously the ring $R[S]$ is isomorphic to the ordinary monoid ring of S over R .

Also, the construction of skew generalized power series rings generalizes another classical ring constructions such as the Mal'cev-Neumann Laurent series rings $((S, \leq)$ a totally ordered group and trivial ω ; see [3, p. 528]), the Mal'cev-Neumann construction of twisted Laurent series rings $((S, \leq)$ a totally ordered group; see [10, p. 242]), and generalized power series rings $R[[S]]$ (trivial ω ; see [42, Section 4]), twisted generalized power series rings (see [14, Section 2], [23]).

Recall that a monoid S is said to be *torsion-free* if for any $n \in \mathbb{N}$ and $s, t \in S$, $s^n = t^n$ implies $s = t$. By use the terminology of [23], an order \leq on a monoid S is said to be *subtotal* (see [42]) if for any $s, t \in S$ there exists $n \in \mathbb{N}$ such that $s^n \leq t^n$ or $t^n \leq s^n$. Clearly, every total order \leq is subtotal. Furthermore, Example 3.8 in [25] show that in general a subtotal order need not be total. If (S, \cdot, \leq) is an abelian ordered group, then the order \leq is subtotal if and only if for every $s \in S$ there exists $n \in \mathbb{N}$ such that $s^n \geq 1$ or $s^n \leq 1$.

It is easy to see that if (S, \cdot, \leq) is an ordered torsion-free commutative monoid such that \leq is subtotal, then the binary relation \preceq on S defined by

$$s \preceq t \text{ if and only if } s^n \leq t^n \text{ for some } n \in \mathbb{N}$$

is a total order on S and (S, \cdot, \preceq) is an ordered monoid. The order \preceq will be called *the total order associated with \leq* . Clearly, $s \leq t$ implies $s \preceq t$ for any $s, t \in S$, and thus by [23, Proposition 1.1], if a subset T of S is artinian and narrow with respect to \leq , then T is well-ordered with respect to \preceq . Hence for any $f \in R[[S, \omega]] \setminus \{0\}$ there exists a smallest element s_0 of $\text{supp}(f)$ with respect to \preceq , which will be denoted by $\pi(f)$.

To study when the skew generalized power series ring $R[[S, \omega]]$ is a clean ring, we will need the following results on units which plays a key role in this paper.

Proposition 2.1 ([23, Proposition 2.2]). *Let R be a ring, (S, \leq) a strictly ordered monoid, $\omega : S \rightarrow \text{End}(R)$ a monoid homomorphism and $A = R[[S, \omega]]$. Let $f \in A$ and assume that there exists a smallest element s_0 in $\text{supp}(f)$. If $s_0 \in U(S)$ and $f(s_0) \in U(R)$, then $f \in U(A)$.*

Proposition 2.2 ([23, Lemma 2.5]). *Let R be a ring, (S, \cdot, \leq) an ordered abelian torsion-free group such that \leq is subtotal, $\omega : S \rightarrow \text{End}(R)$ a monoid homomorphism, $A = R[[S, \omega]]$ and \preceq the total order associated with \leq . If $f \in A \setminus \{0\}$ and for the smallest element s_0 of $\text{supp}(f)$ with respect to \preceq we have $f(s_0) \in U(R)$, then $f \in U(A)$.*

According to Krempa [8], an endomorphism α of a ring R is said to be *rigid* if $a\alpha(a) = 0$ implies $a = 0$, for $a \in R$. A ring R is said to be α -*rigid* if there exists a rigid endomorphism α of R .

In [6], the authors introduced α -compatible rings and studied their properties. An endomorphism α of a ring R is said to be *compatible* (and the ring R is called an α -*compatible ring*) if for each $a, b \in R$, $ab = 0$ if and only if

$a\alpha(b) = 0$. Basic properties of rigid and compatible endomorphisms, proved by Hashemi and Moussavi in [6, Lemmas 2.2 and 2.1] are summarized in the following lemma:

Lemma 2.3. *Let α be an endomorphism of a ring R . Then:*

- (i) *if α is compatible, then α is injective;*
- (ii) *α is compatible if and only if for all $a, b \in R$, $\alpha(a)b = 0 \Leftrightarrow ab = 0$;*
- (iii) *the following conditions are equivalent:*
 - (1) *α is rigid;*
 - (2) *α is compatible and R is reduced;*
 - (3) *for every $a \in R$, $\alpha(a)a = 0$ implies that $a = 0$.*

Marks, Mazurek and Ziembowski in [18] extended these notions as follows:

Definition 2.4 ([18]). Let R be a ring, (S, \leq) a strictly ordered monoid and $\omega : S \rightarrow \text{End}(R)$ a monoid homomorphism. The ring R is said to be S -compatible (resp. S -rigid) if ω_s is compatible (resp. rigid) for every $s \in S$.

A ring R is called 2-primal if $\mathbf{P}(R) = \text{nil}(R)$. It is obvious that commutative rings and reduced rings are 2-primal. Shin in [44, Proposition 1.11] showed that a ring R is 2-primal if and only if every minimal prime ideal P of R is completely prime (i.e., R/P is a domain). Also he proved that the minimal-prime spectrum of a 2-primal ring is a Hausdorff space with a basis of closed-and-open sets [44, Proposition 4.7] (for further information on 2-primal rings, see [15], [16], [37] and the references therein).

In the proof of the next results we will need the following theorem. Some characterizations of the Jacobson radical of skew generalized power series rings prove in [21].

