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CONVERGENCE PROPERTIES FOR THE PARTIAL SUMS OF WIDELY ORTHANT DEPENDENT RANDOM VARIABLES UNDER SOME INTEGRABLE ASSUMPTIONS AND THEIR APPLICATIONS

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Abstract. Widely orthant dependence (WOD, in short) is a special dependence structure. In this paper, by using the probability inequalities and moment inequalities for WOD random variables, we study the L_p convergence and complete convergence for the partial sums respectively under the conditions of RCI(α), SRCI(α) and R-h-integrability. We also give an application to nonparametric regression models based on WOD errors by using the L_p convergence that we obtained. Finally we carry out some simulations to verify the validity of our theoretical results.

1. Introduction

It is well known that the probability limit theorem and its applications for independent random variables have been studied by many authors, while the assumption of independence is not reasonable in real practice. If the independent case is classical in the literature, the treatment of dependent random variables is more recent. In this article, we are interested in WOD random variables and further study the limiting behavior for the partial sums of WOD random variables under some special integrable assumptions.

1.1. The concepts of widely orthant dependent random variables

The widely orthant dependence structure was introduced by Wang et al. [24] as follows.

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Definition 1.1. For the random variables $\{X_n, n \geq 1\}$, if there exists a finite real sequence $\{g_U(n), n \geq 1\}$ satisfying for each $n \geq 1$ and for all $x_i \in (-\infty, \infty)$, $1 \leq i \leq n$,

$$P(X_1 > x_1, X_2 > x_2, \ldots, X_n > x_n) \leq g_U(n) \prod_{i=1}^{n} P(X_i > x_i),$$

then we say that the $\{X_n, n \geq 1\}$ are widely upper orthant dependent (WUOD, in short); if there exists a finite real sequence $\{g_L(n), n \geq 1\}$ satisfying for each $n \geq 1$ and for all $x_i \in (-\infty, \infty)$, $1 \leq i \leq n$,

$$P(X_1 \leq x_1, X_2 \leq x_2, \ldots, X_n \leq x_n) \leq g_L(n) \prod_{i=1}^{n} P(X_i \leq x_i),$$

then we say that the $\{X_n, n \geq 1\}$ are widely lower orthant dependent (WLOD, in short); if they are both WUOD and WLOD, then we say that the $\{X_n, n \geq 1\}$ are widely orthant dependent (WOD, in short), and $g_U(n), g_L(n), n \geq 1$, are called dominating coefficients.

An array $\{X_{ni}, i \geq 1, n \geq 1\}$ of random variables is called rowwise WOD if for each $n \geq 1$, $\{X_{ni}, i \geq 1\}$ is a sequence of WOD random variables.

Recall that when $g_L(n) = g_U(n) = M$ for some constant $M \geq 1$, the random variables $\{X_n, n \geq 1\}$ are called extended negatively upper orthant dependent (ENUOD, in short) and extended negatively lower orthant dependent (ENLOD, in short), respectively. If they are both ENUOD and ENLOD, then we say that the random variables $\{X_n, n \geq 1\}$ are extended negatively orthant dependent (ENOD, in short), which was proposed by Liu [15]. When $g_L(n) = g_U(n) = 1$ for any $n \geq 1$, the random variables $\{X_n, n \geq 1\}$ are called negatively upper orthant dependent (NUOD, in short) and negatively lower orthant dependent (NLOD, in short), respectively. If they are both NUOD and NLOD, then we say that the random variables $\{X_n, n \geq 1\}$ are negatively orthant dependent (NOD, in short), the concept of which was introduced by Joag-Dev and Proschan [12], and they further pointed out that NA random variables are NOD. Hu [11] introduced the concept of negatively superadditive dependence (NSD, in short) and gave an example illustrating that NSD does not imply NA. Hu [11] posed an open problem whether NA implies NSD. Christofides and Vaggelatou [6] solved this open problem and indicated that NA implies NSD. In addition, Hu [11] pointed out that NSD implies NOD (see Property 2 of Hu [11]). From the statements above, we can see that the class of WOD random variables contains END random variables, NOD random variables, NSD random variables, NA random variables and independent random variables as special cases. Hence, studying the probability limit behavior of WOD random variables and its applications are of great interest.

Since the concept of WOD random variables was introduced by Wang et al. [24], many interesting results and applications have been obtained. See, for example, Wang et al. [24] provided some examples which show that the class of

WOD random variables contains some common negatively dependent random variables, some positively dependent random variables and some others; in addition, they studied the uniform asymptotics for the finite-time ruin probability of a new dependent risk model with a constant interest rate; Wang et al. [28] studied the asymptotics of the finite-time ruin probability for a generalized renewal risk model with independent strong subexponential claim sizes and widely lower orthant dependent inter-occurrence times; Chen et al. [4] gave a new type of Nagaev's inequality for WOD random variables and gave some applications including the strong law of large numbers, the complete convergence, the a.s. elementary renewal theorem and the weighted elementary renewal theorem; Wang [25] provided the upper and lower bounds of large deviations for WOD random variables; Xi et al. [31] showed some convergence properties for partial sums of WOD random variables and presented some statistical applications; Wu et al. [30] made further research on complete moment convergence for WOD random variables under some mild conditions; Chen et al. [3] extended the Spitzer's law to a version under the WOD random variables; Ding et al. [7] gave some consistency results of wavelet estimators for the nonparametric regression models under WOD random errors; Shen and Wu [20] established the complete q-th moment convergence for WOD random variables and provided some statistical applications, and so forth.

The main purpose of the paper is to present some limiting behaviors for the partial sums of WOD random variables under some integrable assumptions by using some probability inequalities and moment inequalities. We further study the L_p convergence and complete convergence for arrays of rowwise WOD random variables under the conditions of RCI(α), SRCI(α) and R-h-integrability. In addition, we will apply the L_p convergence to nonparametric regression models and investigate the mean consistency for the nonparametric regression estimator based on WOD errors.

The following concept of stochastic domination will be used in this work.

Definition 1.2. An array $\{X_{ni}, i \geq 1, n \geq 1\}$ of random variables is said to be stochastically dominated by a random variable X if there exists a positive constant C such that

$$P(|X_{ni}| > x) \leq CP(|X| > x)$$

for all $x \geq 0, i \geq 1$ and $n \geq 1$.

1.2. Some concepts of integrability

Complete convergence and strong law of large numbers for weighted sums or partial sums of random variables play important roles in the area of limit theorems in probability theory and mathematical statistics. Conditions of independence and identical distribution of random variables are basic in historic results due to Bernoulli, Borel or Kolmogorov. Since then, many attempts have been made to relax these strong conditions. For example, independence has been relaxed to pairwise independence or pairwise negative quadrant dependence or, even replaced by conditions of dependence such as mixing, martingale, positive dependence or negative dependence. In order to relax the identical distribution, several other conditions have been considered, such as stochastic domination by an integrable random variable. The classical notion of uniform integrability of a sequence $\{X_n, n \geq 1\}$ of integrable random variables is defined through the following condition:

$$\lim_{a\to\infty} \sup_{n\geq 1} E|X_n|I(|X_n| > a) = 0.$$

Chandra and Goswami [2] introduced the following two special kinds of uniform integrability, which are weaker than the classical one.

