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MULTILINEAR CALDERÓN-ZYGMUND OPERATORS AND THEIR COMMUTATORS ON CENTRAL MORREY SPACES WITH VARIABLE EXPONENT

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Abstract. In this paper, we establish the boundedness of the m-linear Calderón-Zygmund operators on product of central Morrey spaces with variable exponent. The corresponding boundedness properties of their commutators with λ-central BMO symbols are also considered. Finally, we prove that the multilinear commutators of Calderón-Zygmund singular integrals introduced by Pérez and Trujillo-González are bounded on central Morrey spaces with variable exponent. Our results improve and generalize some previous classical results to the variable exponent setting.

1. Introduction

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space. We denote by $\mathcal{S}(\mathbb{R}^n)$ the Schwartz class and by $\mathcal{S}'(\mathbb{R}^n)$ the set of all tempered distributions on $\mathbb{R}^n$. Let $K(x, y_1, \dots, y_m)$ be a locally integrable function defined away from the diagonal $x = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$, which satisfies the size estimate

$$|K(x, y_1, \dots, y_m)| \le \frac{A}{(\sum_{k=1}^m |x - y_k|)^{mn}} \tag{1}$$

for some $A > 0$ and all $(x, y_1, \dots, y_m) \in (\mathbb{R}^n)^{m+1}$ with $x \ne y_k$ for some k. Furthermore, assume that for some $\varepsilon > 0$ we have the smoothness estimates

$$|K(x, y_1, \dots, y_m) - K(x', y_1, \dots, y_m)| \le \frac{A|x - x'|^\varepsilon}{(\sum_{k=1}^m |x - y_k|)^{mn+\varepsilon}} \tag{2}$$

whenever $|x - x'| \le \frac{1}{2}\max\{|x - y_1|, \dots, |x - y_m|\}$, and also that

$$|K(x, y_1, \dots, y_i, \dots, y_m) - K(x, y_1, \dots, y_i', \dots, y_m)| \le \frac{A|y_i - y_i'|^\varepsilon}{(\sum_{k=1}^m |x - y_k|)^{mn+\varepsilon}} \tag{3}$$

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whenever $|y_i - y_i'| \le \frac{1}{2}\max\{|x-y_1|, \dots, |x-y_m|\}$ for all $1 \le i \le m$.

Kernels satisfying conditions (1), (2) and (3) are called m-linear Calderón-Zygmund kernels and are denoted by $m - CZK(A, \varepsilon)$. A m-linear operator $T : \mathcal{S}(\mathbb{R}^n) \times \cdots \times \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is said to be associated with K if

$$(4)\quad T(f_1, \dots, f_m)(x) = \int_{(\mathbb{R}^n)^m} K(x, y_1, \dots, y_m) f_1(y_1) \cdots f_m(y_m) dy_1 \cdots dy_m,$$

whenever $f_1, \dots, f_m \in L_c^\infty$, the space of compactly supported bounded functions on $\mathbb{R}^n$, and $x \notin \bigcap_{j=1}^m \mathrm{supp} f_j$. Let T be as in (4) with an $m - CZK(A, \varepsilon)$ kernel. If T is bounded from $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for some $1 < p_1, \dots, p_m < \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, then T is called an m-linear Calderón-Zygmund operator.

A function $b \in L_{\mathrm{loc}}(\mathbb{R}^n)$, the set of all complex-valued locally integrable functions on $\mathbb{R}^n$, is said to belong to $\mathrm{BMO}(\mathbb{R}^n)$, if

$$\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} := \sup_{x \in B} \frac{1}{|B|} \int_B |b(y) - b_B| \, dy < \infty,$$

where the supremum is taken over all balls in $\mathbb{R}^n$, $b_B = |B|^{-1} \int_B f(x) dx$ and $|B|$ is the Lebesgue measure of B. Let $\vec{f} = (f_1, \dots, f_m)$, $\vec{b} = (b_1, \dots, b_m)$ and $b_i \in \mathrm{BMO}(\mathbb{R}^n)$ for $i \in \{1, \dots, m\}$. If $f_1, \dots, f_m \in L_c^\infty$ and $x \notin \bigcap_{j=1}^m \mathrm{supp} f_j$, then the commutator of m-linear Calderón-Zygmund operator T and $\vec{b}$ is defined by

$$(5)\quad [\vec{b}, T]\vec{f}(x) = \int_{(\mathbb{R}^n)^m} \prod_{i=1}^m (b_i(x) - b_i(y_i)) f_i(y_i) K(x, y_1, \dots, y_m) dy_1 \cdots dy_m.$$

Multilinear Calderón-Zygmund operators were introduced by Coifman and Meyer [4,5] and were systematically studied by Grafakos and Torres [14]. They presented a systematic treatment of m-linear Calderón-Zygmund operators and obtained strong type and endpoint weak type estimates of such operators on classical Lebesgue spaces. Since then, the research on these operators is quite active. For instance, Lerner et al. [19] proved the boundedness of m-linear Calderón-Zygmund operators on weighted Lebesgue spaces. Fu, Lin and Lu [12] established the boundedness properties of these operators on the product of central Morrey spaces.

