LOCALIZATION OF THE VORTICITY DIRECTION CONDITIONS FOR THE 3D SHEAR THICKENING FLUIDS

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Abstract. It is obtained that a localization of the vorticity direction coherence conditions for the regularity of the 3D shear thickening fluids to an arbitrarily small space-time cylinder. It implies the regularity of any geometrically constrained weak solution of the system considered independently of the type of the spatial domain or the boundary conditions.

1. Introduction

We focus on the local regularity of solution to a non-Newtonian incompressible fluid which is governed by the following system

$$(1) \qquad \begin{cases} u_t + u \cdot \nabla u - \operatorname{div} \boldsymbol{\sigma} + \nabla \pi = 0, & \text{in } \Omega \times (0, \infty), \\ \operatorname{div} u = 0, & \text{in } \Omega \times (0, \infty), \\ u(0, x) = u_0(x), & \text{in } \Omega, \end{cases}$$

where $u = (u_1, u_2, u_3)^\top$ denotes the unknown velocity of the fluid and π the pressure, and

$$\boldsymbol{\sigma} = |D(u)|^{p-2} D(u), \quad D(u) = \frac{1}{2} \left(\nabla u + (\nabla u)^\top \right).$$

We don't impose the boundary condition since we consider the local property of the system, the following result holds for any boundary conditions.

When $p = 2$, the system reduces to the classical Navier-Stokes equations. It is well known that the Navier-Stokes equations has a global weak solution, see [14]. A remarkable and classical sufficient condition for uniqueness and regularity is the so-called Prodi-Serrin condition, which was obtained by Prodi [18] and Serrin [19, 20]. The geometric approach to study the Navier-Stokes equations was pioneered in [7]. They proved that Lipschitz regularity of $\sin \theta(x, y, t)$ can control the evolution of the enstrophy, where we denote by $\theta(x, y, t)$ the angle between the unit vectors of direction of vorticity at locations x, and y

at time t. Later, Beirão da Veiga and Berselli [4] showed that regularity still holds in the whole space by replacing Lipschitz continuity by $\frac{1}{2}$-Hölder continuity. For a more general class of conditions, see [10]. In the aforementioned results, the geometric conditions were assumed uniformly on a time interval and uniformly in the region of intense vorticity throughout the whole space $\mathbb{R}^3$. In [11], it was shown that it is possible to localize the conditions on coherence of the vorticity direction derived in [10]. Later, Grujić [9] extended the result to the cases when the spatial domain is not the whole space. For the cases when the spatial domain is not the whole space, it is worth noting that Beirão da Veiga [2,3,5] obtained a seminal interesting results under the slip or non-slip boundary conditions.

When $p \neq 2$, it was shown that the system (1) had a global weak solution, see [13,15] for the periodic boundary condition, and [17] for the whole space. When $p > \frac{8}{5}$, Wolf [22] showed the existence of weak solutions with Dirichlet boundary condition. When $\frac{7}{5} < p < 2$, the short time existence results of strong solutions with the periodic boundary condition or in the whole space is obtained in [6,8]. When $p > 2$, the short time existence results of strong solutions is obtained in [1]. When $p \geq \frac{11}{5}$, the global existence of strong solutions is shown in [13] with the periodic boundary condition, see also [15,16]. It is natural to ask whether the strong solution is global when $2 < p < \frac{11}{5}$. In [1], authors gave a Serrin's type regularity criteria for ∇u. In [23], the author of the present paper proved that a weak solution to the non-Newtonian incompressible fluid in the whole space is strong if the direction of the vorticity is $\frac{11-5p}{2}$-Hölder continuous with respect to the space variables when $2 < p < \frac{11}{5}$. This paper aims to establish a local result as Grujić [9]. We will obtain the following result.

Theorem 1.1. *Let* $2 < p < \frac{11}{5}$. *Suppose that* $\Omega \subset \mathbb{R}^3$ *is open and* u *is a weak solution to* (1) *in* $[0, T)$. *Fix a point* (x_0, t_0) *in* $\Omega \times (0, T)$, *and let* $0 < R < 1$ *be such that the open parabolic cylinder* $Q_{2R}(x_0, t_0) = B(x_0, 2R) \times (t_0 - (2R)^2, t_0)$ *is contained in* $\Omega \times (0, T)$.