Definition 1.3. For $\alpha \in (0, \infty)$, a sequence $\{X_n, n \geq 1\}$ of random variables is said to be residually Cesàro α-integrable ($RCI(\alpha)$, in short) if

$$\sup_{n\geq 1} \frac{1}{n}\sum_{i=1}^{n} E|X_i| < \infty \text{ and } \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} E(|X_i| - i^{\alpha})I(|X_i| > i^{\alpha}) = 0.$$

Definition 1.4. For $\alpha \in (0, \infty)$, a sequence $\{X_n, n \geq 1\}$ of random variables is said to be strong residually Cesàro α-integrable ($SRCI(\alpha)$, in short) if

$$\sup_{n\geq 1} \frac{1}{n}\sum_{i=1}^{n} E|X_i| < \infty \text{ and } \sum_{n=1}^{\infty} \frac{1}{n} E(|X_n| - n^{\alpha})I(|X_n| > n^{\alpha}) < \infty.$$

Wang and Hu [26] introduced a much weaker concept of uniform integrability named R-h-integrability as follows.

Definition 1.5. The array $\{X_{ni}, u_n \leq i \leq v_n, n \geq 1\}$ is said to be residually h-integrable (R-h-integrable, in short) with exponent $p > 0$ if

$$\sup_{n\geq 1} \frac{1}{k_n}\sum_{i=u_n}^{v_n} E|X_{ni}|^p < \infty$$

and

$$\lim_{n\to\infty} \frac{1}{k_n}\sum_{i=u_n}^{v_n} E\left(|X_{ni}| - h^{1/p}(n)\right)^p I\left(|X_{ni}|^p > h(n)\right) = 0.$$

Under the condition of R-h-integrability with exponent p, Wang and Hu [26] established some weak laws of large numbers for arrays of dependent random variables. Noting that

$$\left(|X_{ni}| - h^{1/p}(n)\right)^p I(|X_{ni}|^p > h(n)) \leq |X_{ni}|^p I(|X_{ni}|^p > h(n)),$$

the concept of R-h-integrability with exponent p is weaker than h-integrability with exponent p, which was introduced by Sung et al. [22].

Throughout the paper, let $\{X_n, n \geq 1\}$ be a sequence of WOD random variables with dominating coefficients $g_U(n)$, $g_L(n)$, $n \geq 1$. Let $\{X_{ni}, u_n \leq i \leq v_n, n \geq 1\}$ be an array of rowwise WOD random variables, where $\{u_n, n \geq 1\}$ and $\{v_n, n \geq 1\}$ are two sequences of integers (not necessary positive or finite) such that $v_n > u_n$ for all $n \geq 1$ and $v_n - u_n \to \infty$ as $n \to \infty$. Let $\{k_n, n \geq 1\}$ be a sequence of positive numbers such that $k_n \to \infty$ as $n \to \infty$, and $\{h(n), n \geq 1\}$ be an increasing sequence of positive constants with $h(n) \uparrow \infty$ as $n \uparrow \infty$. Denote $g(n) = \max\{g_U(n), g_L(n)\}$ and $S_n = \sum_{i=1}^{n} X_i$. Let C denote a positive constant, which can be different in various places. $a_n = O(b_n)$ means $a_n \leq C b_n$.

1.3. Some lemmas

In this subsection, we will present some important lemmas, which play important roles in proving the main results.

The first one is a basic property for WOD random variables, which can be found in Wang et al. [27] for instance.

Lemma 1.1. *Let $X_1, X_2, \ldots, X_n$ be WOD and $f_1, f_2, \ldots, f_n$ be all nondecreasing (or all nonincreasing). Then the random variables $f_1(X_1)$, $f_2(X_2)$, $\ldots$, $f_n(X_n)$ are also WOD.*

The next one is about the Marcinkiewicz-Zygmund type moment inequality and Rosenthal type moment inequality for WOD random variables, which can be found in Wang et al. [27].

Lemma 1.2. *Let $p \geq 1$ and $\{X_n, n \geq 1\}$ be a sequence of WOD random variables with $EX_n = 0$ and $E|X_n|^p < \infty$ for each $n \geq 1$. Then there exist positive constants $C_1(p)$ and $C_2(p)$ depending only on p such that*

$$E\left|\sum_{i=1}^{n} X_i\right|^p \leq [C_1(p) + C_2(p)g(n)] \sum_{i=1}^{n} E|X_i|^p \quad \text{for } 1 \leq p \leq 2$$

and

$$E\left|\sum_{i=1}^{n} X_i\right|^p \leq C_1(p) \sum_{i=1}^{n} E|X_i|^p + C_2(p)g(n) \left(\sum_{i=1}^{n} EX_i^2\right)^{p/2} \quad \text{for } p \geq 2.$$

Using Lemma 1.2, we can get the following lemma by the same argument as that in Theorem 2.3.1 of Stout [21].

Lemma 1.3. *Let $p \geq 1$ and $\{X_n, n \geq 1\}$ be a sequence of WOD random variables with $EX_n = 0$ and $E|X_n|^p < \infty$ for any $n \geq 1$. Then there exist two positive constants $C_1(p)$ and $C_2(p)$ depending only on p such that*

$$E\left(\max_{1 \leq j \leq n} \left|\sum_{i=1}^{j} X_i\right|^p\right) \leq C_1(p)(\log n)^p g(n) \sum_{i=1}^{n} E|X_i|^p \quad \text{for } 1 < p \leq 2$$

and

$$E\left(\max_{1 \leq j \leq n} \left|\sum_{i=1}^{j} X_i\right|^p\right) \leq C_1(p)(\log n)^p \sum_{i=1}^{n} E|X_i|^p$$

$$+ C_2(p)(\log n)^p g(n) \left(\sum_{i=1}^{n} EX_i^2 \right)^{p/2} \quad \text{for } p \geq 2,$$

where $\log n = \ln \max(n, e)$.

The following lemma can be found in Remark 3 of Landers and Rogge [13].

Lemma 1.4. *For sequences* $\{a_n, n \geq 1\}$ *and* $\{b_n, n \geq 1\}$ *of nonnegative real numbers, if* $\sup_{n\geq 1} n^{-1} \sum_{i=1}^{n} a_i < \infty$*, then*

$$\sum_{i=1}^{n} a_i b_i \leq \left(\sup_{m \geq 1} m^{-1} \sum_{i=1}^{m} a_i \right) \sum_{i=1}^{n} b_i$$

for each $n \geq 1$.

The last one is a fundamental inequality for stochastic domination. For the proof, one can refer to Wu [29].

Lemma 1.5. *Assume that* $\{X_{ni}, i \geq 1, n \geq 1\}$ *is an array of random variables stochastically dominated by a random variable* X*. Then for all* $\alpha > 0$ *and* $b > 0$*, there exist two positive constants* C_1 *and* C_2 *such that*

$$E|X_{ni}|^\alpha I(|X_{ni}| \leq b) \leq C_1 \left[E|X|^\alpha I(|X| \leq b) + b^\alpha P(|X| > b) \right]$$

and

$$E|X_{ni}|^\alpha I(|X_{ni}| > b) \leq C_2 E|X|^\alpha I(|X| > b).$$

Consequently, $E|X_{ni}|^\alpha \leq CE|X|^\alpha$.

2. L_p convergence for the partial sums under the condition of RCI(α)

In this section, we will investigate the L_p convergence for arrays of row-wise WOD random variables under the condition of RCI(α) by using the Marcinkiewicz-Zygmund type moment inequality and Rosenthal type moment inequality.

The first theorem deals with the case $1 \leq p < 2$.

