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DIRECT SUM FOR BASIC COHOMOLOGY OF CODIMENSION FOUR TAUT RIEMANNIAN FOLIATION

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Abstract. We discuss the decomposition of degree two basic cohomology for codimension four taut Riemannian foliation according to the holonomy invariant transversal almost complex structure J, and show that J is C^∞ pure and full. In addition, we obtain an estimate of the dimension of basic J-anti-invariant subgroup. These are the foliated version for the corresponding results of T. Draghici et al. [3].

1. Introduction

In order to study S. K. Donaldson's tamed to compatible question [2], T.-J. Li and W. Zhang [7] defined two subgroups $H^+_J(M)$, $H^-_J(M)$ of the real degree 2 de Rham cohomology group $H^2(M;\mathbb{R})$ for a compact almost complex manifold (M,J). They are the sets of cohomology classes which can be represented by J-invariant and J-antiinvariant real 2-forms respectively. Later, T. Draghici, T.-J. Li and W. Zhang showed in [3] that in dimension four

$$H^2(M;\mathbb{R}) = H^+_J(M) \oplus H^-_J(M),$$

and they call such almost complex structure J to be C^∞-pure and full. This is specifical for four dimension, since A. Fino and A. Tomassini's Example 3.3 in [4] gives a six dimensional almost complex manifold (M,J) with J being not C^∞-pure, and higher dimensional non-C^∞-pure examples can be obtained by producting it with another almost complex manifold (see Remark 2.7 in [3]). It becomes nature to ask when will the almost complex structure be C^∞-pure and full on higher dimension. This article lays the groundwork for the case in which the higher dimensional manifold admits a codimension four taut Riemannian foliation $\mathcal{F}$. The main result is Theorem 4.1 which basically says a transversal almost complex structure J on a codimension four taut Riemannian foliated

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manifold satisfying $\theta(V)J = 0,\ \forall\ V \in \Gamma T\mathcal{F}$ is C^∞-pure and full in the sense of Definition 4.

The structure of this article is the following: Section 2 are notions of transverse structures, basic forms, characteristic form and filtrations needed later. We consider the compatibility of transversal almost complex structure with a taut Riemannian metric in Section 3. After these preliminaries, basic J-(anti)invariant cohomology groups naturally come out and so is C^∞-pureness and fullness of J. After some lemmas similar to those in [3], we are able to proof the decomposition of the real degree 2 de Rham cohomology group in Section 4. The last section provides bounds on the dimension of J-(anti)invariant cohomology groups.

2. Taut Riemannian foliation

Let's first recall some definitions and results in foliations, the below in this section is referred to [9]. Let M be a closed oriented smooth manifold of dimension $n = p + q$ endowed with a codimension q foliation $\mathcal{F}$. The integrable subbundle $T\mathcal{F} \subset TM$ is given by vectors tangent to plaques, then we further have the rank q normal bundle defined as the quotient bundle $Q = TM/T\mathcal{F}$ and the projection

$$\begin{aligned} \pi:\ TM &\to Q \\ Y\ &\mapsto \pi(Y) \end{aligned}$$

denoted by $\overline{Y} = \pi(Y)$.

Define the $\Gamma T\mathcal{F}$-action on ΓQ as

$$\theta(V)s = \overline{[V, Y_s]} \quad \text{for } V \in \Gamma T\mathcal{F},\ s \in \Gamma Q,$$

where $Y_s \in \Gamma TM$ is any choice with $\overline{Y}_s = s$. It can be checked that the definition $\theta(V)s$ is independent of the choice of Y_s.

Consider a Riemannian metric $g = g_{T\mathcal{F}} \oplus g_{T\mathcal{F}^\perp}$ on M splitting TM orthogonally as $TM = T\mathcal{F} \oplus T\mathcal{F}^\perp$, which means there is a bundle map $\sigma : Q \overset{\cong}{\to} T\mathcal{F}^\perp \subset TM$ splitting the exact sequence

$$0 \to T\mathcal{F} \to TM \to Q \to 0,$$

i.e., satisfying $\pi \circ \sigma$=identity. This induces a metric on Q by $g_Q = \sigma^* g_{T\mathcal{F}^\perp}$, then the splitting map $\sigma :\ (Q, g_Q) \to (T\mathcal{F}^\perp, g_{T\mathcal{F}^\perp})$ is a metric isomorphism.