Recently, following the fundamental work of Kováčik and Rákosník [18] on variable Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$ and variable Sobolev spaces $W^{k,p(\cdot)}(\mathbb{R}^n)$ (Here and below the exponent $p(\cdot)$ is a function and not a constant), various function spaces with variable exponent, such as variable Morrey spaces, variable Hardy spaces and variable Besov-type spaces etc., have been widely studied by a large number of authors, see [1,9,11,23,32–34]. These spaces are of interest in their own right, and also have applications to fluid dynamics, image restoration and PDEs with non-standard growth conditions, we refer to [3,15,25] and their references for more information.

The $L^{p(\cdot)}(\mathbb{R}^n)$-boundedness of m-linear Calderón-Zygmund operators and their commutators are independently obtained by Huang and Xu [16] and Tao et al. [29]. Moreover, Tao et al. [29] proved the boundedness of these operators on product of variable Morrey spaces over bounded domains. Subsequently, Wang and Xu [31] further established the boundedness of these operators on variable Morrey spaces on unbounded domains. More boundedness properties of these operators on variable exponent function spaces were also given by Lu and Zhu [22] and Cruz-Uribe et al. [8].

Lu and Yang [20, 21] first introduced the central bounded mean oscillation space $\mathrm{CBMO}_q(\mathbb{R}^n)$, $1 < q < \infty$, which satisfies the following condition:

$$\|f\|_{\mathrm{CBMO}_q(\mathbb{R}^n)} := \sup_{R>0} \left(\frac{1}{|B(0,R)|} \int_{B(0,R)} |f(y) - f_{B(0,R)}|^q \, dy \right)^{1/q} < \infty.$$

The space $\mathrm{CBMO}_q(\mathbb{R}^n)$ can be regarded as a local version of $\mathrm{BMO}(\mathbb{R}^n)$ at the origin. However, the space $\mathrm{CBMO}_q(\mathbb{R}^n)$ depends on q. Obviously, if $q_1 < q_2$, then $\mathrm{CBMO}_{q_2}(\mathbb{R}^n) \subset \mathrm{CBMO}_{q_1}(\mathbb{R}^n)$. Therefore, there is no analogy of the well-known John-Nirenberg inequality of $\mathrm{BMO}(\mathbb{R}^n)$ for the space $\mathrm{CBMO}_q(\mathbb{R}^n)$. One can imagine that the behavior of $\mathrm{CBMO}_q(\mathbb{R}^n)$ may be quite different from that of $\mathrm{BMO}(\mathbb{R}^n)$.

Alvarez, Guzmán-Partida and Lakey [2] studied the relationship between central BMO spaces and Morrey spaces. Furthermore, they introduced λ-central bounded mean oscillation spaces and central Morrey spaces, respectively.

Definition 1. Let $\lambda < 1/n$ and $1 < q < \infty$. The λ-central bounded mean oscillation space $\mathrm{CBMO}^{q,\lambda}(\mathbb{R}^n)$ consists of all $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$\|f\|_{\mathrm{CBMO}^{q,\lambda}(\mathbb{R}^n)} := \sup_{R>0} \left(\frac{1}{|B(0,R)|^{1+\lambda q}} \int_{B(0,R)} |f(y) - f_{B(0,R)}|^q \, dy \right)^{1/q} < \infty. \tag{6}$$

Remark 1.1. If two functions which differ by a constant are regarded as a function in the space $\mathrm{CBMO}^{q,\lambda}(\mathbb{R}^n)$, then $\mathrm{CBMO}^{q,\lambda}(\mathbb{R}^n)$ becomes a Banach space. The space $\mathrm{CBMO}^{q,\lambda}(\mathbb{R}^n)$ when $\lambda = 0$ is just the space $\mathrm{CBMO}_q(\mathbb{R}^n)$ defined above. Apparently, (6) is equivalent to the following condition:

$$\sup_{R>0} \inf_{c\in\mathbb{C}} \left(\frac{1}{|B(0,R)|^{1+\lambda q}} \int_{B(0,R)} |f(y) - c|^q \, dy \right)^{1/q} < \infty.$$