Theorem 2.1. *Let* $1 \leq p < 2$ *and* $\{X_n, n \geq 1\}$ *be a sequence of WOD random variables. Suppose that the following conditions hold true*:

(i) $\{|X_n|^p, n \geq 1\}$ *is* $RCI(\alpha)$ *for some* $\alpha \in (0, \frac{1}{p})$;

(ii) $g(n) = O(n^\delta)$ *for some* $0 \leq \delta < (2-p)(\frac{1}{p} - \alpha)$;

(iii) $\lim_{n\to\infty} g(n) \left(\frac{1}{n} \sum_{i=1}^{n} E[(|X_i|^p - i^\alpha) I(|X_i|^p > i^\alpha)] \right) = 0.$

Then

$$n^{-1/p} |S_n - ES_n| \to 0 \text{ in } L_p. \tag{2.1}$$

Proof. For fixed $n \geq 1$, denote

$$Y_n = -n^\alpha I(X_n < n^\alpha) + X_n I(X_n \leq n^\alpha) + n^\alpha I(X_n > n^\alpha),$$

$$Z_n = X_n - Y_n,$$

and

$$S_n^{(1)} = \sum_{i=1}^n Y_i,\ S_n^{(2)} = \sum_{i=1}^n Z_i.$$

It is easy to see that $|Y_n| = \min\{|X_n|, n^\alpha\}$, $|Z_n| = (|X_n| - n^\alpha)I(|X_n| > n^\alpha)$ and $|Z_n|^p \leq (|X_n|^p - n^\alpha)I(|X_n|^p > n^\alpha)$ for all $p \geq 1$. By Lemma 1.1, $\{Y_n - EY_n, n \geq 1\}$ and $\{Z_n - EZ_n, n \geq 1\}$ are both sequences of zero mean WOD random variables.

Note that $S_n - ES_n = S_n^{(1)} - ES_n^{(1)} + S_n^{(2)} - ES_n^{(2)}$. To prove (2.1), it suffices to show

$$n^{-1/p}|S_n^{(1)} - ES_n^{(1)}| \to 0 \text{ in } L_2 \tag{2.2}$$

and

$$n^{-1/p}|S_n^{(2)} - ES_n^{(2)}| \to 0 \text{ in } L_p. \tag{2.3}$$

Using Lemma 1.2 and the definition of RCI(α) of the sequence $\{|X_n|^p, n \geq 1\}$, we obtain

$$\begin{aligned}
n^{-2/p}E|S_n^{(1)} - ES_n^{(1)}|^2 &= n^{-2/p}E\left|\sum_{i=1}^n (Y_i - EY_i)\right|^2 \\
&\leq n^{-2/p}[C_1(p) + C_2(p)g(n)]\sum_{i=1}^n E\left|Y_i - EY_i\right|^2 \\
&\leq Cn^{-2/p}g(n)\sum_{i=1}^n EY_i^2 \\
&\leq Cn^{-2/p+(2-p)\alpha} \cdot n^\delta \sum_{i=1}^n E|Y_i|^p \\
&\leq Cn^{-(2-p)(\frac{1}{p}-\alpha)+\delta}\left(\sup_{n\geq 1} n^{-1}\sum_{i=1}^n E|X_i|^p\right).
\end{aligned}$$

The expression above clearly goes to 0 as $n \to \infty$.

Using Lemma 1.2 again and condition (iii), we have

$$\begin{aligned}
n^{-1}E\left|S_n^{(2)} - ES_n^{(2)}\right|^p &= n^{-1}E\left|\sum_{i=1}^n (Z_i - EZ_i)\right|^p \\
&\leq n^{-1}[C_1(p) + C_2(p)g(n)]\sum_{i=1}^n E|Z_i - EZ_i|^p
\end{aligned}$$

$$\leq Cg(n)\left(\frac{1}{n}\sum_{i=1}^{n}E[(|X_i|^p - i^{\alpha})I(|X_i|^p > i^{\alpha})]\right) \to 0$$

as $n \to \infty$.

This completes the proof of the theorem. □

Remark 2.1. When $g(n) = O(1)$, the third condition (iii) of Theorem 2.1 is equivalent to the second condition of the definition of RCI(α).

Next, we consider the case $p \geq 2$.

Theorem 2.2. *Let $p \geq 2$ and $\{X_n, n \geq 1\}$ be a sequence of WOD random variables with*

$$\sup_{n\geq 1}\frac{1}{n}\sum_{i=1}^{n}E|X_i|^p < \infty.$$

If $g(n) = O(n^{\delta})$ holds for some $0 \leq \delta < p(\theta - 1/2)$ and $\theta > 1/2$, then

$$n^{-\theta}|S_n - ES_n| \to 0 \quad in \quad L_p. \tag{2.4}$$

Proof. By Lemma 1.2 and C_r-inequality, we obtain

$$\begin{aligned}E\left(n^{-\theta}|S_n - ES_n|\right)^p &\leq C_1 n^{-p\theta}\sum_{i=1}^{n}E|X_i|^p + C_2 n^{-p\theta}g(n)\left(\sum_{i=1}^{n}E|X_i|^2\right)^{p/2}\\ &\leq C_1 n^{-p\theta}\sum_{i=1}^{n}E|X_i|^p + C_2 n^{-p\theta+(p/2)-1}g(n)\sum_{i=1}^{n}(EX_i{}^2)^{p/2}\\ &\leq Cn^{-p\theta+(p/2)-1+\delta}\sum_{i=1}^{n}E|X_i|^p\\ &\leq Cn^{-p(\theta-1/2)+\delta}\left(\sup_{n\geq 1}\frac{1}{n}\sum_{i=1}^{n}E|X_i|^p\right) \to 0 \quad \text{as} \quad n \to \infty,\end{aligned}$$

which yields (2.4). The proof is completed. □

3. Complete convergence for the maximum of the partial sums under the condition of SRCI(α)

A sequence $\{X_n, n \geq 1\}$ of random variables is said to converge completely to a constant a if for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty}P(|X_n - a| > \varepsilon) < \infty.$$

In this case we write $X_n \to a$ completely. It is easily seen that complete convergence implies almost sure convergence in view of the Borel-Cantelli lemma.

The condition of SRCI(α) is a "strong" version of the condition of RCI(α). In this section, we will show that each of the theorems in the previous section has a corresponding "strong" analogue in the sense of complete convergence.

Theorem 3.1. *Let $1 < p < 2$, and $\{X_n, n \geq 1\}$ be a sequence of WOD random variables. If $\{|X_n|^p, n \geq 1\}$ is SRCI(α) for some $\alpha \in (0, 1/(2-p))$ and $g(n) = O(n^\delta)$ for some $0 \leq \delta < \min\{p-1, 1-(2-p)\alpha\}$, then*

$$n^{-1} \max_{1\leq i\leq n} |S_i - ES_i| \to 0 \text{ completely.} \tag{3.1}$$

Proof. For each $n \geq 1$, let $m = m_n$ be the integer such that

$$2^{m-1} < n \leq 2^m.$$

Observe that

$$\begin{aligned} n^{-1} \max_{1\leq i\leq n} |S_i - ES_i| &\leq (2^{m-1})^{-1} \max_{1\leq i\leq 2^m} |S_i - ES_i| \\ &= 2 \cdot 2^{-m} \max_{1\leq i\leq 2^m} |S_i - ES_i|. \end{aligned}$$

Hence, to prove (3.1), it suffices to show

$$2^{-m} \max_{1\leq i\leq 2^m} |S_i - ES_i| \to 0 \text{ completely.}$$

Let $Y_n, Z_n, S_n^{(1)}, S_n^{(2)}$ be defined as in the proof of Theorem 2.1. We first prove

$$2^{-m} \max_{1\leq i\leq 2^m} |S_i^{(2)} - ES_i^{(2)}| \to 0 \text{ completely.} \tag{3.2}$$