Theorem 2.5 ([21]). *Let R be a 2-primal ring, (S, \leq) a nontrivial ordered group and $\omega : S \rightarrow \text{Aut}(R)$ a group homomorphism. Assume that \leq is total or (S, \leq) is an abelian torsion-free group such that \leq is subtotal, and $A = R[[S, \omega]]$. Suppose that R is S -compatible and $\mathbf{P}(R)$ is a nilpotent ideal of R . Then $J(A)$ is a nilpotent ideal of A and coincides with $\mathbf{P}(R)[[S, \omega]]$.*

3. Clean rings of skew generalized power series

In this section we will characterize the cleanness of a skew generalized power series ring $R[[S, \omega]]$ under various assumptions on R , S and ω . We also study when $R[[S, \omega]]$ is (uniquely) clean. We start with the following lemma, which plays a key role in the sequel.

Lemma 3.1. *Let R be a ring, (S, \leq) a strictly ordered monoid, and $\omega : S \rightarrow \text{End}(R)$ a monoid homomorphism. If R is S -compatible and e is an idempotent of R , then we have the following statements:*

- (1) $\omega_s(e) = e$ for all $s \in S$;
- (2) $c_e R[[S, \omega]] c_e = (e R e)[[S, \omega]]$.

Proof. (1) Since $e(1 - e) = (1 - e)e = 0$ and R is S -compatible, it concludes that $e = e\omega_s(e)$ and $\omega_s(e) = e\omega_s(e)$ for all $s \in S$. Hence $\omega_s(e) = e$ for all $s \in S$.

(2) By an easy computation and using part (1) the result proves. \square

Recall from [17] that an ordered monoid (S, \leq) is called *quasitotally ordered* (and that \leq is a quasitotal order on S) if \leq can be refined to an order \preceq with respect to which S is a strictly totally ordered monoid. The class of quasitotally ordered monoids is quite large and important. For example, this class includes the totally ordered monoids, submonoids of a free group, and torsion-free nilpotent groups (see [30, Lemma 13.1.6 and Corollary 13.2.8]). Also, every commutative, torsion-free, and cancellative ordered monoid is quasitotally ordered monoid (e.g. see [39, 3.3]).

Proposition 3.2. *Let R be a ring, (S, \leq) a quasitotally ordered monoid, and $\omega : S \rightarrow \text{End}(R)$ a monoid homomorphism. Assume that R is S -rigid. If f is an idempotent of $R[[S, \omega]]$, then $f(1)$ is an idempotent of R and $f = \mathbf{c}_{f(1)}$.*

Proof. Suppose that f is a nonzero idempotent of $R[[S, \omega]]$. By hypothesis, the order (S, \leq) can be refined to a strict total order \preceq on S . It implies that there exists $u_0 \in \text{supp}(f)$ such that u_0 is a minimal element of $\text{supp}(f)$ under the total order \preceq . For any $(u, v) \in X_{u_0^2}(f, f)$, $u_0 \preceq u$, $u_0 \preceq v$. If $u_0 \prec u$, since \preceq is a strict order, $u_0^2 \prec uu_0 \preceq uv = u_0^2$, a contradiction. Thus $u = u_0$. Similarly, $v = u_0$. Hence:

$$(3.1) \quad f^2(u_0^2) = \sum_{(u,v) \in X_{u_0^2}(f,f)} f(u)\omega_u(f(v)) = f(u_0)\omega_{u_0}(f(u_0)).$$

Assume that $u_0 \prec 1$. Since \preceq is a strict order, it follows that $u_0^2 \prec u_0$. Hence the minimality of $\text{supp}(f)$ implies that $f(u_0^2) = 0$. From $f^2 = f$ and Equation 3.1, we infer that $f(u_0)\omega_{u_0}(f(u_0)) = 0$. Since R is S -rigid, we obtain $f(u_0) = 0$ which contradicts to the fact that u_0 is a minimal element of $\text{supp}(f)$. Hence $1 \preceq u_0$.

Suppose that there exists $1 \prec s_0$ such that $f(s_0) \neq 0$. We can assume that s_0 is the smallest with the condition under the total order \preceq . Therefore $f(s) = 0$ for all $1 \prec s \prec s_0$. From $f^2 = f$, it implies that

$$f(1)^2 = f(1) \quad \text{and} \quad f(s_0) = f(1)f(s_0) + f(s_0)\omega_{s_0}(f(1)).$$

Since $f(1)$ is an idempotent element of the ring R , from Lemma 3.1(1) we infer $f(s_0) = f(1)f(s_0) + f(s_0)f(1)$. Multiplying the last equation on the left by $f(1)$ we have $f(1)f(s_0) = f(1)f(s_0) + f(1)f(s_0)f(1)$ and thus $f(1)f(s_0)f(1) = 0$. Since R is S -rigid, $f(1)f(s_0) = f(s_0)\omega_{s_0}(f(1)) = 0$ and $f(s_0) = 0$ follows, which is a contradiction. Consequently we have $f(s) = 0$ for all $s \in S \setminus \{1\}$. Thus $f = \mathbf{c}_{f(1)}$, as desired. \square

If (S, \leq) is a totally ordered monoid, then the *positive cone* of S is denoted by S^+ , i.e., $S^+ = \{s \in S : s \geq 1\}$. Since S^+ is a submonoid of S , for any ring

R and $\omega : S \rightarrow \text{End}(R)$ a monoid homomorphism the skew generalized power series ring $R[[S^+, \omega]]$ is a subring of the ring $R[[S, \omega]]$.