Suppose that there exist two positive constants K, M *such that the following condition holds,*

$$|\sin \theta(x, y, t)| \leq K|x - y|^{\frac{4+5p-3p^2}{2p}}$$

for all (x, t), (y, t) *in* $Q_{2R} \cap \{|\omega| > M\}$.

Then the localized enstrophy remains uniformly bounded up to $t = t_0$, *i.e.,*

$$\sup_{t \in (t_0 - R^2, t_0)} \int_{B(x_0, R)} |\omega|^2(x, t)dx < \infty,$$

where $\omega = \nabla \times u$.

Remark 1.2. In [23], we proved that if $|\sin \theta(x, y, t)| \leq K|x - y|^{\frac{11-5p}{2}}$, then a weak solution is strong. Noting that $\frac{11-5p}{2} < \frac{4+5p-3p^2}{2p}$ for $2 < p < \frac{11}{5}$, hence Theorem 1.1 seems not to be optimal. However, we believe, this is a technical difficulty, not an essential difficulty.

2. Preliminary results

2.1. Some useful inequalities

Lemma 2.1 ([12] Korn inequality). *Let* $\mathbf{u} \in W^{1,p}(\mathbb{R}^3)$ *and* $p > 1$. *Then*

$$\|\nabla\mathbf{u}\|_p \leq c\|D(\mathbf{u})\|_p,$$

where the constant c *does not depend on* $\mathbf{u}$.

Lemma 2.2 ([21] Hardy-Littlewood-Sobolev theorem). *Set*

$$(I_\lambda f)(x) = \int_{\mathbb{R}^n} |x - y|^{-n+\lambda} f(y)dy,$$

and let $0 < \lambda < n$, $1 \leq p < q < \infty$, $\frac{1}{q} = \frac{1}{p} - \frac{\lambda}{n}$. *Then, for* $p > 1$, *we have*

$$\|I_\lambda(f)\|_q \leq c\|f\|_p,$$

where c *is a positive constant depending only* p, q, n.

2.2. Localization formula

Let $\psi(x,t) = \phi(x)\eta(t)$ be a smooth cut-off function on $Q_{2r}(x_0, t_0)$ satisfying supp $\phi \subset B(x_0, 2r)$, $\phi = 1$ on $B(x_0, r)$, $\frac{|\nabla\phi|}{\phi^\rho} \leq \frac{c}{r}$ for some $\rho \in (0,1)$, $0 \leq \phi \leq 1$ and supp $\eta \subset (t_0 - (2r)^2, t_0]$, $\eta = 1$ on $[t_0 - r^2, t_0]$, $|\eta'| \leq \frac{c}{r^2}$, $0 \leq \eta \leq 1$.

From Section 3 of [9], one has

$$(2) \qquad \phi^2(x)(\omega \cdot \nabla)u \cdot \omega(x) = VST_{loc} + LOT,$$

where

$$VST_{loc} = -cP.V. \int_{B(x_0, 2r)} (\omega(x) \times \omega(y)) \cdot G_\omega(x,y)\phi(y)\phi(x)dy$$

with

$$(G_\omega(x,y))_k = \frac{\partial^2}{\partial x_i \partial y_k} \frac{1}{|x - y|} \omega_i(x)$$

and

$$LOT = \left(\frac{\partial}{\partial x_i} J - \frac{\partial\phi}{\partial x_i} u_j \right) \phi(x)\omega_i(x)\omega_j(x)$$

with

$$J = c \int_{B(x_0, 2r)} \frac{1}{|x-y|} (2\nabla\phi \cdot \nabla u_j + \Delta\phi u_j) \, dy - c \int_{B(x_0, 2r)} \epsilon_{jkl} \frac{1}{|x-y|} \frac{\partial}{\partial y_k} \phi \, \omega_l dy.$$

3. The proof of Theorem 1.1

Proof. In the following proof, for convenience, we set $D = D(u)$ and $D_{ij} = \frac{\partial_i u_j + \partial_j u_i}{2}$. Applying the inner product $\nabla \times (\psi^2 \nabla \times u)$ to (1), we have

$$\int_\Omega \left(u_t + u \cdot \nabla u - \text{div}\left(|D|^{p-2}D \right) \right) \cdot \left(\nabla \times (\psi^2 \nabla \times u) \right) dx = 0.$$