Noting that

$$\sum_{n=1}^{\infty} P(|X_n - a| > \varepsilon) \leq \sum_{n=1}^{\infty} \frac{E|X_n - a|^k}{\varepsilon^k},$$

it suffices to show

$$\sum_{m=0}^{\infty} E\left(2^{-m} \max_{1\leq i\leq 2^m} \left|S_i^{(2)} - ES_i^{(2)}\right|\right)^p < \infty.$$

By Lemma 1.3, we get

$$\begin{aligned} &\sum_{m=0}^{\infty} E\left(2^{-m} \max_{1\leq i\leq 2^m} \left|S_i^{(2)} - ES_i^{(2)}\right|\right)^p \\ &\leq C \sum_{m=0}^{\infty} 2^{-mp} \cdot g(2^m) \log^p(2^m) \sum_{i=1}^{2^m} E|Z_i - EZ_i|^p \\ &\leq C \sum_{m=0}^{\infty} 2^{-mp} \cdot m^p \cdot 2^{m\delta} \sum_{i=1}^{2^m} E|Z_i|^p \\ &\leq C \sum_{m=0}^{\infty} m^p 2^{-mp+m\delta} \sum_{i=1}^{2^m} E|Z_i|^p \\ &= C \sum_{i=1}^{\infty} E|Z_i|^p \sum_{[m:2^m \geq i]} m^p 2^{-mp+m\delta} \end{aligned}$$

$$\leq C\sum_{i=1}^{\infty}\frac{\ln^2 i-\ln i+1}{i^{p-\delta}}E|Z_i|^p$$

$$\leq C\sum_{i=1}^{\infty}i^{-1}E[(|X_i|^p-i^{\alpha})I(|X_i|^p>i^{\alpha})]<\infty, \tag{3.3}$$

which implies (3.2).

Next, we show that

$$2^{-m}\max_{1\leq i\leq 2^m}|S_i^{(1)}-ES_i^{(1)}|\to 0 \text{ completely.} \tag{3.4}$$

Similar to the proof of (3.3) and noting that $|Y_i|=\min\{|X_i|,i^{\alpha}\}$, we have by Lemmas 1.3 and 1.4 that

$$\begin{aligned}
&\sum_{m=0}^{\infty}E\left(2^{-m}\max_{1\leq i\leq 2^m}\left|S_i^{(1)}-ES_i^{(1)}\right|\right)^2\\
&\leq C\sum_{m=0}^{\infty}2^{-2m}\cdot\log^2(2^m)\cdot g(2^m)\sum_{i=1}^{2^m}EY_i^2\\
&\leq C\sum_{m=0}^{\infty}2^{-2m}m^2 2^{m\delta}\sum_{i=1}^{2^m}i^{(2-p)\alpha}E|X_i|^p\\
&\leq C\sum_{m=0}^{\infty}m^2 2^{-2m+m\delta}\sum_{i=1}^{2^m}i^{(2-p)\alpha}\left(\sup_{k\geq 1}k^{-1}\sum_{i=1}^{k}E|X_i|^p\right)\\
&\leq C\sum_{m=0}^{\infty}2^{-2m+m\delta}m^2\sum_{i=1}^{2^m}i^{(2-p)\alpha}\\
&\leq C\sum_{i=1}^{\infty}i^{(2-p)\alpha}\sum_{[m:2^m\geq i]}m^2 2^{-2m+m\delta}\\
&\leq C\sum_{i=1}^{\infty}[(\ln i)^2+\ln i-1]i^{(2-p)\alpha+\delta-2}<\infty,
\end{aligned}$$

which yields (3.4). This completes the proof of the theorem. □

For the case $p\geq 2$, we have the following theorem.

Theorem 3.2. *Let $p\geq 2$, $\theta>1/2$, and $\{X_n,n\geq 1\}$ be a sequence of WOD random variables with $g(n)=O(n^{\delta})$ for some $0\leq\delta<(\theta-1/2)p$. If*

$$\sup_{n\geq 1}\frac{1}{n}\sum_{i=1}^{n}E|X_i|^p<\infty, \tag{3.5}$$

then

$$n^{-\theta}\max_{1\leq i\leq n}|S_i-ES_i|\to 0 \quad \textit{completely.} \tag{3.6}$$

Proof. Let $m_n, n \geq 1$ be defined as in the proof of Theorem 3.1. To prove (3.6), it suffices to show

$$2^{-m\theta} \max_{1 \leq i \leq 2^m} |S_i - ES_i| \to 0 \text{ completely.}$$

Indeed, by Lemma 1.3 and C_r inequality, we obtain

$$\sum_{m=0}^{\infty} E\left(2^{-m\theta} \max_{1\leq i\leq 2^m} |S_i - ES_i|\right)^p = \sum_{m=0}^{\infty} 2^{-mp\theta} E\left(\max_{1\leq i\leq 2^m} |S_i - ES_i|\right)^p$$

$$\leq \sum_{m=0}^{\infty} 2^{-mp\theta}\left[C_1(p)\log^p(2^m)\sum_{i=1}^{2^m} E|X_i|^p + C_2(p)\log^p(2^m)g(2^m)\left(\sum_{i=1}^{2^m} EX_i^2\right)^{p/2}\right]$$

$$\leq C\sum_{m=0}^{\infty} m^p \cdot 2^{-mp\theta+m\delta}\left[\sum_{i=1}^{2^m} E|X_i|^p + \left(\sum_{i=1}^{2^m} EX_i^2\right)^{p/2}\right]$$

$$\leq C\sum_{m=0}^{\infty} m^p \cdot 2^{-mp\theta-m+mp/2+m\delta}\sum_{i=1}^{2^m} E|X_i|^p$$

$$= C\sum_{i=1}^{\infty} E|X_i|^p \sum_{[m:2^m\geq i]} m^p \cdot 2^{-mp\theta-m+mp/2+m\delta}$$

$$\leq C\sum_{i=1}^{\infty}\left(\int_{\frac{\ln i}{\ln 2}}^{\infty} m^p \cdot 2^{-mp\theta-m+mp/2+m\delta} dm\right) E|X_i|^p$$

$$\leq C\left(\sup_{n\geq 1}\frac{1}{n}\sum_{i=1}^{n} E|X_i|^p\right)\sum_{i=1}^{\infty}\left(\int_{\frac{\ln i}{\ln 2}}^{\infty} m^p \cdot 2^{-mp\theta-m+mp/2+m\delta} dm\right)$$

$$\leq C\sum_{i=1}^{\infty}\frac{(\ln i)^2 - \ln i + 1}{i^{p\theta+1-p/2-\delta}} < \infty,$$

which yields (3.6). This completes the proof of the theorem. □

4. L_p convergence for the partial sums under the condition of R-h-integrability

Inspired by Shen et al. [19], we get the following result on L_p convergence for arrays of rowwise WOD random variables.

Theorem 4.1. *Let* $1 \leq p < 2$, *and* $\{X_{ni}, u_n \leq i \leq v_n, n \geq 1\}$ *be an array of rowwise WOD random variables. Let* $k_n \to \infty$ *and* $h(n) \uparrow \infty$ *as* $n \to \infty$. *Suppose that the following conditions hold*:

(i) $\sup_{n\geq 1} \frac{1}{k_n} \sum_{i=u_n}^{v_n} E|X_{ni}|^p < \infty$;

(ii) $\lim_{n\to\infty} g(v_n - u_n + 1)\frac{1}{k_n}\sum_{i=u_n}^{v_n} E(|X_{ni}| - h^{1/p}(n))^p I(|X_{ni}|^p > h(n)) = 0$;

(iii) $\lim_{n\to\infty}\left(\frac{h(n)}{k_n}\right)^{\frac{2-p}{p}} g(v_n - u_n + 1) = 0.$

Then

$$\frac{1}{k_n^{1/p}}\sum_{i=u_n}^{v_n}(X_{ni} - EX_{ni}) \to 0 \quad in \quad L_p,$$

and, hence, in probability as $n \to \infty$.