Lemma 3.3. *Let R be a ring, (S, \leq) a quasitotally ordered monoid such that for any $s \in S$ with $s \leq 1$ we have $s \in U(S)$, and let $\omega : S \rightarrow \text{End}(R)$ be a monoid homomorphism. Then for any $f \in R[[S, \omega]]$ there exist $u \in U(S)$ and $g \in R[[S^+, \omega]]$ such that $f = g\mathbf{e}_u$.*

Proof. Our proof follows the method employed in [26, Lemma 2.3]. By hypothesis, the order (S, \leq) can be refined to a strict total order \preceq on S . If $f = 0$, then we can put $u = 1$ and $g = 0$. Thus we assume that $f \neq 0$. Since $\text{supp}(f)$ is a non-empty artinian and narrow subset of S , the set of minimal elements of $\text{supp}(f)$ is finite and non-empty. Thus there exists a minimal element of $\text{supp}(f)$ under the total order \preceq , which will be denoted by s . If $s \succeq 1$, then $\text{supp}(f) \subseteq S^+$, so $f \in R[[S^+, \omega]]$ and we can set $u = 1$ and $g = f$. We are left with the case where $s \prec 1$. Then $s \in U(S)$. Furthermore, if t is any element of $\text{supp}(f)$, then $s \preceq t$, so $1 \preceq ts^{-1}$ and thus $ts^{-1} \in S^+$. Hence, for the function $g : S \rightarrow R$ defined by

$$g(x) = f(xs) \quad \text{for any } x \in S$$

we have $\text{supp}(g) \subseteq \text{supp}(f) \cdot s^{-1} \subseteq S^+$, which implies that $g \in R[[S^+, \omega]]$. Moreover, for any $x \in S$ we have

$$f(x) = g(xs^{-1}) = g(xs^{-1})\omega_{xs^{-1}}(\mathbf{e}_s(s)) = g\mathbf{e}_s(x),$$

which shows that $f = g\mathbf{e}_s$. Hence we can put $u = s$, and the result follows. \square

A ring R is (*von Neumann*) *regular* (resp. *unit-regular*) if $a \in aRa$ (resp. $a \in aU(R)a$) for all $a \in R$. A ring R is *semiregular* if $R/J(R)$ is regular and idempotents lift modulo $J(R)$. Recall that a monoid S is *cyclic* if for some $s \in S$ we have $S = \{s^n : n \in \mathbb{N} \cup \{0\}\}$.

Proposition 3.4. *Let R be a ring, (S, \cdot, \leq) a quasitotally ordered cyclic group generated by s , and let $\omega : S \rightarrow \text{End}(R)$ be a group homomorphism. Assume that R is S -rigid.*

- (1) *If $\mathbf{c}_a + \mathbf{c}_b\mathbf{e}_s$ is a unit of $R[[S, \omega]]$, then there exists $c \in R$ such that $a = aca$.*
- (2) *If $R[[S, \omega]]$ is a clean ring, then R is (*von Neumann*) regular.*

Proof. (1) By hypothesis, the order (S, \leq) can be refined to a strict total order \preceq on S . Since S is a cyclic group, Lemma 3.3 implies that there exist $n \in \mathbb{N} \cup \{0\}$ and $f \in R[[S^+, \omega]]$ such that

$$(3.2) \quad (\mathbf{c}_a + \mathbf{c}_b\mathbf{e}_s)f = \mathbf{e}_{s^n},$$

where S^+ is the positive cone of S . Without loss of generality, we can assume that $1 \prec s$. Then computing both sides of Equation (3.2) at 1 we obtain:

$$(3.3) \quad af(1) + b\omega_s(f(s^{-1})) = \mathbf{e}_{s^n}(1).$$

Since $s^{-1} \prec 1$ and $f \in R[[S^+, \omega]]$, $f(s^{-1}) = 0$. Then Equation (3.3) becomes:

$$(3.4) \quad af(1) = \mathbf{e}_{s^n}(1).$$

If $n = 0$, then Equation (3.4) implies that $af(1) = 1$ and so $a = af(1)a$. Thus, we will assume that $1 \leq n$. Then Equation (3.4) becomes:

$$(3.5) \quad af(1) = 0.$$

Indeed, after computing both sides of Equation (3.2) at s we obtain:

$$(3.6) \quad af(s) + b\omega_s(f(1)) = 0.$$

Multiplying Equation (3.6) by a from the left yields $a^2f(s) + ab\omega_s(f(1)) = 0$. Since R is S -rigid, from Equation (3.5), we conclude $ab\omega_s(f(1)) = 0$. This implies $a^2f(s) = 0$. Since R is reduced, we have

$$(3.7) \quad af(s) = 0.$$

We see easily by induction and an argument similar above that $af(1) = af(s) = \dots = af(s^{n-1}) = 0$. Since R is S -rigid, we have $\omega_s(f(s^{n-1}))a = 0$. On the other hand, computing both sides of Equation (3.2) at s^n we obtain:

$$(3.8) \quad af(s^n) + b\omega_s(f(s^{n-1})) = 1.$$

Now multiplying Equation (3.8) by a from the right gives $a = af(s^n)a$.

(2) Let $a \in R$. Then $-\mathbf{c}_a\mathbf{e}_{s^{-1}} = e + u$ where $e^2 = e \in R[[S, \omega]]$ and u is a unit of $R[[S, \omega]]$. By Proposition 3.2, there exists $b \in R$ such that $e = \mathbf{c}_b$. So $\mathbf{c}_a\mathbf{e}_{s^{-1}} + \mathbf{c}_b$ is a unit of $R[[S, \omega]]$, and hence $\mathbf{c}_a + \mathbf{c}_b\mathbf{e}_s$ is a unit of $R[[S, \omega]]$. From part (1) it follows that there exists $c \in R$ such that $a = aca$. This proves that R is (von Neumann) regular, and the proof is complete. \square

In the proof of the next result we will need the following criterion for cleanliness of a ring which is due to Zhou and Ziemkowski [46].