Suppose ∇^M is the Levi-Civita connection induced by the Riemannian metric g on M. For $s \in \Gamma Q$, define

$$\nabla_X s = \begin{cases} \pi\left[X, \sigma(s)\right] & \text{for } X \in \Gamma T\mathcal{F}, \\ \pi\left(\nabla^M_X \sigma(s)\right) & \text{for } X \in \Gamma T\mathcal{F}^\perp, \end{cases}$$

then ∇ is an adapted connection in Q, which means ∇ restricting along $T\mathcal{F}$ is the partial Bott connection.

Consider the Q-valued bilinear form on $T\mathcal{F}$, i.e., $\alpha : T\mathcal{F} \otimes T\mathcal{F} \to Q$ given by

$$\alpha(U, V) = \pi\left(\nabla_U^M V\right) \qquad \text{for } U, V \in \Gamma T\mathcal{F}.$$

A calculation shows for $s \in \Gamma Q$,

$$(\theta(Y) g_{T\mathcal{F}})(U, V) = -2g(Y, \alpha(U, V)). \tag{1}$$

The Weingarten map $W(s) : T\mathcal{F} \to T\mathcal{F}$ is defined by

$$g_Q(\alpha(U, V), s) = g(W(s)U, V).$$

Then $\mathrm{Tr}W \in \Gamma Q^*$, and it can be extended to a 1-form $\kappa \in \Omega^1(M)$ by setting $\kappa(V) = 0$ for $V \in \Gamma T\mathcal{F}$, where we have used the identification $T\mathcal{F}^\perp \cong Q$. We call κ the mean curvature 1-form of $\mathcal{F}$ on (M, g).

Recall that a Riemannian foliation is a foliation $\mathcal{F}$ with a holonomy invariant transversal metric g_Q on Q, i.e.,

$$\theta(V) g_Q = 0, \ \forall \ V \in \Gamma T\mathcal{F}.$$

The metric g on $(M, \mathcal{F})$ is called bundle-like if the induced metric g_Q is holonomy invariant, i.e., $\theta(V) g_Q = 0$ for all $V \in \Gamma T\mathcal{F}$, and a Riemannian foliation $\mathcal{F}$ is called taut if there exists a bundle-like metric for which the mean curvature 1-form $\kappa = 0$.

A differential form $\alpha \in \Omega^r(M)$ is basic, if

$$i(V)\alpha = 0, \ \theta(V)\alpha, \forall \ V \in \Gamma T\mathcal{F}.$$

Denote by $\Omega_B^* = \Omega_B^*(\mathcal{F})$ the set of all basic forms, and the exterior differential $d_B = d|_{\Omega_B}$. By Cartan's magic formula, it can be checked that (Ω_B^*, d_B) forms a sub-complex of the de Rham complex (Ω^*, d). The corresponding cohomology

$$H_B^*(\mathcal{F}) = H_B^*(\mathcal{F}; \mathbb{R})$$

is called the basic cohomology of $\mathcal{F}$.

If $T\mathcal{F}$ is oriented, the foliation $\mathcal{F}$ with dimension p is then said to be tangentially oriented. The p-form $\chi_\mathcal{F}$ defined by

$$\chi_\mathcal{F}(Y_1, \ldots, Y_p) = \det\left(g(Y_i, E_j)_{ij}\right), \forall \ Y_1, \ldots, Y_p \in \Gamma TM,$$

is called the characteristic form of $\mathcal{F}$, where $\{E_1, E_2, \ldots, E_p\}$ is a local oriented orthonormal frame of $T\mathcal{F}$.

Consider the multiplicative filtration of the de Rham complex $\Omega^* = \Omega^*(M)$ as follows

$$F^r\Omega^m = \{\alpha \in \Omega^m \mid i(V_1) \cdots i(V_{m-r+1})\alpha = 0 \text{ for } V_1, \ldots, V_{m-r+1} \in \Gamma T\mathcal{F}\}.$$

Obviously,

$$F^0\Omega^m = \Omega^m \text{ and } \quad F^{m+1}\Omega^m = 0.$$

Furthermore, for the foliation $(M, \mathcal{F})$, we have

$$F^{q+1}\Omega^m = 0 \quad (q = \mathrm{codim}\,\mathcal{F}). \tag{2}$$

3. Holonomy invariant transversal almost complex structure

If the foliation $\mathcal{F}$ is of even codimension, and there exists almost complex structure J on Q, i.e., an endomorphism $J : Q \to Q$ such that $J^2 = -Id_Q$, then extend J onto TM by setting $JX = 0$ for $X \in T\mathcal{F}$. Such J is called the transversal almost complex structure.