Proof. For fixed $n \geq 1$, denote for $u_n \leq i \leq v_n$ that

$$\begin{aligned} Y_{ni} = & -h^{1/p}(n)I(X_{ni} < -h^{1/p}(n)) + X_{ni}I(|X_{ni}| \leq h^{1/p}(n)) \\ & + h^{1/p}(n)I(X_{ni} > h^{1/p}(n)), \\ Z_{ni} = & \, X_{ni} - Y_{ni} \\ = & \, (X_{ni} + h^{1/p}(n))I(X_{ni} < -h^{1/p}(n)) + (X_{ni} - h^{1/p}(n))I(X_{ni} > h^{1/p}(n)), \\ S_n = & \, \frac{1}{k_n^{1/p}}\sum_{i=u_n}^{v_n}(Y_{ni} - EY_{ni}), \ T_n = \frac{1}{k_n^{1/p}}\sum_{i=u_n}^{v_n}(Z_{ni} - EZ_{ni}). \end{aligned}$$

Noting that

$$\frac{1}{k_n^{1/p}}\sum_{i=u_n}^{v_n}(X_{ni} - EX_{ni}) = S_n + T_n, \ n \geq 1,$$

we have by C_r-inequality that

$$E\left|\frac{1}{k_n^{1/p}}\sum_{i=u_n}^{v_n}(X_{ni} - EX_{ni})\right|^p \leq CE|S_n|^p + CE|T_n|^p.$$

To prove $\frac{1}{k_n^{1/p}}\sum_{i=u_n}^{v_n}(X_{ni} - EX_{ni}) \to 0$ in L_p, we only need to show $E|S_n|^p \to 0$ and $E|T_n|^p \to 0$ as $n \to \infty$, where $1 \leq p < 2$.

Firstly, we will show that $E|S_n|^p \to 0$ as $n \to \infty$. Noting that $1 \leq p < 2$, it suffices to show $ES_n^2 \to 0$ as $n \to \infty$.

For fixed $n \geq 1$, we have by Lemma 1.2 and conditions (i) and (iii) that

$$\begin{aligned} ES_n^2 &= E\left|\frac{1}{k_n^{1/p}}\sum_{i=u_n}^{v_n}(Y_{ni} - EY_{ni})\right|^2 \\ &\leq \frac{C}{k_n^{2/p}} g(v_n - u_n + 1)\sum_{i=u_n}^{v_n} EY_{ni}^2 \\ &\leq \frac{C}{k_n^{2/p}}\,[h(n)]^{(2-p)/p}\, g(v_n - u_n + 1)\sum_{i=u_n}^{v_n} E|Y_{ni}|^p \\ &\leq C\left(\frac{h(n)}{k_n}\right)^{(2-p)/p} g(v_n - u_n + 1)\left(\sup_{n\geq 1}\frac{1}{k_n}\sum_{i=u_n}^{v_n} E|X_{ni}|^p\right) \\ &\to 0 \ \text{ as } \ n \to \infty, \end{aligned}$$

which implies that $ES_n^2 \to 0$ as $n \to \infty$, and thus, $E|S_n|^p \to 0$ as $n \to \infty$.

Next, we will show that $E|T_n|^p \to 0$ as $n \to \infty$. Noting that

$$|Z_{ni}| = \left(|X_{ni}| - h^{1/p}(n)\right) I\left(|X_{ni}| > h^{1/p}(n)\right),$$

we have by Lemma 1.2 and condition (ii) that

$$\begin{aligned} E|T_n|^p &= E\left|\frac{1}{k_n^{1/p}}\sum_{i=u_n}^{v_n}(Z_{ni} - EZ_{ni})\right|^p \\ &\le \frac{C}{k_n} g(v_n - u_n + 1)\sum_{i=u_n}^{v_n} E|Z_{ni} - EZ_{ni}|^p \\ &\le \frac{C}{k_n} g(v_n - u_n + 1)\sum_{i=u_n}^{v_n} E|Z_{ni}|^p \\ &\le \frac{C}{k_n} g(v_n - u_n + 1)\sum_{i=u_n}^{v_n} E\left(|X_{ni}| - h^{1/p}(n)\right)^p I\left(|X_{ni}|^p > h(n)\right) \\ &\to 0 \quad \text{as} \quad n \to \infty, \end{aligned}$$

which implies that $E|T_n|^p \to 0$ as $n \to \infty$. This completes the proof of the theorem. □

Remark 4.1. If we take $g(n) = O(1)$, then conditions (i) and (ii) of Theorem 4.1 are actually the two conditions of the definition of R-h integrability. In this case, WOD reduces to END. On the other hand, Shen et al. [19] obtained the similar conclusion as Theorem 4.1 for END random variables. In other words, Theorem 4.1 extends the relevant conclusion in Shen et al. [19].

By Theorem 4.1, we can get the following corollary, which will be applied to nonparametric regression models.

Corollary 4.1. *Let $1 \le p < 2$, $\{X_{ni}, u_n \le i \le v_n, n \ge 1\}$ be an array of rowwise WOD random variables and $\{a_{ni}, u_n \le i \le v_n, n \ge 1\}$ be an array of constants. Suppose that $h(n) \uparrow \infty$, and*

(i) $\sup_{n\ge1}\sum_{i=u_n}^{v_n} |a_{ni}|^p E|X_{ni}|^p < \infty$;

(ii) $\lim_{n\to\infty} g(v_n - u_n + 1)\sum_{i=u_n}^{v_n} |a_{ni}|^p E|X_{ni}|^p I\left(|X_{ni}|^p > h(n)\right) = 0$;

(iii) $\lim_{n\to\infty} g(v_n - u_n + 1)\left(h(n)\cdot \sup_{u_n\le i\le v_n} |a_{ni}|^p\right)^{(2-p)/p} = 0.$

Then

$$\sum_{i=u_n}^{v_n} a_{ni}(X_{ni} - EX_{ni}) \to 0 \quad in \quad L_p,$$

and hence, in probability as $n \to \infty$.

Proof. Denote $\frac{1}{k_n} = \sup_{u_n \le i \le v_n} |a_{ni}|^p$. It follows from (iii) that $k_n \to \infty$ as $n \to \infty$. Without loss of generality, we assume that $a_{ni} \ge 0$ for all $u_n \le i \le v_n$ and $n \ge 1$. Otherwise, we will use a_{ni}^+ and a_{ni}^- instead of a_{ni} respectively and note that $a_{ni} = a_{ni}^+ - a_{ni}^-$. Hence, it is clearly that $\{k_n^{1/p} a_{ni} X_{ni}, u_n \le i \le v_n, n \ge 1\}$ is still an array of rowwise WOD random variables by Lemma 1.1.