Lemma 3.5 ([46, Lemma 2.3]). *Let $a \in R$ and $e^2 = e \in R$ such that both $ea(1 - e)$ and $(1 - e)ae$ are contained in $J(R)$. If ea is clean in eRe and $(1 - e)a(1 - e)$ is clean in $(1 - e)R(1 - e)$, then a is clean in R .*

The following proposition provides us with a method of constructing clean rings.

Theorem 3.6. *Let R be a 2-primal ring, (S, \leq) a nontrivial ordered group and $\omega : S \rightarrow \text{Aut}(R)$ a group homomorphism. Assume that \leq is total or (S, \leq) is an abelian torsion-free group such that \leq is subtotal. Suppose that R is S -compatible and $J(R)$ is a nilpotent ideal of R . If R is semiregular, then the skew generalized power series ring $R[[S, \omega]]$ is clean.*

Proof. We set $A := R[[S, \omega]]$ and $\widehat{R} := R/J(R)$. We only consider the case (S, \leq) is an abelian torsion-free group such that \leq is subtotal because the proof of the other case is similar. Suppose that \preceq is the total order associated with \leq . Suppose that f is a nonzero element of A . We will show that f is clean in A . Since $\text{supp}(f)$ is a non-empty artinian and narrow subset of S , the

set of minimal elements of $\text{supp}(f)$ is finite and non-empty. Thus there exists a minimal element of $\text{supp}(f)$ under the total order \preceq . Then by Lemma 3.3, there exist $s \in S$ and $g \in R[[S^+, \omega]]$ such that $f = g\mathbf{e}_s$, where S^+ is the positive cone of S . Without loss of generality, we can assume that $1 \preceq s$. Since $\text{supp}(g)$ is a non-empty artinian and narrow subset of S , the set $\{t \in \text{supp}(g) : 1 \preceq t \preceq s\}$ is finite, say equal to $\{t_1 = 1, t_2, t_3, \dots, t_k = s\}$. Since $J(R)$ is nilpotent and R is 2-primal, from Theorem 2.5 it implies that $J(A) = J(R)[[S, \omega]]$. As R is 2-primal, the assumption implies that \widehat{R} is strongly regular. It follows that $\widehat{g(t_1)} = \widehat{e_0} \widehat{u_0}$, where $\widehat{e_0}$ is a central idempotent of \widehat{R} and $\widehat{u_0}$ is a unit of \widehat{R} . Since $J(R)$ is nil, [10, Theorem 21.28] concludes that idempotents of \widehat{R} can be lifted to idempotents of R , so we can assume that $e_0^2 = e_0$. Therefore, $g(t_1) = e_0 u_0 e_0 + j_0$ and so $e_0 g(t_1) e_0 = e_0 u_0 e_0 + e_0 j_0 e_0$, where $e_0 u_0 e_0$ is a unit of $e_0 R e_0$ and $j_0 \in J(R)$. Applying Lemma 3.1, we obtain

$$\begin{aligned} \mathbf{c}_{e_0} f \mathbf{c}_{e_0} &= \mathbf{c}_{e_0} g \mathbf{c}_{e_0} \mathbf{e}_s, \\ (e_0 R e_0)[[S, \omega]] &= \mathbf{c}_{e_0} A \mathbf{c}_{e_0}. \end{aligned}$$

As $\mathbf{c}_{e_0} g \mathbf{c}_{e_0}(t_1) = e_0 u_0 e_0 + e_0 j_0 e_0$ is a unit of the ring $e_0 R e_0$, from Proposition 2.2 it follows that $\mathbf{c}_{e_0} f \mathbf{c}_{e_0}$ is a unit (and hence a clean element of the ring $(e_0 R e_0)[[S, \omega]] = \mathbf{c}_{e_0} A \mathbf{c}_{e_0}$). Also, since $\widehat{e_0}$ is a central element in \widehat{R} and R is an S -compatible ring, Lemma 3.1(2) implies that $\mathbf{c}_{e_0} f - f \mathbf{c}_{e_0} \in J(R)[[S, \omega]] = J(A)$. Therefore, we have

$$\mathbf{c}_{e_0} f \mathbf{c}_{(1-e_0)}, \mathbf{c}_{(1-e_0)} f \mathbf{c}_{e_0} \in J(A).$$

By Lemma 3.5, to prove that f is clean in A it suffices to show that

$$\mathbf{c}_{(1-e_0)} f \mathbf{c}_{(1-e_0)}$$

is clean in $((1-e_0)R(1-e_0)[[S, \omega]] = \mathbf{c}_{(1-e_0)} A \mathbf{c}_{(1-e_0)})$. Note that

$$f_1 := \mathbf{c}_{(1-e_0)} f \mathbf{c}_{(1-e_0)} = \mathbf{c}_{(1-e_0)} g \mathbf{c}_{(1-e_0)} \mathbf{e}_s,$$

and also we have $\mathbf{c}_{(1-e_0)} g \mathbf{c}_{(1-e_0)}(t_1) = (1-e_0)j_0(1-e_0)$.