After integrating by parts, we can obtain that

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega}\psi^2|\omega|^2dx + \int_{\Omega}\psi^2\nabla\left(|D|^{p-2}D_{ij}\right)\cdot\nabla D_{ij}dx$$

$$= \int_{\Omega}\psi\partial_t\psi|\omega|^2dx + \int_{\Omega}\partial_j\left(|D|^{p-2}D_{ij}\right)\partial_k u_i\partial_k\psi^2dx$$

$$- \int_{\Omega}\partial_k\left(|D|^{p-2}D_{ij}\right)\partial_k u_i\partial_j\psi^2dx + \int_{\Omega}\text{div}\left(|D|^{p-2}D\right)\cdot\left((\nabla\psi^2)\times\omega\right)dx$$

$$- \int_{\Omega}u\cdot\nabla\omega\cdot\psi^2\omega dx + \int_{\Omega}\omega\cdot\nabla u\cdot\psi^2\omega dx\,,$$

which yields

$$\frac{1}{2}\int_{B(x_0,2r)}\phi^2(x)|\omega|^2(x,s)dx + \int_{Q_{2r}^s}\psi^2\nabla\left(|D|^{p-2}D_{ij}\right)\cdot\nabla D_{ij}dxdt$$

$$= \int_{Q_{2r}^s}\psi\partial_t\psi|\omega|^2dxdt + \int_{\Omega}\partial_j\left(|D|^{p-2}D_{ij}\right)\partial_k u_i\partial_k\psi^2dxdt$$

$$- \int_{Q_{2r}^s}\partial_k\left(|D|^{p-2}D_{ij}\right)\partial_k u_i\partial_j\psi^2dx + \int_{Q_{2r}^s}\text{div}\left(|D|^{p-2}D\right)\cdot\left((\nabla\psi^2)\times\omega\right)dxdt$$

$$+ \int_{Q_{2r}^s}u\cdot\nabla\omega\cdot\psi^2\omega dxdt + \int_{Q_{2r}^s}\omega\cdot\nabla u\cdot\psi^2\omega dxdt$$

$$=: T_1 + \cdots + T_6,$$

where $Q_{2r}^s = B(x_0,2r)\times(t_0-(2r)^2,s)$ for a fixed s in (t_0-2r^2,t_0). Now, we estimate T_i $(i=1,2,\ldots,6)$.

Control of T_1. It is easy to see that

$$|T_1| \le c(r)\int_{Q_{2r}^s}|\omega|^2dxdt\,.$$

Control of $T_2+T_3+T_4$. One has

$$|T_2+T_3+T_4| \le c\int_{Q_{2r}^s}\psi^{1+\rho}|D|^{p-2}|\nabla D||\nabla u|dxdt$$

$$\le \epsilon\int_{Q_{2r}^s}\psi^2|D|^{p-2}|\nabla D|^2dxdt + c\int_{Q_{2r}^s}|\nabla u|^pdxdt\,.$$

Control of T_5. According to the calculation of (2.5) in [11], one has

$$|T_5| \le \epsilon^m\int_{Q_{2r}^s}|u|^m|\psi\omega|^2dxdt + c(r)\int_{Q_{2r}^s}|\omega|^2dxdt \quad \text{for any } m\in(1,2].$$

Notice that

$$
\left(\int_{B_{2r}} |\psi\omega|^{3p} dx \right)^{\frac{1}{3}} \leq c \left(\int_{B_{2r}} |\psi\nabla u|^{3p} dx \right)^{\frac{1}{3}}
$$

$$
= c \left(\int_{B_{2r}} |\nabla(\psi u) - \nabla\psi\, u|^{3p} dx \right)^{\frac{1}{3}}
$$

(3)
$$
\leq c \left(\int_{B_{2r}} |\psi D|^{3p} dx \right)^{\frac{1}{3}} + c \left(\int_{B_{2r}} |u|^{3p} dx \right)^{\frac{1}{3}}
$$

$$
\leq c \left(\int_{B_{2r}} (\psi |D|^{\frac{p}{2}})^6 dx \right)^{\frac{1}{3}} + c \left(\int_{B_{2r}} |u|^{\frac{3p}{3-p}} dx \right)^{\frac{3-p}{3}}
$$

$$
\leq c \int_{B_{2r}} |\nabla(\psi |D|^{\frac{p}{2}})|^2 dx + c \int_{B_{2r}} (|u|^p + |\nabla u|^p) dx
$$

$$
\leq c \int_{B_{2r}} \psi^2 |D|^{p-2} |\nabla D|^2 dx + c \int_{B_{2r}} (|u|^p + |\nabla u|^p) dx,
$$

where we have used Lemma 2.1. One can easily obtain that

$$
\int_{Q_{2r}^s} |u|^m |\psi\omega|^2 dx dt \leq \int_{t_0-2r^2}^s \left(\int_{B_{2r}} |u|^{\frac{3pm}{3p-2}} dx \right)^{1-\frac{2}{3p}} \left(\int_{B_{2r}} |\psi\omega|^{3p} dx \right)^{\frac{2}{3p}} dt\,.
$$