Taking $k_n^{1/p} a_{ni} X_{ni}$ instead of X_{ni} in Theorem 4.1, we have by condition (i) that

$$\sup_{n\ge 1} \frac{1}{k_n} \sum_{i=u_n}^{v_n} E|k_n^{1/p} a_{ni} X_{ni}|^p = \sup_{n\ge 1} \sum_{i=u_n}^{v_n} |a_{ni}|^p E|X_{ni}|^p < \infty.$$

It follows by condition (ii) that

$$\begin{aligned}
&\lim_{n\to\infty} g(v_n - u_n + 1) \frac{1}{k_n} \sum_{i=u_n}^{v_n} E\left(|k_n^{1/p} a_{ni} X_{ni}| - h^{1/p}(n)\right)^p I(|k_n^{1/p} a_{ni} X_{ni}|^p > h(n)) \\
\le\ & \lim_{n\to\infty} g(v_n - u_n + 1) \frac{1}{k_n} \sum_{i=u_n}^{v_n} E|k_n^{1/p} a_{ni} X_{ni}|^p I(|k_n^{1/p} a_{ni} X_{ni}|^p > h(n)) \\
\le\ & \lim_{n\to\infty} g(v_n - u_n + 1) \sum_{i=u_n}^{v_n} |a_{ni}|^p E|X_{ni}|^p I(|X_{ni}|^p > h(n)) = 0.
\end{aligned}$$

Hence, the desired result follows from the statements above and Theorem 4.1 immediately. The proof is completed. □

5. An application to nonparametric regression models

In Section 4, we have established the L_p convergence for arrays of rowwise WOD random variables under some uniformly integrable conditions. In this section, we will present an application to nonparametric regression models based on WOD errors by using the L_p convergence obtained in Section 4.

Consider the following nonparametric regression model:

$$Y_{ni} = f(x_{ni}) + \varepsilon_{ni},\ i = 1, 2, \ldots, n,\ n \ge 1, \tag{5.1}$$

where $\{x_{ni}, i = 1, 2, \ldots, n\}$ are known fixed design points from A, and $A \subset \mathbb{R}^m$ is a given compact set for some $m \ge 1$, $f(\cdot)$ is an unknown regression function defined on A, and $\varepsilon_{n1}, \varepsilon_{n2}, \ldots, \varepsilon_{nn}$ are random errors for each $n \ge 1$. As an estimator of $f(\cdot)$, we consider the weighted regression estimator as follows:

$$f_n(x) = \sum_{i=1}^{n} W_{ni}(x) Y_{ni},\ x \in A \subset \mathbb{R}^m, \tag{5.2}$$

where $W_{ni}(x) = W_{ni}(x : x_{n1}, x_{n2}, \ldots, x_{nn}),\ i = 1, 2, \ldots, n$ are the weight functions.

The above weighted regression estimator for nonparametric regression models was first adapted by Georgiev [9] to the fixed design case. Since then, many authors were devoted to studying the asymptotic properties of $f_n(x)$ and providing many interesting results. We refer the readers to Roussas [16], Fan

[8], Roussas et al. [17], Tran et al. [23], Liang and Jing [14], Wang et al. [27], Chen et al. [5], Shen [18], Bruno et al. [1] and Grama et al. [10] for instance. The purpose of this section is to further investigate the mean consistency and weak consistency for the estimator $f_n(x)$ in the nonparametric regression model based on WOD errors by using the results obtained in Section 4.

5.1. Theoretical results

In this subsection, let $c(f)$ denote the set of continuity points of the function f on A. The symbol $\|x\|$ denotes the Euclidean norm. For any fixed design point $x \in A$, the following assumptions on weight functions $W_{ni}(x)$ will be used:

(H_1) $\sum_{i=1}^{n} W_{ni}(x) \to 1$ as $n \to \infty$;

(H_2) $\sum_{i=1}^{n} |W_{ni}(x)| \leq C < \infty$ for all n;

(H_3) $\sum_{i=1}^{n} |W_{ni}(x)| \cdot |f(x_{ni}) - f(x)| I(\|x_{ni} - x\| > \eta) \to 0$ as $n \to \infty$ for all $\eta > 0$.

We point out that the design assumptions (H_1)-(H_3) are regular conditions for nonparametric regression models and are very general. For more details, one can refer to Liang and Jing [14] and Wang et al. [27] for instance. Based on the assumptions above, we present the following result for the nonparametric regression estimator $f_n(x)$.

Theorem 5.1. *Let $\{\varepsilon_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of rowwise WOD random variables with mean zero which is stochastically dominated by a random variable X with $E|X|^p < \infty$ for some $1 < p < 2$. Suppose that the conditions (H_1)-(H_3) hold, and*

$$\max_{1 \leq i \leq n} |W_{ni}(x)| = O(n^{-u}) \quad \text{for some} \quad 0 < u \leq 1. \tag{5.3}$$

If $g(n) = O(n^{\delta})$ for some $0 \leq \delta < \min\{u(p-1), u(2-p)\}$, then for all $x \in c(f)$,

$$f_n(x) \to f(x) \quad \text{in} \quad L_p, \tag{5.4}$$

and thus,

$$f_n(x) \to f(x) \quad \text{in probability.}$$

Proof. For $\eta > 0$ and $x \in c(f)$, we obtain

$$\begin{aligned} |Ef_n(x) - f(x)| \leq & \sum_{i=1}^{n} |W_{ni}(x)| \cdot |f(x_{ni}) - f(x)| I(\|x_{ni} - x\| \leq \eta) \\ & + \sum_{i=1}^{n} |W_{ni}(x)| \cdot |f(x_{ni}) - f(x)| I(\|x_{ni} - x\| > \eta) \end{aligned}$$

$$+ |f(x)| \cdot \left| \sum_{i=1}^{n} W_{ni}(x) - 1 \right|. \tag{5.5}$$

It follows from $x \in c(f)$ that for all $\varepsilon > 0$, there exists a constant $\lambda > 0$ such that for all x' which satisfies $\|x' - x\| < \lambda$, we have $|f(x') - f(x)| < \varepsilon$. Taking $0 < \eta < \lambda$ in (5.5), we obtain that

$$\begin{aligned} |Ef_n(x) - f(x)| \leq \varepsilon \sum_{i=1}^{n} |W_{ni}(x)| \\ &+ \sum_{i=1}^{n} |W_{ni}(x)| \cdot |f(x_{ni}) - f(x)| I(\|x_{ni} - x\| > \eta) \\ &+ |f(x)| \cdot \left| \sum_{i=1}^{n} W_{ni}(x) - 1 \right|. \end{aligned}$$

By assumptions (H_1)-(H_3) and the arbitrariness of $\varepsilon > 0$, we have that for all $x \in c(f)$,

$$\lim_{n \to \infty} Ef_n(x) = f(x).$$

Note that

$$E|f_n(x) - f(x)|^p \leq 2^{p-1} E|f_n(x) - Ef_n(x)|^p + 2^{p-1} |Ef_n(x) - f(x)|^p.$$

Hence, to prove (5.4), it suffices to show

$$E|f_n(x) - Ef_n(x)|^p = E \left| \sum_{i=1}^{n} W_{ni}(x) \varepsilon_{ni} \right|^p \to 0 \quad \text{as} \quad n \to \infty. \tag{5.6}$$