Since $R_1 := (1-e_0)R(1-e_0)$ is again 2-primal and S -compatible, strongly regular modulo $J(R_1)$ with $J(R_1)$ nil and $\mathbf{P}(R_1)$ is a nilpotent ideal of R_1 as argued above we have $(1-e_0)g(t_2)(1-e_0) = e_1 u_1 e_1 + j_1$, where e_1 is an idempotent of R_1 which is central modulo $J(R_1)$, $e_1 u_1 e_1$ is a unit of $e_1 R_1 e_1$ and $j_1 \in J(R_1)$. Applying Lemma 3.1, we obtain $\mathbf{c}_{e_1} f_1 \mathbf{c}_{e_1} = \mathbf{c}_{e_1} g \mathbf{c}_{e_1} \mathbf{e}_s$. We set:

$$g_1(t) := \begin{cases} g(t) & t_1 \preceq t \prec t_2 \\ 0 & \text{otherwise,} \end{cases} \quad g_2(t) := \begin{cases} g(t) & t_2 \preceq t \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that $g_1, g_2 \in A$ and also we have

$$(3.9) \quad \mathbf{c}_{e_1} f_1 \mathbf{c}_{e_1} = \mathbf{c}_{e_1} g_1 \mathbf{c}_{e_1} \mathbf{e}_s + \mathbf{c}_{e_1} g_2 \mathbf{c}_{e_1} \mathbf{e}_s.$$

Since $\mathbf{c}_{e_1} g_1 \mathbf{c}_{e_1}(t_1) = e_1 j_0 e_1 \in e_1 J(R_1) e_1$, we have

$$\mathbf{c}_{e_1} g_1 \mathbf{c}_{e_1} \mathbf{e}_s \in J(\mathbf{c}_{e_1} R_1 [[S, \omega]] \mathbf{c}_{e_1}).$$

Furthermore, since $\mathbf{c}_{e_1}g_2\mathbf{c}_{e_1}(t_2) = e_1u_1e_1 + e_1j_1e_1$ is a unit of the ring $e_1R_1e_1$, Proposition 2.2 implies that $\mathbf{c}_{e_1}g_2\mathbf{c}_{e_1}\mathbf{e}_s$ is a unit of the ring $(e_1R_1e_1)[[S, \omega]]$. It follows from Equation (3.9) that $\mathbf{c}_{e_1}f_1\mathbf{c}_{e_1}$ is a unit (and hence a clean element of $\mathbf{c}_{e_1}R_1[[S, \omega]]\mathbf{c}_{e_1}$). Since $e_1a - ae_1 \in J(R_1)$ for all $a \in R_1$, we have

$$\mathbf{c}_{e_1}f_1\mathbf{c}_{(1-e_0-e_1)}, \mathbf{c}_{(1-e_0-e_1)}f_1\mathbf{c}_{e_1} \in J(R_1[[S, \omega]]).$$

Therefore to show that f_1 is clean in $R_1[[S, \omega]]$, by Lemma 3.5 it suffices to prove that $f_2 := \mathbf{c}_{(1-e_0-e_1)}f_1\mathbf{c}_{(1-e_0-e_1)}$ is clean in $\mathbf{c}_{(1-e_0-e_1)}R[[S, \omega]]\mathbf{c}_{(1-e_0-e_1)}$.

Consider that $(1 - e_0 - \cdots - e_{i-1})g(t_i)(1 - e_0 - \cdots - e_{i-1}) = e_iu_ie_i + j_i$ such that e_i is an idempotent of $R_i := (1 - e_0 - \cdots - e_{i-1})R(1 - e_0 - \cdots - e_{i-1})$ which is central modulo $J(R_i)$, $e_iu_ie_i$ is a unit of $e_iR_ie_i$ and $j_i \in J(R_i)$ for $i = 1, 2, \dots, k-1$. We set $f_i := \mathbf{c}_{(1-e_0-e_1-\cdots-e_{i-1})}f\mathbf{c}_{(1-e_0-e_1-\cdots-e_{i-1})}$ for $i = 1, 2, \dots, k-1$.

A similar argument as above yields to show that f_i is clean in $R_i[[S, \omega]]$, by Lemma 3.5 it suffices to prove that f_{i+1} is a clean element in $R_{i+1}[[S, \omega]]$ for $i = 1, 2, \dots, k-1$. Now we set:

$$g_1(t) := \begin{cases} g(t) & t_1 \preceq t \preceq t_{k-1} \\ 0 & \text{otherwise,} \end{cases} \quad g_2(t) := \begin{cases} g(t) & s \preceq t \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that $g_1, g_2 \in A$ and also we have $f_k = h_1 + h_2$, where

$$(3.10) \quad h_1 = \mathbf{c}_{(1-e_0-e_1-\cdots-e_{k-1})}g_1\mathbf{c}_{(1-e_0-e_1-\cdots-e_{k-1})}\mathbf{e}_s,$$

$$(3.11) \quad h_2 = \mathbf{c}_{(1-e_0-e_1-\cdots-e_{k-1})}g_2\mathbf{c}_{(1-e_0-e_1-\cdots-e_{k-1})}\mathbf{e}_s.$$

Since for $i = 1, 2, \dots, k-1$, $\mathbf{c}_{(1-e_0-e_1-\cdots-e_{k-1})}g_1\mathbf{c}_{(1-e_0-e_1-\cdots-e_{k-1})}(t_i) \in J(R_k)$ it follows that $h_1 \in J(R_k)[[S, \omega]] = J(R_k[[S, \omega]])$. Furthermore, since $\text{supp}(h_2) \subseteq S^+$, we have $h_2 \in [[S^+, \omega]]$. As R_k is clean, [35, Theorem 3.2(8)] implies that $R_k[[S^+, \omega]]$ is clean. But h_2 is an element of the ring $R_k[[S^+, \omega]]$. Hence h_2 is a clean element of $R_k[[S^+, \omega]]$. Therefore h_2 is a clean element of $R_k[[S, \omega]]$. Thus $h_2 = w + e$, where w is a unit of $R_k[[S, \omega]]$ and e is an idempotent of $R_k[[S, \omega]]$. On the other hand, $h_1 \in J(R_k[[S, \omega]])$. Therefore, $h_1 + w$ is a unit of $R_k[[S, \omega]]$, and so $f_k = (h_1 + w) + e$ is a clean element of $R_k[[S, \omega]]$. Hence by Lemma 3.5, we infer that f is clean in A , and the proof is complete. \square