Definition 2. Let $\lambda \in \mathbb{R}$ and $1 < q < \infty$. The central Morrey space $B^{q,\lambda}(\mathbb{R}^n)$ is defined by

$$\|f\|_{B^{q,\lambda}(\mathbb{R}^n)} := \sup_{R>0} \left(\frac{1}{|B(0,R)|^{1+\lambda q}} \int_{B(0,R)} |f(y)|^q \, dy \right)^{1/q} < \infty. \tag{7}$$

Fu, Lin and Lu [12] established λ-central BMO estimates for a class of multisublinear operators on the product of central Morrey spaces. As its special cases, the corresponding results of m-ilinear Calderón-Zygmund operators can be deduced. Moreover, the boundedness on the product of λ-central Morrey spaces for commutators generated by m-linear Calderón-Zygmund operators and λ-central BMO functions are also obtained in the paper [28]. In 2019, Fu et al. [13] introduced the λ-central BMO spaces and the central Morrey spaces with variable exponent and proved the boundedness of the fractional singular integrals and its commutator on those spaces. Inspired by the works above, the first aim of the present paper is to study the boundedness of m-linear Calderón-Zygmund operators and their commutators on the product of λ-central Morrey spaces with variable exponent.

On the other hand, Pérez and Trujillo-González [24] introduced another kind of multilinear commutators defined by

$$T_{\vec{b}}f(x) = \int_{\mathbb{R}^n} \prod_{i=1}^{m} (b_i(x) - b_i(y)) K(x,y) f(y) dy, \tag{8}$$

where T is a standard Calderón-Zygmund singular integral operator, $\vec{b} = (b_1, \ldots, b_m)$ with $b_i \in \mathrm{BMO}(\mathbb{R}^n)$, $i = 1, \ldots, m$, $K(x,y)$ is a standard Calderón-Zygmund kernel, namely, there exists $\sigma > 0$ such that, for all distinct $x, y \in \mathbb{R}^n$ and all z with $2|x-z| < |x-y|$, it verifies

$$|K(x,y)| \leq \frac{C}{|x-y|^n}, \tag{9}$$

$$|K(x,y) - K(z,y)| + |K(y,x) - K(y,z)| \leq C \frac{|x-z|^\sigma}{|x-y|^{n+\sigma}}. \tag{10}$$

Pérez and Trujillo-González in [24] proved that $T_{\vec{b}}$ is bounded on $L^p(\omega)$ for any $\omega \in A_p$, $1 < p < \infty$, where A_p denotes Muckenhoupt's weight class (see [26] for the definition). Later, Tan et al. [27] generalized its boundedness to weighted Lebesgue spaces with variable exponent. Recently, Tao and Shi [28] established the boundedness on central Morrey spaces $B^{q,\lambda}(\mathbb{R}^n)$ for the multilinear commutators $T_{\vec{b}}$ with vector symbol $\vec{b} = (b_1, b_2, \ldots, b_m)$, where each b_i is a λ-central BMO function. These results inspire us to ask whether the multilinear operators $T_{\vec{b}}$ have the similar mapping properties in variable exponent central Morrey spaces? The second aim of this paper is to give an affirmative answer to this question.

As usual, the symbol $\mathbb{N}$ denotes the set of all natural numbers. We denote $B(0,R) := \{y \in \mathbb{R}^n : |y| < r\}$ simply by B, and $B(0,kR)$ by kB for $k \in \mathbb{N}$. f_B denotes the integral average of f on B, i.e., $f_B = \frac{1}{|B|}\int_B f(x)dx$. χ_E is a characteristic function of a measurable set $E \subset \mathbb{R}^n$. $|E|$ denotes the Lebesgue measure of a measurable set E. $p'(\cdot)$ means the conjugate exponent defined by $1/p(\cdot) + 1/p'(\cdot) = 1$. The letter C stands for a positive constant that is independent of the essential variables and not necessarily the same in each occurrence.

2. Preliminaries and main results

In this section, we recall some basic properties of Lebesgue spaces with variable exponent, and state some results as needed. We refer to [6, 10] for more details.