We will apply Corollary 4.1 with $X_{ni} = \varepsilon_{ni}, a_{ni} = W_{ni}(x), u_n = 1, v_n = n$ and $h(n) = n^a$, where $0 < a < p(u - \frac{\delta}{2-p})$. By $E|X|^p < \infty$ and (5.3), we have

$$\begin{aligned} \sup_{n \geq 1} \sum_{i=u_n}^{v_n} |a_{ni}|^p E|X_{ni}|^p &= \sup_{n \geq 1} \sum_{i=1}^{n} |W_{ni}(x)|^p E|\varepsilon_{ni}|^p \\ &\leq C \sup_{n \geq 1} \left(\max_{1 \leq i \leq n} |W_{ni}(x)| \right)^{p-1} \sum_{i=1}^{n} |W_{ni}(x)| \cdot E|X|^p \\ &\leq C \sup_{n \geq 1} n^{-u(p-1)} \leq C < \infty, \end{aligned}$$

$$\begin{aligned} 0 &\leq g(n) \sum_{i=1}^{n} |W_{ni}(x)|^p E|\varepsilon_{ni}|^p I(|\varepsilon_{ni}|^p > h(n)) \\ &\leq Cg(n) \sum_{i=1}^{n} |W_{ni}(x)|^p E|X|^p \\ &\leq Cn^{\delta} \left(\max_{n \geq 1} |W_{ni}(x)|^{p-1} \right) \sum_{i=1}^{n} |W_{ni}(x)| \cdot E|X|^p \end{aligned}$$

$$\leq Cn^{-u(p-1)+\delta} \to 0 \quad \text{as} \quad n \to \infty,$$

and

$$\left(h(n) \sup_{u_n \leq i \leq v_n} |a_{ni}|^p\right)^{(2-p)/p} g(n) \leq Cn^{a(2-p)/p+\delta} \left(\sup_{1\leq i\leq n} |W_{ni}(x)|^p\right)^{(2-p)/p}$$
$$\leq Cn^{[a-p(u-\frac{\delta}{2-p})]\frac{2-p}{p}} \to 0 \quad \text{as} \quad n \to \infty.$$

Thus, the conditions (i)-(iii) in Corollary 4.1 are satisfied. By Corollary 4.1, we can obtain the desired result (5.6), and thus (5.4) holds. The proof is completed. □

5.2. The nearest neighbor weights

In this subsection, we will illustrate that the designed assumptions (H_1)-(H_3) and (5.3) are satisfied for the nearest neighbor weights. Assume that $A = [0, 1]$ and take $x_{ni} = \frac{i}{n}$, $i = 1, 2, \ldots, n$. For any $x \in A$, we rewrite $|x_{n1} - x|, |x_{n2} - x|, \ldots, |x_{nn} - x|$ as follows:

$$|x^{(n)}_{R_1(x)} - x| \leq |x^{(n)}_{R_2(x)} - x| \leq \cdots \leq |x^{(n)}_{R_n(x)} - x|.$$

If $|x_{ni} - x| = |x_{nj} - x|$, then $|x_{ni} - x|$ is permuted before $|x_{nj} - x|$ when $x_{ni} < x_{nj}$.

Let $1 \leq k_n \leq n$, the nearest neighbor weight function estimator of $f(x)$ in model (5.1) is defined as follows:

$$\widetilde{f}_n(x) = \sum_{i=1}^{n} \widetilde{W}_{ni}(x) Y_{ni},$$

where

$$\widetilde{W}_{ni}(x) = \begin{cases} \frac{1}{k_n}, & \text{if } |x_{ni} - x| \leq |x^{(n)}_{R_{k_n}(x)} - x|, \\ 0, & \text{otherwise.} \end{cases}$$

Assume that $f(x)$ is continuous on compact set A. For simplicity, let $k_n = \lfloor n^a \rfloor$ for some $0 < a < 1$.

It is easily checked that for any $x \in [0, 1]$, $\widetilde{W}_{ni}(x)$ satisfies assumptions (H_1)-(H_3) and (5.3).

For (H_1), it is easy to get that

$$\sum_{i=1}^{n} \widetilde{W}_{ni}(x) = \sum_{i=1}^{n} \widetilde{W}_{nR_i(x)}(x) = \sum_{i=1}^{k_n} \frac{1}{k_n} = 1.$$

We can easily get (H_2) by $\widetilde{W}_{ni}(x) \geq 0$.

Next, we prove (H_3). Actually, it follows from the definition of $\widetilde{W}_{ni}(x)$ that

$$\sum_{i=1}^{n} |\widetilde{W}_{ni}(x)| \cdot |f(x_{ni}) - f(x)| I(\|x_{ni} - x\| > \eta)$$

$$\leq C\sum_{i=1}^{n}\frac{(x_{ni}-x)^2|\widetilde{W}_{ni}(x)|}{\eta^2}$$
$$\leq C\sum_{i=1}^{k_n}\frac{(\frac{i}{n})^2}{k_n}$$
$$\leq Cn^{2a-2}\to 0 \text{ as } n\to\infty.$$

Finally, we verify (5.3). By the definition of $\widetilde{W}_{ni}(x)$, we have that

$$\max_{1\leq i\leq n}|\widetilde{W}_{ni}(x)|=\frac{1}{k_n}\leq Cn^{-a}:=Cn^{-u},$$

where $u=a$.

From the statements above, we can see that all conditions of Theorem 5.1 are satisfied.

5.3. Simulation

In this subsection, we will provide some numerical simulations to verify the validity of our theoretical result.

The data are generated from model (5.1). For any fixed $n\geq 3$, let random vector $(\xi_{n1},\xi_{n2},\dots,\xi_{nn})\sim N_n(\mathbf{0},\mathbf{\Sigma})$, where $\mathbf{0}$ represents zero vector and

$$\mathbf{\Sigma}=\begin{pmatrix} 1+\rho^2 & -\rho & 0 & \cdots & 0 & 0 & 0\\ -\rho & 1+\rho^2 & -\rho & \cdots & 0 & 0 & 0\\ 0 & -\rho & 1+\rho^2 & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 1+\rho^2 & -\rho & 0\\ 0 & 0 & 0 & \cdots & -\rho & 1+\rho^2 & -\rho\\ 0 & 0 & 0 & \cdots & 0 & -\rho & 1+\rho^2 \end{pmatrix}_{n\times n},\quad 0<\rho<1.$$

TABLE 1. $E(f)$ for $f(x)=x^{1.5}$ and $f(x)=\sin x$

$f(x)$	x	n=50	n=100	n=200	n=500
$x^{1.5}$	0.25	0.01879762	0.01094119	0.005982647	0.003000944
	0.5	0.01842202	0.0104473	0.005595755	0.002911286
	0.75	0.02424447	0.0105015	0.006491024	0.003075828
$\sin x$	0.25	0.02164711	0.009789948	0.005561452	0.002931394
	0.5	0.01696706	0.01019933	0.005751109	0.002931394
	0.75	0.02465604	0.01128112	0.006392818	0.002988757

By Joag-Dev and Proschan [12], it can be seen that $(\xi_{n1},\xi_{n2},\dots,\xi_{nn})$ is an NA vector for each $n\geq 3$ with finite moment of any order, and thus is a WOD vector. We choose casually that $\rho=0.6$, $u=a=2/3$, $k_n=\lfloor n^{2/3}\rfloor$ and $p=12/7$ in Theorem 5.1. Taking the points $x=0.25,0.5,0.75$ and the sample sizes n as $n=50,100,200,500$, respectively, we use R software to compute $(\widetilde{f}_n(x)-f(x))^p$ for 1000 times and use their arithmetic mean, denote by $E(f)$,

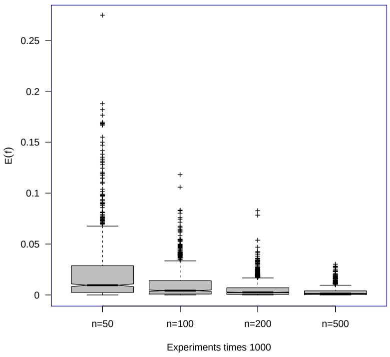

Figure 1: Boxplots of E(f) with x=0.25 and f(x)=x^(1.5)