Recall that a module ${}_R M$ has the (full) exchange property if for every module ${}_R A$ and any two decompositions $A = M' \oplus N = \bigoplus_{i \in I} A_i$ with $M' \cong M$, there exist submodules $A'_i \subseteq A_i$ such that $A = M' \oplus (\bigoplus_{i \in I} A'_i)$. A module ${}_R M$ has the *finite exchange property* if the above condition is satisfied whenever the index set I is finite. Exchange rings were introduced by Warfield [45] via the exchange property of modules. By [27], every clean ring is an exchange ring and it is shown in [27, Proposition 1.8] that a ring with central idempotents is clean if and only if it is an exchange ring [45]. A ring R is said to be *strongly regular* if for any $a \in R$ there exists $b \in R$ such that $a = a^2b$. Strongly regular rings are exactly von Neumann regular rings in which all idempotents are central.

The following theorem provides a characterization of the cleanness of a skew generalized power series ring $R[[S, \omega]]$ in the case where (S, \leq) an ordered cyclic group and \leq is total or (S, \leq) is an abelian torsion-free group such that \leq is subtotal.

Theorem 3.7. *Let R be a 2-primal ring, (S, \leq) an ordered cyclic group and $\omega : S \rightarrow \text{Aut}(R)$ a group homomorphism. Assume that \leq is total or (S, \leq) is an abelian torsion-free group such that \leq is subtotal. Suppose that R is S -compatible and $\mathbf{P}(R)$ is a nilpotent ideal of R . Then the following statements are equivalent:*

- (1) $R[[S, \omega]]$ is a clean ring;
- (2) $R[[S, \omega]]/J(R[[S, \omega]])$ is a clean ring;
- (3) $R[[S, \omega]]$ is an exchange ring;
- (4) $R[[S, \omega]]/J(R[[S, \omega]])$ is an exchange ring;
- (5) R is semiregular with $J(R)$ nil;
- (6) $R/J(R)$ is strongly regular with $J(R)$ nil.

Proof. We set $A := R[[S, \omega]]$, $\overline{R} := R/\mathbf{P}(R)$ and $\widehat{R} := R/J(R)$.

(1) \Rightarrow (2). The result follows from [1, Proposition 7], since R is a clean ring if and only if \widehat{R} is a clean ring and all idempotents of the ring \widehat{R} can be lifted to idempotents of the ring R .

(2) \Rightarrow (4) and (1) \Rightarrow (3). Trivial.

(3) \Rightarrow (4). The result follows from [27, Corollary 2.4], since R is an exchange ring if and only if \widehat{R} is an exchange ring and all idempotents of the ring \widehat{R} can be lifted to idempotents of R .

(4) \Rightarrow (5). By Theorem 2.5, $J(A) = \mathbf{P}(R)[[S, \omega]]$. So there exists a natural ring isomorphism

$$\overline{R}[[S, \overline{\omega}]] \cong A/(\mathbf{P}(R)[[S, \omega]]) = A/J(A)$$

is an exchange ring, where $\overline{\omega} : S \rightarrow \text{Aut}(R/\mathbf{P}(R))$ is the induced group homomorphism (i.e., $\overline{\omega}_s(r + \mathbf{P}(R)) = \omega_s(r) + \mathbf{P}(R)$ for any $s \in S$ and $r \in R$). The hypothesis implies that $\mathbf{P}(R) = \text{nil}(R)$, and so \overline{R} is a reduced ring. We prove that \overline{R} is $(S, \overline{\omega})$ -rigid. To prove this, consider any $\overline{a} \in \overline{R}$ and all $s \in S$ with $\overline{a} \cdot \overline{\omega}_s(\overline{a}) = \overline{0}$. So $a\omega_s(a) \in \mathbf{P}(R)$. Therefore $a^2 \in \text{nil}(R)$, by Lemma 2.3(ii) and hence $\overline{a} = \overline{0}$. Thus \overline{R} is an $(S, \overline{\omega})$ -rigid ring. Hence all idempotents of $\overline{R}[[S, \overline{\omega}]]$ are central by Proposition 3.2. Therefore $\overline{R}[[S, \overline{\omega}]]$ is a clean ring by [27, Proposition 1.8]. Hence \overline{R} is regular by Proposition 3.4. It follows that $J(R) \subseteq \mathbf{P}(R)$. So $J(R) = \text{nil}(R)$, which is nil.

(5) \Rightarrow (6). The hypothesis implies that $J(R) = \text{nil}(R)$, and so \widehat{R} is a strongly regular ring.

(6) \Rightarrow (1). This follows from Theorem 3.6. \square

The following corollaries are immediate consequences of Theorem 3.7.

Corollary 3.8. *Let R be a domain, (S, \leq) an ordered cyclic group and $\omega : S \rightarrow \text{Aut}(R)$ a group homomorphism. Assume that \leq is total or (S, \leq) is an*

abelian torsion-free group such that \leq is subtotal. Then $R[[S, \omega]]$ is a clean ring if and only if R is a division ring.

