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ON THE SOLVABILITY OF A FINITE GROUP BY THE SUM OF SUBGROUP ORDERS

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Abstract. Let G be a finite group and $\sigma_1(G) = \frac{1}{|G|}\sum_{H\leq G}|H|$. Under some restrictions on the number of conjugacy classes of (non-normal) maximal subgroups of G, we prove that if $\sigma_1(G) < \frac{117}{20}$, then G is solvable. This partially solves an open problem posed in [9].

1. Introduction

Given a finite group G, we consider the function

$$\sigma_1(G) = \frac{1}{|G|}\sum_{H\leq G}|H|$$

studied in our previous papers [6, 9]. Recall some basic properties of σ_1:

- if G is cyclic of order n and $\sigma(n)$ denotes the sum of all divisors of n, then $\sigma_1(G) = \frac{\sigma(n)}{n}$;
- σ_1 is multiplicative, i.e., if G_i, $i = 1, 2, \ldots, m$, are finite groups of coprime orders, then $\sigma_1(\prod_{i=1}^m G_i) = \prod_{i=1}^m \sigma_1(G_i)$;
- $\sigma_1(G) \geq \sigma_1(G/H) + \frac{1}{(G:H)}(\sigma_1(H) - 1) \geq \sigma_1(G/H)$ for all $H \trianglelefteq G$.

Let $k(G)$ and $k'(G)$ be the numbers of conjugacy classes of maximal subgroups of G and of non-normal maximal subgroups of G, respectively. The starting point for our discussion is given by the open problem in [9], which asks to study whether there is a constant $c \in (2, \infty)$ such that $\sigma_1(G) < c$ implies the solvability of G. In the current note, we will show that if $k(G) \leq 3$ or $k'(G) \neq 3$, then such a constant is $\frac{117}{20}$.

For the proof of our results, we need the following three theorems from [3] (see Theorems 2 and 1, respectively) and [7] (see Exercise 7 of Section 10.5).

Theorem A. *A finite group G with $k'(G) \leq 2$ is always solvable. In particular, a finite group G with $k(G) \leq 2$ is always solvable.*

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Theorem B. *A finite group G with $k(G) = 3$ is non-solvable if and only if either $G/\Phi(G) \cong \mathrm{PSL}(2,7)$ or $G/\Phi(G) \cong \mathrm{PSL}(2,2^p)$, where p is a prime.*

Theorem C. *A finite group with an abelian maximal subgroup is always solvable.*

We note that similar problems for some other functions related to the structure of a finite group G, for example for the function $\psi(G) = \sum_{x \in G} o(x)$ (where $o(x)$ denotes the order of the element x), have been recently investigated by many authors (see e.g. [1, 2, 5]).

Most of our notation is standard and will not be repeated here. Basic definitions and results on groups can be found in [7]. For subgroup lattice concepts we refer the reader to [8].

2. Main results

We start with an easy but important lemma.

Lemma 2.1. *Let G be a finite group and $[M]$ be a conjugacy class of non-normal maximal subgroups of G. Then*

$$\sum_{H \in [M]} |H| = |G|.$$

Proof. Since $M \subseteq N_G(M)$ and M is not normal, we have $N_G(M) = M$. Therefore

$$|[M]| = (G : N_G(M)) = (G : M),$$

which leads to

$$\sum_{H \in [M]} |H| = |M||[M]| = |M|(G : M) = |G|,$$

as desired. □

We are now able to prove our first main result.

Theorem 2.2. *Let G be a finite group with $k'(G) \neq 3$. If $\sigma_1(G) < \frac{117}{20}$, then G is solvable.*

Proof. For $k'(G) \leq 2$ the conclusion follows by Theorem A.

Assume that $k'(G) \geq 4$ and $\sigma_1(G) < \frac{117}{20}$, but G is not solvable. Then it has no cyclic maximal subgroup by Theorem C. Let $[M_i]$, $i = 1, 2, 3, 4$, be four distinct conjugacy classes of non-normal maximal subgroups of G. We infer that

$$\sigma_1(G) \geq \frac{1}{|G|} \left(|G| + \sum_{i=1}^{4} \sum_{H \in [M_i]} |H| + \sum_{H \leq G,\, H = \text{cyclic}} |H| \right).$$

From Lemma 2.1 we have

$$\sum_{i=1}^{4} \sum_{H \in [M_i]} |H| = \sum_{i=1}^{4} |G| = 4|G|.$$

Also, Theorem 2 of [3] shows that

$$\sum_{H \leq G,\, H = \text{cyclic}} |H| = \sum_{a \in G} \frac{o(a)}{\phi(o(a))} \geq \sum_{a \in G} 1 = |G|.$$

Then

$$\sigma_1(G) \geq \frac{1}{|G|}(|G| + 4|G| + |G|) = 6 > \frac{117}{20},$$

a contradiction. □

Next we will focus on proving our second main result. The following lemma will be helpful to us.

Lemma 2.3. *We have*:

a) $\sigma_1(\mathrm{PSL}(2,7)) > \frac{117}{20}$;

b) $\sigma_1(\mathrm{PSL}(2,2^p)) \geq \frac{117}{20}$, *and the equality holds if and only if* $p = 2$.