Set $m = 2 - \frac{4}{3p}$, noting that $u \in L^\infty(0, T; L^2(\Omega))$ (u is a weak solution of (1)), by (3) one gets

$$
\int_{Q_{2r}^s} |u|^m |\psi\omega|^2 dx dt
$$

$$
\leq c \int_{t_0-2r^2}^s \left(\int_{B_{2r}} |\psi\omega|^{3p} dx \right)^{\frac{1}{3}} dt + c
$$

$$
\leq c \int_{Q_{2r}^s} \psi^2 |D|^{p-2} |\nabla D|^2 dx dt + c \int_{Q_{2r}^s} (|u|^p + |\nabla u|^p) dx dt + c.
$$

Hence

$$
|T_5| \leq c \int_{Q_{2r}^s} \psi^2 |D|^{p-2} |\nabla D|^2 dx dt + c \int_{Q_{2r}^s} (|u|^p + |\nabla u|^p) dx dt
$$

$$
+ c(r) \int_{Q_{2r}^s} |\omega|^2 dx dt + c\,.
$$

Control of T_6. First, one has

$$
T_6 \leq \left| \int_{Q_{2r}^s \cap \{|\omega| < M\}} \omega \cdot \nabla u \cdot \psi^2 \omega\, dx dt \right| + \left| \int_{Q_{2r}^s \cap \{|\omega| > M\}} \omega \cdot \nabla u \cdot \psi^2 \omega\, dx dt \right|
$$

$$
= T_6^l + T_6^h\,.
$$

It is easy to get that

$$T_6^l \le c \int_{Q_r} |\nabla u|^p dx dt + c\,.$$

For T_3^h, one can deduce from (2) that

$$T_6^h \le c \int_{Q_{2r}^s} \int_{B(x_0,2r)} \frac{1}{|x-y|^{3-\lambda}} \psi(y,t)|\omega|(y,t)\psi(x,t)|\omega|(x,t)|\omega|(x,t)dy dx dt$$
$$+ \text{the lower order terms} = I + I_{LOT}\,,$$

where $\lambda = \frac{4+5p-3p^2}{2p}$. By Hölder's inequality and Lemma 2.2, one has

$$I \le \int_{t_0-2r^2}^s \|\psi\omega(t)\|_{L^\alpha(\mathbb{R}^3)} \|\psi\omega(t)\|_{L^\beta(\mathbb{R}^3)} \|\omega(t)\|_{L^p(B(x_0,2r))} dt$$

$$\le \int_{t_0-2r^2}^{t_0} \|\psi\omega(t)\|_{L^{\tilde\alpha}(\mathbb{R}^3)} \|\psi\omega(t)\|_{L^\beta(\mathbb{R}^3)} \|\omega(t)\|_{L^p(B(x_0,2r))} dt\,,$$

where

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{p} = 1\,, \quad \frac{1}{\tilde\alpha} = \frac{1}{\alpha} - \frac{\lambda}{3}\,,$$

which gives

$$\frac{1}{\tilde\alpha} + \frac{1}{\beta} + \frac{1}{p} = 1 + \frac{\lambda}{3}\,.$$

Choosing θ_1, θ_2 be the two parameters such that

$$\frac{1-\theta_1}{2} + \frac{\theta_1}{3p} = \frac{1}{\tilde\alpha}\,, \quad \frac{1-\theta_2}{2} + \frac{\theta_2}{3p} = \frac{1}{\beta}$$

which gives

$$\theta_1 + \theta_2 = \frac{\frac{1}{p} - \frac{\lambda}{3}}{\frac{1}{2} - \frac{1}{3p}}\,.$$