Proof. It is easy to show that R is S -compatible. Now, the result follows from Theorem 3.7. \square

Let R be a ring, α an automorphism of R , and (S, \leq) an infinite cyclic group generated by x with the ordering $x^m \leq x^n$ if and only if $m \leq n$. By setting $\omega_{x^n} = \alpha^n$ for any $n \in \mathbb{Z}$ we obtain a monoid homomorphism $\omega : S \rightarrow \text{End}(R)$. The ring $R[[S, \omega, \leq]]$ is called the *skew Laurent series ring* and denoted by $R[[x, x^{-1}; \alpha]]$. The following result generalizes [46, Theorem 2.5].

Corollary 3.9. *Let R be a 2-primal ring and α an automorphism of R . Suppose that R is either a ring with ACC on both right and left annihilators, or is left or right Goldie, or has the ACC on ideals, or has right Krull dimension and R is S -compatible. Then the following statements are equivalent:*

- (1) $R[[x, x^{-1}; \alpha]]$ is a clean ring;
- (2) $R[[x, x^{-1}; \alpha]]/J(R[[x, x^{-1}; \alpha]])$ is a clean ring;
- (3) $R[[x, x^{-1}; \alpha]]$ is an exchange ring;
- (4) $R[[x, x^{-1}; \alpha]]/J(R[[x, x^{-1}; \alpha]])$ is an exchange ring;
- (5) R is semiregular with $J(R)$ nil;
- (6) $R/J(R)$ is strongly regular with $J(R)$ nil.

Proof. If R has the ACC on ideals or R is right Goldie or satisfies the ascending chain condition on both right and left annihilators, then by [43, Lemma 2.6.22], [12, Theorem 1] and [7, Theorem 1], $\mathbf{P}(R)$ is nilpotent, respectively. If R has right Krull dimension, then by [13], $\mathbf{P}(R)$ is nilpotent. Also, if R is a ring with ACC on both right and left annihilators, then by [2, Theorem 1.34], $\mathbf{P}(R)$ is nilpotent. Now, the result follows by Theorem 3.7. \square

A ring R is called *strongly π -regular* if for each $a \in R$ there exists $n \geq 1$ such that $a^n \in a^{n+1}R$. A commutative ring R is strongly π -regular if and only if $R/J(R)$ is (strongly) regular with $J(R)$ nil (see [11, Exercise 4.15]).

Corollary 3.10. *Let R be a commutative ring, (S, \leq) a totally ordered cyclic group and $\omega : S \rightarrow \text{Aut}(R)$ a group homomorphism. Assume that R is S -compatible and R is either a ring with ACC on both right and left annihilators, or is left or right Goldie, or has the ACC on ideals, or has right Krull dimension. Then the skew Mal'cev-Neumann series ring $R((S, \omega))$ is a clean ring if and only if R is strongly π -regular.*

Corollary 3.11. *Let R be a reduced ring and (S, \leq) a totally ordered cyclic group. Then the Mal'cev-Neumann series ring $R((S))$ is a clean ring if and only if R is strongly regular.*

We close this paper by investigating the uniquely clean property of a skew generalized power series ring $R[[S, \omega]]$.

Theorem 3.12. *Let R be a ring, (S, \leq) an ordered group and $\omega : S \rightarrow \text{Aut}(R)$ a group homomorphism. Assume that \leq is total or (S, \leq) is an abelian torsion-free group such that \leq is subtotal. Then $R[[S, \omega]]$ is not uniquely clean for any nontrivial ring R .*

Proof. Our proof follows the method employed in [46, Proposition 2.11]. We set $A := R[[S, \omega]]$. Suppose on the contrary that A is uniquely clean. Then $\bar{A} := A/J(A)$ is Boolean by [29, Theorem 20]. So $\bar{1}$ is the only unit of \bar{A} . Let s be any element of S such that $1 < s$. Therefore, $\bar{e}_s = \bar{1}$, that is, $1 - e_s \in J(A)$. From Propositions 2.1 and 2.2 it implies that $1 - e_s$ is a unit of A . This is a contradiction. \square

Corollary 3.13. *The following rings is not uniquely clean for any nontrivial ring R :*

- (a) *The skew Laurent series ring $R[[x, x^{-1}; \sigma]]$, where α is an automorphism of R .*
- (b) *The skew Mal'cev-Neumann series ring $R((S, \omega))$, where (S, \leq) is a totally ordered group and $\omega : S \rightarrow \text{Aut}(R)$ a group homomorphism.*

To characterize skew generalized power series rings that are uniquely clean, we will need the following lemma.

Lemma 3.14. *Let R be a ring, (S, \leq) a positively strictly ordered monoid, and $\omega : S \rightarrow \text{End}(R)$ a monoid homomorphism. If e is an idempotent of $R[[S, \omega]]$ and $f(1)$ is a central element of R such that $\omega_s(f(1)) = f(1)$ for all $s \in S$, then $e = c_{f(1)}$.*

Proof. The proof is similar to Proposition 3.2. \square

Below we provide a characterization of uniquely clean skew generalized power series rings $R[[S, \omega]]$ in the case where (S, \leq) is a positively strictly ordered monoid. Recall from [24] that an endomorphism σ of a ring R is *idempotent-stabilizing* if $\sigma(e) = e$ for every idempotent e of R . It is easy to show that every σ -compatible ring is idempotent-stabilizing. The following theorem generalizes [29, Example 9].