Let $p(\cdot) : \mathbb{R}^n \to [1, \infty)$ be a measurable function. The Lebesgue space with variable exponent $L^{p(\cdot)}(\mathbb{R}^n)$ consists of all measurable functions f on $\mathbb{R}^n$ such that

$$I_{p(\cdot)}(f) := \int_{\mathbb{R}^n} |f(x)|^{p(x)} dx < \infty.$$

This becomes a Banach space when endowed with the norm

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \inf\{\lambda > 0 : I_{p(\cdot)}(f/\lambda) \leq 1\}.$$

Given an open set $\Omega \subset \mathbb{R}^n$, the space $L^{p(\cdot)}_{\rm loc}(\Omega)$ is defined by

$$L^{p(\cdot)}_{\rm loc}(\Omega) = \{f : f \in L^{p(\cdot)}(K) \text{ for all compact subsets } K \subset \Omega\}.$$

In the sequel, we define $\mathcal{P}(\mathbb{R}^n)$ to be the set of measurable function $p(\cdot) : \mathbb{R}^n \to [1, \infty)$ such that

$$p_- := \text{ess inf}_{x\in\mathbb{R}^n} p(x) > 1, \quad p_+ := \text{ess sup}_{x\in\mathbb{R}^n} p(x) < \infty,$$

and

$$\mathcal{B}(\mathbb{R}^n) := \{p(\cdot) \in \mathcal{P}(\mathbb{R}^n) : M \text{ is bounded on } L^{p(\cdot)}(\mathbb{R}^n)\},$$

where M is the Hardy-Littlewood maximal operator defined by

$$Mf(x) = \sup_{x\in\mathbb{R}^n, r>0} r^{-n} \int_{B(x,r)} |f(y)|dy.$$

A measurable function $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ is called globally log-Hölder continuous if it satisfies

$$|p(x) - p(y)| \leq \frac{C}{-\log(|x-y|)}, \quad x, y \in \mathbb{R}^n, \quad |x-y| \leq 1/2, \tag{11}$$

$$|p(x) - p(y)| \leq \frac{C}{\log(e + |x|)}, \quad x, y \in \mathbb{R}^n, \quad |y| \geq |x|. \tag{12}$$

The set of $p(\cdot)$ satisfying (11) and (12) is denoted by $LH(\mathbb{R}^n)$. Cruz-Uribe et al. [7, Theorem 1.1] showed that if $p(\cdot) \in \mathcal{P}(\mathbb{R}^n) \bigcap LH(\mathbb{R}^n)$, then the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, namely, $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$.

When $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $g \in L^{p'(\cdot)}(\mathbb{R}^n)$, the generalized Hölder inequality holds in the form

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \leq r_p \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)}, \tag{13}$$

with $r_p = 1 + 1/p_- - 1/p_+$, see [18, Theorem 2.1].

The subsequent Lemmas 2.1 and 2.2 are due to Izuki [17, Page 203].

Lemma 2.1. *Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then we have*

$$\frac{1}{|B|}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \leq C.$$

Lemma 2.2. *Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then we have for all measurable subsets $E \subset B$,*

$$\frac{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_E\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \leq C\frac{|B|}{|E|}, \ \frac{\|\chi_E\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \leq C\left(\frac{|E|}{|B|}\right)^{\delta_1}, \ \frac{\|\chi_E\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)}} \leq C\left(\frac{|E|}{|B|}\right)^{\delta_2},$$

where δ_1, δ_2 are constants with $0 < \delta_1, \delta_2 < 1$.

Lemma 2.3. *Let $p(\cdot), q(\cdot), r(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ be such that*

$$\frac{1}{r(x)} = \frac{1}{p(x)} + \frac{1}{q(x)}$$

for almost every $x \in \mathbb{R}^n$. Then we have

$$\|fg\|_{L^{r(\cdot)}(\mathbb{R}^n)} \leq 2\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|g\|_{L^{q(\cdot)}(\mathbb{R}^n)}$$

for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $g \in L^{q(\cdot)}(\mathbb{R}^n)$.

Lemma 2.4. *Suppose that $p(\cdot) \in LH(\mathbb{R}^n)$ and $0 < p_- \leq p(x) \leq p_+ < \infty$.*

(i) *For all cubes (or balls) $|Q| \leq 2^n$ and any $x \in Q$, we have*

$$\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} \approx |Q|^{1/p(x)}.$$

(ii) *For all cubes (or balls) $|Q| \geq 1$, we have*

$$\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} \approx |Q|^{1/p_\infty},$$

where $p_\infty = \lim_{x\to\infty} p(x)$.