By interpolating, one can get

$$I \le \int_{t_0-2r^2}^{t_0} \|\psi\omega(t)\|_{L^2(B(x_0,2r))}^{2-\theta_1-\theta_2} \|\psi\omega(t)\|_{L^{3p}(B(x_0,2r))}^{\theta_1+\theta_2} \|\omega(t)\|_{L^p(B(x_0,2r))} dt$$

$$\le \sup_{t\in(t_0-(2r)^2,t_0)} \|\phi\omega(t)\|_{L^2(B(x_0,2r))}^{2-\theta_1-\theta_2} \left(\int_{t_0-2r^2}^{t_0} \|\psi\omega\|_{L^{3p}(B(x_0,2r))}^{\frac{p(\theta_1+\theta_2)}{p-1}} dt \right)^{\frac{p-1}{p}} \|\omega\|_{L^p(Q_{2r})}\,.$$

Since $\lambda = \frac{4+5p-3p^2}{2p}$, one has $\theta_1 + \theta_2 = p-1$ (hence one can choose $\theta_1 = \theta_2 = \frac{p-1}{2}$), then

$$I \le \sup_{t\in(t_0-(2r)^2,t_0)} \|\phi\omega(t)\|_{L^2(B(x_0,2r))}^{3-p} \left(\int_{t_0-2r^2}^{t_0} \|\psi\omega\|_{L^{3p}(B(x_0,2r))}^p dt \right)^{\frac{p-1}{p}} \|\omega\|_{L^p(Q_{2r})}$$

$$\le c\|\omega\|_{L^p(Q_{2r})} \left(\frac{1}{2} \sup_{t\in(t_0-(2r)^2,t_0)} \|\phi\omega(t)\|_{L^2(B(x_0,2r))}^2 + \left(\int_{t_0-2r^2}^{t_0} \|\psi\omega\|_{L^{3p}(B(x_0,2r))}^p dt \right)^{\frac{2}{p}} \right)$$

$$\le c\|\omega\|_{L^p(Q_{2r})}\left(\frac{1}{2}\sup_{t\in(t_0-(2r)^2,t_0)}\|\phi\omega(t)\|^2_{L^2(B(x_0,2r))}+\int_{t_0-2r^2}^{t_0}\|\psi\omega\|^p_{L^{3p}(B(x_0,2r))}dt+c\right)$$

$$\le c\|\omega\|_{L^p(Q_{2r})}\left(\frac{1}{2}\sup_{t\in(t_0-(2r)^2,t_0)}\|\phi\omega(t)\|^2_{L^2(B(x_0,2r))}+\int_{B_{2r}}\psi^2|D|^{p-2}|\nabla D|^2dx\right)$$

$$+c\|\omega\|_{L^p(Q_{2r})}\int_{B_{2r}}(|u|^p+|\nabla u|^p)dx+c\|\omega\|_{L^p(Q_{2r})},$$

where we have used the estimate (3). On the other hand, as the calculations of pages 868–869 in [9], one can get

$$I^h_{LOT}\le c\int_{t_0-(2r)^2}^{t_0}\|\nabla u(t)\|_{L^p(B(x_0,2r))}\|\psi\omega(t)\|^2_{L^{\frac{6p}{3-p+\lambda}}(B(x_0,2r))}dt$$

$$+c(r,\|\nabla u\|_{L^p(Q_{2r})})$$

$$+c(r)\int_{t_0-(2r)^2}^{t_0}\|\nabla u(t)\|_{L^2(B(x_0,2r))}\|\omega(t)\|_{L^2(B(x_0,2r))}\|\psi\omega(t)\|_{L^2(B(x_0,2r))}dt,$$

where $\lambda=\frac{4+5p-3p^2}{2p}$. Note that

$$\int_{t_0-(2r)^2}^{t_0}\|\nabla u(t)\|_{L^p(B(x_0,2r))}\|\psi\omega(t)\|^2_{L^{\frac{6p}{3-p+\lambda}}(B(x_0,2r))}dt$$

$$\le c\int_{t_0-(2r)^2}^{t_0}\|\nabla u(t)\|_{L^p(B(x_0,2r))}\|\psi\omega(t)\|^{3-p}_{L^2(B(x_0,2r))}\|\psi\omega(t)\|^{p-1}_{L^{3p}(B(x_0,2r))}dt$$

$$\le c\|\nabla u\|_{L^p(Q_{2r})}\left(\frac{1}{2}\sup_{t\in(t_0-(2r)^2,t_0)}\|\phi\omega(t)\|^2_{L^2(B(x_0,2r))}+\int_{B_{2r}}\psi^2|D|^{p-2}|\nabla D|^2dx\right)$$