Lemma 3.1. *For an even codimensional Riemannian foliation* $(M, \mathcal{F})$ *with a taut Riemannian metric* $g = g_{T\mathcal{F}} \oplus g_{T\mathcal{F}^\perp}$, *if there exists a transversal almost complex structure* J *satisfying* $\theta(V)J = 0$ *for any* $V \in \Gamma T\mathcal{F}$ (*we call such* J *to be holonomy invariant*), *then the new metric* g_J *defined by*

$$g_J(X,Y) = \begin{cases} g_{T\mathcal{F}}(X,Y) & \text{for } X, Y \in \Gamma T\mathcal{F} \\ g_{T\mathcal{F}^\perp}(X,Y) + g_{T\mathcal{F}^\perp}(JX, JY) & \text{for } X, Y \in \Gamma T\mathcal{F}^\perp \end{cases}$$

is also taut.

Proof. Since $\theta(V)J = 0$ for any $V \in \Gamma T\mathcal{F}$ and g is bundle-like,

$$(\theta(V)g_{J,Q})(s, s') = (\theta(V)g_Q)(s, s') + (\theta(V)g_Q)(Js, Js') = 0,$$

i.e., g_J is also bundle-like.

For the tautness part, let $e_1, \dots, e_n$ be an orthonormal basis of T_xM such that $e_1, \dots, e_p \in T\mathcal{F}_x$ and $e_{p+1}, \dots, e_n \in T\mathcal{F}_x^\perp$. Then by (1), we have the mean curvature 1-form κ for g,

$$\begin{aligned} \kappa(s)_x &= \operatorname{Tr} W(s)_x \\ &= \sum_{i=1}^{p} g\left(W(s)e_i, e_i\right) \\ &= \sum_{i=1}^{p} g_Q\left(\alpha\left(e_i, e_i\right), s\right) \\ &= -\frac{1}{2} \sum_{i=1}^{p} \left(\theta(s) g_{T\mathcal{F}}\right)(e_i, e_i), \end{aligned}$$

which shows that κ is independent of g_Q.

We denote by κ_J the mean curvature 1-form with respect to g_J. Since g is taut, κ vanishes, and so is κ_J, i.e., g_J is also taut. ☐

In the sequel, we still denote this g_J by g, and call the taut Riemannian metric g compatible with J. In this case, define the 2-form $F(\cdot, \cdot) = g(J\cdot, \cdot)$, then we have that for any $V \in \Gamma T\mathcal{F}$,

$$i(V)F = 0,$$

and

$$[\theta(V)F](s_1, s_2) = [\theta(V)g](Js_1, s_2) = 0, \ \forall\ s_1, s_2 \in Q.$$

Hence, F is a basic 2-form, and $(\mathcal{F}, g, J, F)$ is called a transversal almost Hermitian structure.

4. C^∞-pure and full

For an even codimensional Riemannian foliation $\mathcal{F}$ on M endowed with a transversal almost complex structure J satisfying $\theta(V)J = 0$, $\forall\ V \in T\mathcal{F}$, denote by Λ^2_B the bundle of real basic 2-forms. Since $\theta(V)J = 0$, $\forall\ V \in T\mathcal{F}$, we have a well-defined action of J on Λ^2_B by:

$$\begin{aligned} J :\ & \Lambda^2_B \to \Lambda^2_B \\ & \alpha(\cdot,\cdot) \mapsto \alpha(J\cdot, J\cdot). \end{aligned}$$

Then by the formula:

$$\alpha(\cdot,\cdot) = \frac{\alpha(\cdot,\cdot) + \alpha(J\cdot, J\cdot)}{2} + \frac{\alpha(\cdot,\cdot) - \alpha(J\cdot, J\cdot)}{2},$$

we get a splitting

$$\Lambda^2_B = \Lambda^+_J \oplus \Lambda^-_J,$$

where Λ^+_J is the bundle of J-invariant basic 2-forms, and Λ^-_J is the bundle of J-anti-invariant basic 2-forms.

Let Ω^2_B be the space of basic 2-forms on M, Ω^+_J (Ω^-_J) the space of J-invariant (J-anti-invariant) basic 2-forms.