The proofs of Lemmas 2.3 and 2.4 can be found in [10]. As a consequence of [16, Corollary 2.1] and [7, Theorem 1.2], we obtain the following.

Lemma 2.5. *Let T be a 2-linear Calderón-Zygmund operator. If $p(\cdot)$, $p_1(\cdot)$, $p_2(\cdot) \in \mathcal{P}(\mathbb{R}^n) \bigcap LH(\mathbb{R}^n)$ such that $\frac{1}{p(x)} = \frac{1}{p_1(x)} + \frac{1}{p_2(x)}$, then we have*

$$\|T(f_1, f_2)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C\|f_1\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|f_2\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}$$

with the constant $C > 0$ independent of f_1 and f_2.

Now we recall that the λ-central bounded mean oscillation space and the central Morrey space in [13] are defined as follows.

Definition 3. Let $\lambda < 1/n$ and $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. The λ-central bounded mean oscillation space with variable exponent $\mathrm{CBMO}^{q(\cdot),\lambda}(\mathbb{R}^n)$ is defined by

$$\mathrm{CBMO}^{q(\cdot),\lambda}(\mathbb{R}^n) = \left\{f \in L^{q(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{\mathrm{CBMO}^{q(\cdot),\lambda}(\mathbb{R}^n)} < \infty\right\},$$

where

$$\|f\|_{\mathrm{CBMO}^{q(\cdot),\lambda}(\mathbb{R}^n)} = \sup_{R>0} \frac{\|(f - f_{B(0,R)})\chi_{B(0,R)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{|B(0,R)|^\lambda\|\chi_{B(0,R)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}.$$

Definition 4. Let $\lambda \in \mathbb{R}$ and $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. The central Morrey space with variable exponent $\dot{B}^{q(\cdot),\lambda}(\mathbb{R}^n)$ is defined by

$$\dot{B}^{q(\cdot),\lambda}(\mathbb{R}^n) = \Big\{ f \in L^{q(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{\dot{B}^{q(\cdot),\lambda}(\mathbb{R}^n)} < \infty \Big\},$$

where

$$\|f\|_{\dot{B}^{q(\cdot),\lambda}(\mathbb{R}^n)} = \sup_{R>0} \frac{\|f\chi_{B(0,R)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{|B(0,R)|^{\lambda}\|\chi_{B(0,R)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}.$$

Remark 2.6. If $q(\cdot) = q$ is constant, then we can easily get Definition 1 and Definition 2, respectively. When $\lambda = 0$, the space $\mathrm{CBMO}^{q(\cdot),\lambda}(\mathbb{R}^n)$ is just the space $\mathrm{CBMO}^{q(\cdot)}(\mathbb{R}^n)$ defined in [30].

Remark 2.7. Denote by $\mathrm{CMO}^{q(\cdot),\lambda}(\mathbb{R}^n)$ and $B^{q(\cdot),\lambda}(\mathbb{R}^n)$ the inhomogeneous versions of the λ-central bounded mean oscillation space and the central Morrey space with variable exponent, which are defined respectively by taking the supremum over $R \geq 1$ in Definition 3 and Definition 4 instead of $R > 0$ there. Obviously, $\mathrm{CBMO}^{q(\cdot),\lambda}(\mathbb{R}^n) \subset \mathrm{CMO}^{q(\cdot),\lambda}(\mathbb{R}^n)$ for $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $\lambda < 1/n$, and $\dot{B}^{q(\cdot),\lambda}(\mathbb{R}^n) \subset B^{q(\cdot),\lambda}(\mathbb{R}^n)$ for $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $\lambda \in \mathbb{R}$.

Our main results can be stated as follows.

Theorem 2.8. *Let T be an m-linear Calderón-Zygmund operator defined as in* (4). *Suppose that $q(\cdot), p_i(\cdot) \in \mathcal{P}(\mathbb{R}^n) \bigcap LH(\mathbb{R}^n)$, $\lambda_i < 0$, $i = 1, \ldots, m$, $\lambda = \sum_{i=1}^m \lambda_i$ and $\frac{1}{q(\cdot)} = \sum_{i=1}^m \frac{1}{p_i(\cdot)}$. Then T is bounded from $\dot{B}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n) \times \cdots \times \dot{B}^{p_m(\cdot),\lambda_m}(\mathbb{R}^n)$ to $\dot{B}^{q(\cdot),\lambda}(\mathbb{R}^n)$ and satisfies the following inequality*:

$$\|T(f_1, \ldots, f_m)\|_{\dot{B}^{q(\cdot),\lambda}(\mathbb{R}^n)} \leq C \prod_{i=1}^m \|f_i\|_{\dot{B}^{p_i(\cdot),\lambda_i}(\mathbb{R}^n)}.$$