Proof. a) By using GAP, we get $\sigma_1(\mathrm{PSL}(2,7)) = \frac{1499}{168} > \frac{117}{20}$, as desired.

b) Assume first that $p \geq 3$ and let $q = 2^p$. Then, by [4], $\mathrm{PSL}(2,q)$ has:

- one subgroup of order 1, namely the trivial subgroup;
- one subgroup of order $q^3 - q$, namely $\mathrm{PSL}(2,q)$;
- three conjugacy classes of maximal subgroups $[M_i]$, $i = 1, 2, 3$;
- $\frac{q(q+1)}{2}$ cyclic subgroups of order m, for every divisor $m \neq 1$ of $q - 1$;
- $\frac{q(q-1)}{2}$ cyclic subgroups of order m, for every divisor $m \neq 1$ of $q + 1$;
- $(q+1)\binom{p}{i}_2$ elementary abelian subgroups of order 2^i, for every $i = 1, 2, \ldots, p$, where

$$\binom{p}{i}_2 = \frac{(2^p - 1)\cdots(2-1)}{(2^i - 1)\cdots(2-1)(2^{p-i} - 1)\cdots(2-1)}$$

is the Gaussian binomial coefficient.

Note that every M_i is non-normal because $\mathrm{PSL}(2,q)$ is a simple group. Thus, by Lemma 2.1, we have

$$\sum_{H \in [M_i]} |H| = |\mathrm{PSL}(2,q)| = q^3 - q.$$

It follows that

$$\begin{aligned}\sigma_1(\mathrm{PSL}(2,q)) &\geq \frac{1}{q^3 - q}\left[1 + 4(q^3 - q) + \frac{q(q+1)}{2}\sum_{m \mid q-1,\, m \neq 1} m\right.\\ &\qquad \left. + \frac{q(q-1)}{2}\sum_{m \mid q+1,\, m \neq 1} m + (q+1)\sum_{i=1}^{p}\binom{p}{i}_2 2^i\right]\\ &\geq \frac{1}{q^3 - q}\left[1 + 4(q^3 - q) + \frac{q(q+1)}{2}(q-1)\right.\end{aligned}$$

$$
\begin{aligned}
&+\frac{q(q-1)}{2}(q+1)+(q+1)\left(2\binom{p}{1}_2+4\binom{p}{2}_2+8\binom{p}{3}_2\right)\Bigg]\\
&=5+\frac{1+(q+1)(q-1)\dfrac{q^2+8q+22}{21}}{q^3-q}\\
&>5+\frac{(q+1)(q-1)\dfrac{q^2+8q+22}{21}}{q^3-q}\\
&=5+\frac{q^2+8q+22}{21q}>\frac{117}{20} \text{ for } q\geq 8,
\end{aligned}
$$

as desired.

Assume now that $p=2$. Then $\mathrm{PSL}(2,2^p)\cong A_5$ and we can easily check that $\sigma_1(A_5)=\frac{117}{20}$, completing the proof. □

Theorem 2.4. *Let G be a finite group with $k(G)=3$. If $\sigma_1(G)<\frac{117}{20}$, then G is solvable.*

Proof. Assume that the statement is false and let G be a counterexample of minimal order. Then Theorem B leads to

$$G/\Phi(G)\cong \mathrm{PSL}(2,7) \text{ or } G/\Phi(G)\cong \mathrm{PSL}(2,2^p), \text{ where } p \text{ is a prime.}$$

If $\Phi(G)\neq 1$, then

$$\sigma_1(G/\Phi(G))\leq\sigma_1(G)<\frac{117}{20}$$

implies that $G/\Phi(G)$ is solvable by the minimality of $|G|$. Since $\Phi(G)$ is nilpotent, and consequently solvable, it follows that G is also solvable, contradicting our assumption. Thus $\Phi(G)=1$, that is

$$G\cong \mathrm{PSL}(2,7) \text{ or } G\cong \mathrm{PSL}(2,2^p) \text{ for a certain prime } p,$$

and Lemma 2.3 implies that $\sigma_1(G)\geq\frac{117}{20}$, a contradiction. □

Using Theorem B and Lemma 3, we also infer the following characterization of A_5.

Theorem 2.5. *Let G be a non-solvable finite group with $k(G)=3$. If $\sigma_1(G)=\frac{117}{20}$, then $G\cong A_5$.*

Proof. Under our hypotheses, we have $G/\Phi(G)\cong\mathrm{PSL}(2,7)$ or $G/\Phi(G)\cong\mathrm{PSL}(2,2^p)$, where p is a prime, by Theorem B.

If $p>2$, then from Lemma 2.3 it follows that $\sigma_1(G/\Phi(G))>\frac{117}{20}$. Therefore

$$\frac{117}{20}=\sigma_1(G)\geq\sigma_1(G/\Phi(G))>\frac{117}{20},$$

a contradiction. Thus $p=2$, that is $G/\Phi(G)\cong A_5$, and so $\sigma_1(G)=\sigma_1(G/\Phi(G))$. On the other hand, we know that

$$\sigma_1(G)\geq\sigma_1(G/\Phi(G))+\frac{1}{(G:\Phi(G))}(\sigma_1(\Phi(G))-1).$$

Then $\sigma_1(\Phi(G)) = 1$, i.e., $\Phi(G)$ is trivial. Consequently, $G \cong A_5$, completing the proof. □

Finally, we formulate a natural problem concerning our study.

Open problem. Let G be an *arbitrary* finite group with $\sigma_1(G) < \frac{117}{20}$. Is it true that G is solvable?

Note that if the condition $\sigma_1(G) < \frac{117}{20}$ does not imply the solvability of G, then a counterexample G of minimal order would be a just non-solvable group with $k'(G) = 3$ by Theorem 2.2 and $k(G) \geq 4$ by Theorem 2.4. Thus G would contain at least one normal maximal subgroup. Also, G would be a Fitting-free group, that is it has no non-trivial solvable normal subgroup, or equivalently it has no non-trivial abelian normal subgroup.

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