Definition. Let $\mathcal{Z}^2_B$ be the space of basic closed 2-forms on M, and let $\mathcal{Z}^\pm_J = \mathcal{Z}^2_B \cap \Omega^\pm_J$. Define

$$H^\pm_J(\mathcal{F}) = \left\{\mathfrak{a} \in H^2_B(\mathcal{F};\mathbb{R}) \mid \exists \alpha \in \mathcal{Z}^\pm_J \text{ such that } [\alpha] = \mathfrak{a}\right\},$$

and the dimension of $H^\pm_J(\mathcal{F})$ are denoted by $h^\pm_J$ respectively.

It is obvious that

$$H^+_J(\mathcal{F}) + H^-_J(\mathcal{F}) \subseteq H^2_B(\mathcal{F};\mathbb{R}).$$

Definition. J is said to be C^∞-pure if $H^+_J(\mathcal{F}) \cap H^-_J(\mathcal{F}) = 0$, and is said to be C^∞-full if $H^+_J(\mathcal{F}) + H^-_J(\mathcal{F}) = H^2_B(\mathcal{F};\mathbb{R})$. J is C^∞-pure and full if $H^+_J(\mathcal{F}) \oplus H^-_J(\mathcal{F}) = H^2_B(\mathcal{F};\mathbb{R})$.

The main result is the following:

Theorem 4.1. *Given a codimension four taut Riemannian foliation $\mathcal{F}$ on a closed smooth manifold M, if J is a transversal almost complex structure satisfying $\theta(V)J = 0$ for any $V \in \Gamma T\mathcal{F}$, then J is C^∞-pure and full.*

Remark 4.2. The condition that $\theta(V)J = 0$ for any $V \in \Gamma T\mathcal{F}$ seems to be necessary. One of the reason is we need this condition to guarantee J preserves basic 2-forms. The other is that for a taut metric, we can easily construct a J compatible taut metric and the corresponding transversal fundamental 2-form will be a basic form.

Remark 4.3. For a K-contact manifold (M, ξ, η, ϕ, g), we have proved that ϕ is C^∞-pure and full [10]. For the characteristic foliation $\mathcal{F}_\xi$, g is taut and $\theta(\xi)\phi = 0$, so this can be considered as a special case of Theorem 4.1.

In order to prove Theorem 4.1, we do some preparation. Let g be a bundle-like metric inducing g_Q on Q. Define the Hodge star operator:

$$\bar{*} : \Omega^r_B(\mathcal{F}) \to \Omega^{q-r}_B(\mathcal{F})$$

as follows:

$$\bar{*}\alpha = (-1)^{p(q-r)} * (\alpha \wedge \chi_{\mathcal{F}}).$$

The relation between $\bar{*}$ and the Hodge star operator $*$ with respect to g is [9]

$$*\alpha = \bar{*}\alpha \wedge \chi_{\mathcal{F}},$$

where $\chi_{\mathcal{F}}$ is the characteristic p-form of $\mathcal{F}$ defined in Section 2.

The scalar product in $\Omega^r_B(\mathcal{F})$ is defined by

$$\langle \alpha, \beta \rangle_B = \int_M \alpha \wedge \bar{*}\beta \wedge \chi_{\mathcal{F}},$$

which is just the restriction of the usual scalar product on $\Omega^r(M)$ to the subspace $\Omega^r_B(\mathcal{F})$ [9].

Define the formal adjoint $\delta_B : \ \Omega^r_B(\mathcal{F}) \to \Omega^{r-1}_B(\mathcal{F})$ of $d_B = d : \ \Omega^{r-1}_B(\mathcal{F}) \to \Omega^r_B(\mathcal{F})$ by

$$\langle d_B\alpha, \beta \rangle_B = \langle \alpha, \delta_B\beta \rangle_B \, .$$

It was shown in [6, 9] that, on $\Omega^r_B(\mathcal{F})$

$$\delta_B = (-1)^{q(r+1)+1} \bar{*} \, (d_B - \kappa\wedge) \, \bar{*}.$$

Define the basic Laplacian

$$\Delta_B = d_B\delta_B + \delta_B d_B,$$

then set

$$\mathcal{H}^r_B(\mathcal{F}) = \{\text{the harmonic basic } r\text{-forms } \omega \mid \Delta_B\omega = 0\}.$$

We have the following Theorem 7.22 in [9].