$$+c\|\nabla u\|_{L^p(Q_{2r})}\int_{B_{2r}}(|u|^p+|\nabla u|^p)dx+c\|\nabla u\|_{L^p(Q_{2r})},$$

and

$$\int_{t_0-(2r)^2}^{t_0}\|\nabla u(t)\|_{L^2(B(x_0,2r))}\|\omega(t)\|_{L^2(B(x_0,2r))}\|\psi\omega(t)\|_{L^2(B(x_0,2r))}dt$$

$$\le c\int_{t_0-(2r)^2}^{t_0}\|\nabla u(t)\|_{L^p(B(x_0,2r))}\|\omega(t)\|_{L^p(B(x_0,2r))}\|\psi\omega(t)\|_{L^2(B(x_0,2r))}dt$$

$$\le \epsilon\sup_{t\in(t_0-(2r)^2,t_0)}\|\phi\omega(t)\|^2_{L^2(B(x_0,2r))}+c\|\nabla u\|^4_{L^p(Q_{2r})}.$$

Thus, one obtains

$$I^h_{LOT}\le c\|\nabla u\|_{L^p(Q_{2r})}\left(\frac{1}{2}\sup_{t\in(t_0-(2r)^2,t_0)}\|\phi\omega(t)\|^2_{L^2(B(x_0,2r))}+\int_{B_{2r}}\psi^2|D|^{p-2}|\nabla D|^2dx\right)$$

$$(4)\qquad+c\|\nabla u\|_{L^p(Q_{2r})}\int_{B_{2r}}(|u|^p+|\nabla u|^p)dx+c\|\nabla u\|_{L^p(Q_{2r})}$$

$$+\epsilon\sup_{t\in(t_0-(2r)^2,t_0)}\|\psi\omega(t)\|^2_{L^2(B(x_0,2r))}+c(r)\|\nabla u\|^4_{L^p(Q_{2r})}.$$

Collecting the estimates on T_i $(i = 1, \ldots, 6)$ and (4), letting ϵ be sufficiently small, one can get

$$
(5) \quad \frac{1}{2} \int_{B(x_0,2r)} \phi^2(x)|\omega|^2(x,s)dx + \int_{Q^s_{2r}} \psi^2 \nabla \left(|D|^{p-2}D_{ij}\right) \cdot \nabla D_{ij}dxdt
$$

$$
\leq c(r) \int_{Q^s_{2r}} |\omega|^2 dxdt + c \int_{Q^s_{2r}} (|u|^p + |\nabla u|^p)dxdt
$$

$$
+ c\|\nabla u\|_{L^p(Q_{2r})} \left(\frac{1}{2} \sup_{t \in (t_0-(2r)^2,t_0)} \|\phi\omega(t)\|^2_{L^2(B(x_0,2r))} + \int_{B_{2r}} \psi^2|D|^{p-2}|\nabla D|^2 dx\right)
$$

$$
+ c\|\nabla u\|_{L^p(Q_{2r})} \int_{B_{2r}} (|u|^p + |\nabla u|^p)dx + c(r, \|\nabla u\|_{L^p(Q_{2r})}).
$$

Since u is a weak solution of system (1), i.e.,

$$
u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;W^{1,p}(\Omega)),
$$

as the argument in [9], one can obtain

$$
\int_{B(x_0,2r)} \phi^2(x)|\omega|^2(x,s)dx + \int_{Q^s_{2r}} \psi^2 \nabla \left(|D|^{p-2}D_{ij}\right) \cdot \nabla D_{ij}dxdt
$$

$$
\leq c(r) \int_{Q^s_{2r}} |\omega|^2 dxdt + c \int_{Q_{2r}} (|u|^p + |\nabla u|^p)dxdt + c(r, \|\nabla u\|_{L^p(Q_{2r})})
$$

$$
\leq c(r).
$$

Actually, one has $\|\nabla u\|_{L^p(Q_{2r})} \to 0$, $r \to 0$, hence there exists $\delta > 0$ such that $c\|\nabla u\|_{L^p(Q_{2\rho})} \leq \frac{1}{4}$ for all $\rho \leq \delta$. If $r \leq \delta$, the third term of (5) can be absorbed. If $r > \delta$, one can obtain a desired bound by covering $B_r(x_0,t_0)$ with finitely many balls $B_\delta(z_0,t_0)$ and redoing the proof on each cylinder. Thus, we complete the proof of the theorem. $\qquad\square$

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