Theorem 3.15. *Let R be a ring, (S, \leq) a positively strictly ordered monoid, and $\omega : S \rightarrow \text{End}(R)$ a monoid homomorphism. Then $R[[S, \omega]]$ is uniquely clean if and only if R is uniquely clean and ω_s is idempotent-stabilizing for all $s \in S$.*

Proof. Suppose that R is a uniquely clean ring and ω_s is idempotent-stabilizing for all $s \in S$. Let $f \in R[[S, \omega]]$. By [35, Theorem 3.2(8)], $R[[S, \omega]]$ is a clean ring. Let $f = g_1 + h_1 = g_2 + h_2$ where g_1, g_2 are units in $R[[S, \omega]]$ and h_1, h_2 are idempotents in $R[[S, \omega]]$. Since $h_1(0)$ and $h_2(0)$ are idempotent elements in R , by [29, Lemma 4] it follows that $h_1(0)$ and $h_2(0)$ are central. From Lemma 3.15 we deduce that $h_1 = c_{h_1(0)}$ and $h_2 = c_{h_2(0)}$. Now from $g_1 - g_2 = c_{h_2(0)} - c_{h_1(0)}$, it follows that $g_1(s) = g_2(s)$ for all $s \in S \setminus \{1\}$.

Moreover, we have $g_1(0) + h_1(0) = g_1(0) + h_2(0)$. Since $g_1(0)$ and $g_2(0)$ are unit elements in R and also R is a uniquely clean ring, it implies that $g_1(0) = g_2(0)$. Hence $g_1 = g_2$ and so $h_1 = h_2$. This proves that $R[[S, \omega]]$ is a uniquely clean ring. Conversely, suppose that $R[[S, \omega]]$ is a uniquely clean ring and a is an idempotent in R . From [29, Lemma 4] we infer that the idempotent element \mathbf{c}_a is central in $R[[S, \omega]]$. Thus $\mathbf{c}_a \mathbf{e}_s = \mathbf{e}_s \mathbf{c}_a$ for all $s \in S$ and so $\omega_s(a) = a$ for all $s \in S$. Therefore ω_s is idempotent-stabilizing for all $s \in S$. Furthermore, it is easy to show that the ring R is isomorphic to a factor ring of $R[[S, \omega]]$. By [29, Theorem 22] every factor ring of a uniquely clean ring is again uniquely clean, therefore R is uniquely clean, which completes the proof. \square

The following corollary provides a rich source of examples of uniquely clean rings.

Corollary 3.16. *Let S be a submonoid of $(\mathbb{N} \cup \{0\})^n$ ($n \geq 2$), endowed with the order \leq induced by the product order, or lexicographic order or reverse lexicographic order. Let R be a ring and $\omega : S \rightarrow \text{End}(R)$ a monoid homomorphism. Then $R[[S, \omega]]$ is uniquely clean if and only if R is uniquely clean and ω_s is idempotent-stabilizing for all $s \in S$.*

Corollary 3.17. *Let $(S_1, \leq_1), \dots, (S_n, \leq_n)$ be positively strictly ordered monoids. Denote by $(\text{lex } \leq)$ and $(\text{relex } \leq)$ the lexicographic order, the reverse lexicographic order, respectively, on the ordered monoid $S_1 \times \dots \times S_n$. Then R is a uniquely clean ring if and only if $R[[S_1 \times \dots \times S_n, \text{lex } \leq]]$ is uniquely clean if and only if $R[[S_1 \times \dots \times S_n, \text{relex } \leq]]$ is uniquely clean.*

Let α and β be endomorphisms of R such that $\alpha \circ \beta = \beta \circ \alpha$. Assume that $S = (\mathbb{N} \cup \{0\}) \times (\mathbb{N} \cup \{0\})$ is endowed with the lexicographic order, or the reverse lexicographic order, or the product order of the usual order of $\mathbb{N} \cup \{0\}$, and define $\omega : S \rightarrow \text{End}(R)$ a monoid homomorphism via $\omega(m, n) = \alpha^m \circ \beta^n$ for any $m, n \in \mathbb{N} \cup \{0\}$. Then $R[[S, \omega]] \cong R[[x, y; \alpha, \beta]]$, in which $(ax^m y^n)(bx^p y^q) = a\alpha^m \circ \beta^n(b)x^{m+p}y^{n+q}$ for any $m, n, p, q \in \mathbb{N} \cup \{0\}$.

Corollary 3.18. *Let α and β be endomorphisms of a ring R such that $\alpha \circ \beta = \beta \circ \alpha$. Then the ring $R[[x, y; \alpha, \beta]]$ is uniquely clean if and only if R is uniquely clean and α and β are idempotent-stabilizing.*

Corollary 3.19. *Let R be a ring and let S be any of the additive monoids $\mathbb{Q}^+ = \{a \in \mathbb{Q} \mid a \geq 0\}$ or $\mathbb{R}^+ = \{a \in \mathbb{R} \mid a \geq 0\}$, where \leq is the usual order. Then the ring $R[[S]]$ is uniquely clean if and only if R is uniquely clean.*

Let R be a ring, and consider the multiplicative monoid $\mathbb{N}^{\geq 1}$, endowed with the usual order \leq . Then $A = R[[\mathbb{N}^{\geq 1}]]$ is the ring of arithmetical functions with values in R , endowed with the Dirichlet convolution:

$$fg(n) = \sum_{d|n} f(d)g(n/d) \quad \text{for each } n \geq 1.$$

Corollary 3.20. *Let R be a ring. Then the ring of arithmetical functions $R[[\mathbb{N}^{\geq 1}]]$ is uniquely clean if and only if R is uniquely clean.*

Remark 3.21. The author does not know the answer to the following question: is it true that the assumption that (S, \leq) is an ordered cyclic group is essential in Proposition 3.4 and Theorem 3.7?

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