Theorem 4.4. *Let $\mathcal{F}$ be a transversally oriented Riemannian foliation on a closed manifold (M, g). Assume g to be bundle-like with $\kappa \in \Omega^1_B(\mathcal{F})$. Then there is a decomposition into mutually orthogonal subspaces*

$$\Omega^r_B \cong \operatorname{im} d_B \oplus \operatorname{im} \delta_B \oplus \mathcal{H}^r_B$$

with finite-dimensional $\mathcal{H}^r_B$.

Remark 4.5. The condition $\kappa \in \Omega^1_B(\mathcal{F})$ can be removed by the basic decomposition of general mean curvature 1-form, see [8].

When the taut foliation $\mathcal{F}$ has codimension $q = 4$, we have $\bar{*}^2 =$id on $\Lambda^2 Q^*$, so we get a decomposition

$$\Lambda^2\mathrm{Q}^* = \Lambda^+\mathrm{Q}^* \oplus \Lambda^-\mathrm{Q}^*,$$

where $\Lambda^{\pm}$ are the ± 1-eigenspace of $\bar{*}$. Suppose $\Omega^{\pm}_B$ are the space of sections of $\Lambda^{\pm}\mathrm{Q}^*$, and denote by $\alpha^+, \ \alpha^-$ the selfdual, anti-selfdual components of a basic

2-form α. Furthermore, we have $\Delta_B \bar{*} = \bar{*}\Delta_B$ (note that if $\kappa \neq 0$, Δ_B and $\bar{*}$ do not commute). Hence,

$$H^2(\mathcal{F}, \mathbb{R}) = \mathcal{H}^2_B(\mathcal{F}) = \mathcal{H}^+_B(\mathcal{F}) \oplus \mathcal{H}^-_B(\mathcal{F}), \tag{3}$$

and we denote the dimension of $\mathcal{H}^2_B(\mathcal{F})$, $\mathcal{H}^+_B(\mathcal{F})$, $\mathcal{H}^-_B(\mathcal{F})$ by b^2_B, b^+_B, b^-_B respectively.

For a codimension four transversal almost Hermitian manifold (M, $\mathcal{F}$, J, g, F), we have the following relation

$$\begin{aligned} \Lambda^+_J &= \mathbb{R}F \oplus \Lambda^-_{g_Q}, \Lambda^+_{g_Q} = \mathbb{R}F \oplus \Lambda^-_J; \\ \Lambda^+_J \cap \Lambda^+_{g_Q} &= \mathbb{R}F, \Lambda^-_J \cap \Lambda^-_{g_Q} = 0. \end{aligned} \tag{4}$$

Hence, similar to [3], we have the following two lemmas:

Lemma 4.6. *If $\alpha \in \Omega^+_B$ and $\alpha = \alpha_h + d\theta + \delta\Psi$ is its basic Hodge decomposition, then $(d\theta)^+_B = (\delta\Psi)^+_B$ and $(d\theta)^-_B = -(\delta\Psi)^-_B$. In particular, the basic 2-form*

$$\alpha - 2(d\theta)^+_B = \alpha_h$$

is harmonic and the 2-form

$$\alpha + 2(d\theta)^-_B = \alpha_h + 2d\theta$$

is closed.

Lemma 4.7. *Let $(M^{p+4}, \mathcal{F}, g, J, F)$ be a closed codimension four taut transversal almost Hermitian manifold. Then $\mathcal{Z}^-_J \subset \mathcal{H}^+_{g_Q}$, and $\mathcal{Z}^-_J \subset H^-_J$ is bijective. Furthermore, $H^-_J = \mathcal{Z}^-_J = \mathcal{H}^{+,F^\perp}_{g_Q}$.*

With the above preparation, we can present the proof of the main result.