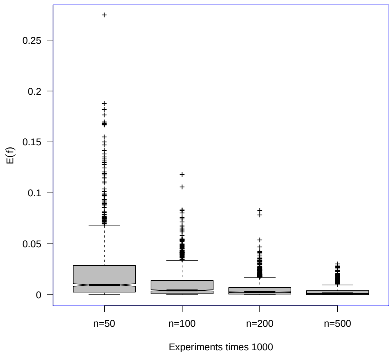

Figure 2: Boxplots of E(f) with x=0.5 and f(x)=x^(1.5)

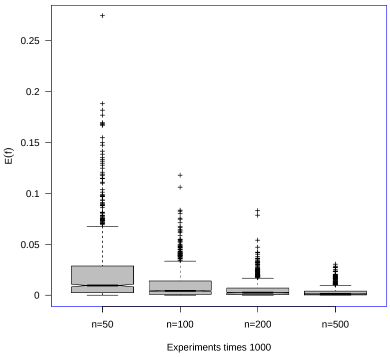

Figure 3: Boxplots of E(f) with x=0.75 and f(x)=x^(1.5)

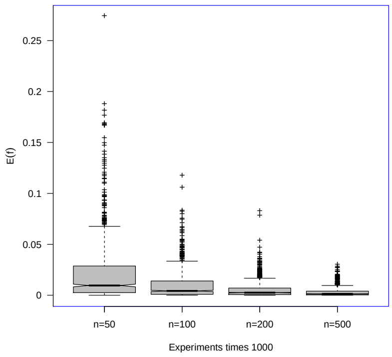

Figure 4: Boxplots of E(f) with x=0.25 and f(x)=sin(x)

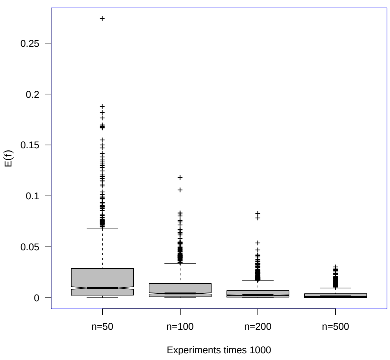

Figure 5: Boxplots of E(f) with x=0.5 and f(x)=sin(x)

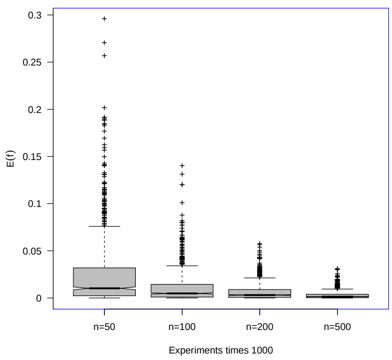

Figure 6: Boxplots of E(f) with x=0.75 and f(x)=sin(x)

to estimate $E(\widetilde{f}_n(x) - f(x))^p$ with $f(x) = x^{1.5}$ and $f(x) = \sin x$, respectively. We obtain the boxplots of $E(f)$ in Figures 1-6 and show the results in Table 1.

Figures 1-3 are the boxplots of $E(f)$ for $f(x) = x^{1.5}$ and Figures 4-6 are the boxplots of $E(f)$ for $f(x) = \sin x$ with $x = 0.25, 0.5, 0.75$, respectively. From Figures 1-6 and Table 1, we can see that no matter $f(x) = x^{1.5}$ or $f(x) = \sin x$,

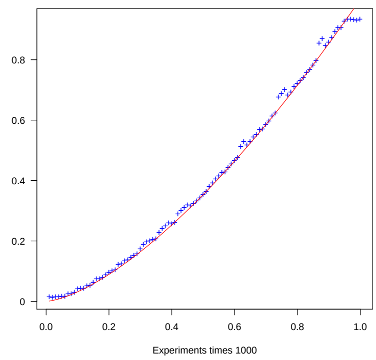

Figure 7: Comparison of $\tilde{f}_n(x)$ and $f(x)=x^{(1.5)}$ with n=200

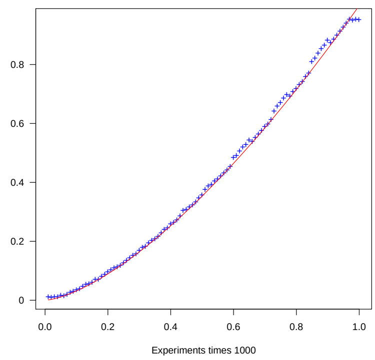

Figure 8: Comparison of $\tilde{f}_n(x)$ and $f(x)=x^{(1.5)}$ with n=500

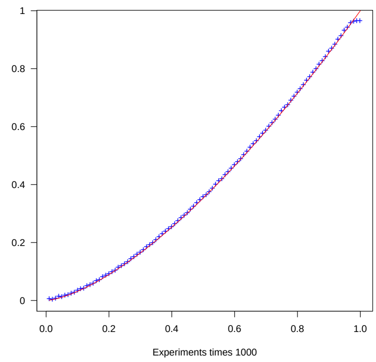

Figure 9: Comparison of $\tilde{f}_n(x)$ and f(x)=x^(1.5) with n=1000

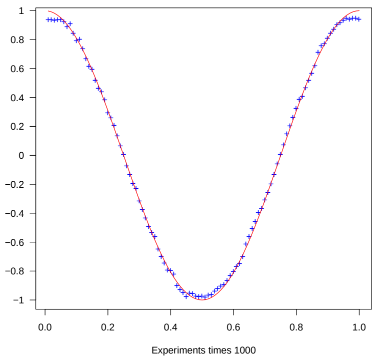

Figure 10: Comparison of $\tilde{f}_n(x)$ and $f(x)=\cos 2*\pi x$ with n=200

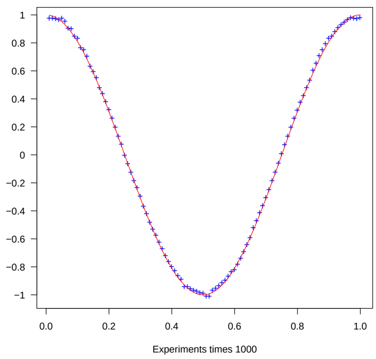

Figure 11: Comparison of $\tilde{f}_n(x)$ and $f(x)=\cos 2*\pi x$ with n=500

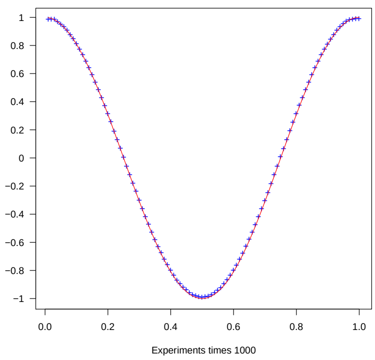

Figure 12: Comparison of $\tilde{f}_n(x)$ and $f(x)=\cos 2*\pi x$ with n=1000

for the fixed points $x = 0.25, 0.5$ or 0.75, the estimation of $E(\widetilde{f}_n(x) - f(x))^p$ decreases markedly to zero as n increases. These simulation results above verify the validity of our theoretical results.

We also present a numerical simulation for the uniformly mean consistency of the nearest neighbor estimator $\widetilde{f}_n(x)$. For $\rho = 0.8$, $k_n = \lfloor n^{4/7} \rfloor$, and every point $x = 0.01, 0.02, \dots, 0.98, 0.99, 1$, we also use R software to compute $\widetilde{f}_n(x)$ for 1000 times and take the mean of 1000 numbers as the final estimation value of $\widetilde{f}_n(x)$ for each point, and then compare the estimations with $f(x) = x^{1.5}$ and $f(x) = \cos 2\pi x$ in Figures 7-12 respectively under different sample sizes.

Figures 7-12 show a very good fit of the uniformly mean consistency result when the sample size n increases. These simulation results above also verify the validity of our theoretical results.

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