Proof of Theorem 4.1. Let g be the J-compatible metric, and F be the basic 2-form. If $\mathfrak{a} \in H^+_J(\mathcal{F}) \cap H^-_J(\mathcal{F})$, let $\alpha' \in \mathcal{Z}^+_J$, $\alpha'' \in \mathcal{Z}^-_J$ be the representative for $\mathfrak{a}$. Then see page 39 in [9],

$$\mathrm{d}\chi_{\mathcal{F}} + \kappa \wedge \chi_{\mathcal{F}} = \varphi_0 \in F^2\Omega^{p+1}.$$

Hence, on a codimension four foliation $(M, \mathcal{F})$, for basic 1-form γ and basic 2-form α'', $\gamma \wedge \alpha'' \wedge \phi_0 \in F^5\Omega^{p+1} = 0$ vanishes. Therefore, by integration by parts, we have

$$\begin{aligned} 0 &= \int_M \alpha' \wedge \alpha'' \wedge \chi_{\mathcal{F}} \\ &= \int_M (\alpha'' + d_B\gamma) \wedge \alpha'' \wedge \chi_{\mathcal{F}} \\ &= \int_M \alpha'' \wedge \alpha'' \wedge \chi_{\mathcal{F}} + \int_M d_B\gamma \wedge \alpha'' \wedge \chi_{\mathcal{F}} \\ &= \int_M \alpha'' \wedge \bar{*}\alpha'' \wedge \chi_{\mathcal{F}} + \int_M \gamma \wedge d_B\alpha'' \wedge \chi_{\mathcal{F}} + \int_M \gamma \wedge \alpha'' \wedge \mathrm{d}\chi_{\mathcal{F}} \end{aligned}$$

$$= \int_M |\alpha''|_g^2 \, dvol + \int_M \gamma \wedge \alpha'' \wedge (\phi_0 - \kappa \wedge \chi_{\mathcal{F}})$$
$$= \int_M |\alpha''|_g^2 \, dvol.$$

Hence, $\alpha'' = 0$, i.e., $\mathfrak{a} = 0$, that's to say $H_J^+(\mathcal{F}) \cap H_J^-(\mathcal{F}) = 0$.

The proof of fullness part is technically almost the same as the proof of Theorem 2.3 in [3]. □

D. Domínguez's remarkable theorem [1] says that for a Riemannian foliation $\mathcal{F}$ on a closed manifold, there always exists a bundle-like metric for $\mathcal{F}$ such that the mean curvature form κ is a basic 1-form. F. Kamber and Ph. Tondeur shows κ should be closed [5]. Furthermore, if $[\kappa] \in H_B^1(\mathcal{F})$ is trivial, then by a suitable conformal change to $g_{T\mathcal{F}}$, the bundle-like metric g can be modified to be a taut metric [5]. Since we have an injective map

$$H_B^1(\mathcal{F}) \to H^1(M),$$

closed and simply connected Riemannian foliation is always taut [9]. Hence, we have the following corollary:

Corollary 4.8. *For a codimension four Riemannian foliation $\mathcal{F}$ on a closed and simply connected smooth manifold M, if J is a transversal almost complex structure satisfying $\theta(V)J = 0$ for any $V \in \Gamma T\mathcal{F}$, then J is C^∞-pure and full.*

5. Bounds on $h_J^\pm$

Under the condition of Theorem 4.1 and by (3), we have

$$h_J^+ + h_J^- = b_2^B = b_B^+ + b_B^-.$$

Furthermore, by relations (4), the following inequalities holds:

$$h_J^+ \geq b_B^-, \; h_J^- \leq b_B^+. \tag{5}$$

This can be strengthened as follows:

Lemma 5.1. *Let $(M, \mathcal{F}, g, J, F)$ be a closed codimension four almost Hermitian taut Riemannian foliation. Assume that the harmonic part F_h of the transversal Hodge decomposition of F is not identically zero. Then*

$$h_J^+ \geq b_B^- + 1, \; h_J^- \leq b_B^+ - 1.$$

Proof. Let $F = F_h + d\theta + \delta\Psi$ be the transversal Hodge decomposition of F, then $F + 2(d\theta)^-$ is a closed J-invariant basic 2-form, and $[F_h + 2d\theta] \in H_B^+ \cap H_B^-$ is nontrivial since F_h is not identically zero. □

A more specific case is when F is closed, i.e., the manifold M in Lemma 5.1 is transversal almost Kähler, we let $\omega = F$.

Theorem 5.2. *If $(M, \mathcal{F}, g, J, \omega)$ is taut transversal almost Kähler of codimension four, then*

$$h_J^+ \geq b_B^- + 1, \; h_J^- \leq b_B^+ - 1.$$

Proof. Since g is taut, $\bar{*}\Delta = \Delta\bar{*}$. Hence, $d\omega = 0$ and $\omega \in \Omega_g^+$ induces that $\delta_B\omega = 0$, i.e., ω is basic harmonic itself. □

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