Theorem 2.9. *Let $[\vec{b}, T]$ be the commutator defined as in* (5) *with vector symbol $\vec{b} = (b_1, \ldots, b_m)$. Suppose that $q(\cdot), p_i(\cdot), s_i(\cdot) \in \mathcal{P}(\mathbb{R}^n) \bigcap LH(\mathbb{R}^n)$, $p_i'(\cdot) < s_i(\cdot)$, $0 < \eta_i < \frac{1}{n}$, $\eta_i + \gamma_i < 0$, $b_i \in \mathrm{CBMO}^{s_i(\cdot),\eta_i}(\mathbb{R}^n)$, $i = 1, \ldots, m$, $\frac{1}{q(\cdot)} = \sum_{i=1}^m \frac{1}{s_i(\cdot)} + \sum_{i=1}^m \frac{1}{p_i(\cdot)}$ and $\lambda = \sum_{i=1}^m \eta_i + \sum_{i=1}^m \gamma_i$. Then $[\vec{b}, T]$ is bounded from $\dot{B}^{p_1(\cdot),\gamma_1}(\mathbb{R}^n) \times \cdots \times \dot{B}^{p_m(\cdot),\gamma_m}(\mathbb{R}^n)$ to $\dot{B}^{q(\cdot),\lambda}(\mathbb{R}^n)$ and satisfies the following inequality*:

$$\|[\vec{b}, T]\vec{f}\|_{\dot{B}^{q(\cdot),\lambda}(\mathbb{R}^n)} \leq C \prod_{i=1}^m \Big(\|b_i\|_{\mathrm{CBMO}^{s_i(\cdot),\eta_i}(\mathbb{R}^n)} \|f_i\|_{\dot{B}^{p_i(\cdot),\gamma_i}(\mathbb{R}^n)} \Big).$$

Theorem 2.10. *Let $T_{\vec{b}}$ be the commutator defined as in* (8) *with vector symbol $\vec{b} = (b_1, \ldots, b_m)$. Suppose that $0 < u_i < \frac{1}{n}$, $p(\cdot), q(\cdot), s_i(\cdot) \in \mathcal{P}(\mathbb{R}^n) \bigcap LH(\mathbb{R}^n)$, $p'(\cdot) < s_i(\cdot)$, $b_i \in \mathrm{CBMO}^{s_i(\cdot),u_i}(\mathbb{R}^n)$, $i = 1, \ldots, m$, $\frac{1}{q(\cdot)} = \sum_{i=1}^m \frac{1}{s_i(\cdot)} + \frac{1}{p(\cdot)}$ and*

$\lambda = \sum_{i=1}^{m} u_i + v < 0$. *Then* $T_{\vec{b}}$ *is bounded from* $\dot{B}^{p(\cdot),v}(\mathbb{R}^n)$ *to* $\dot{B}^{q(\cdot),\lambda}(\mathbb{R}^n)$ *and satisfies the following inequality*:

$$\|T_{\vec{b}}(f)\|_{\dot{B}^{q(\cdot),\lambda}(\mathbb{R}^n)} \leq C \prod_{i=1}^{m} \|b_i\|_{\mathrm{CBMO}^{s_i(\cdot),u_i}(\mathbb{R}^n)} \|f\|_{\dot{B}^{p(\cdot),v}(\mathbb{R}^n)}.$$

Remark 2.11. We remark that Theorem 2.8 can be regarded as a generalization of Corollary 3.1 in [12] to the variable exponent case. Theorems 2.9 and 2.10 also extend, with different methods, the corresponding classical results in [28]. Our approach is mainly based on the theory of variable exponent and on generalization of the λ-central BMO norms. Moreover, our results remain true for the inhomogeneous versions of λ-central BMO spaces and central Morrey spaces with variable exponent.

3. Proof of Theorem 2.8

Without loss of generality, we may assume that $m = 2$. Let f_1, f_2 be functions in $\dot{B}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)$ and $\dot{B}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)$, respectively. For fixed $R > 0$, we simply denote by $B = B(0, R)$. We also denote by $L^{q(\cdot)} = L^{q(\cdot)}(\mathbb{R}^n)$ and $\dot{B}^{p(\cdot),\lambda} = \dot{B}^{p(\cdot),\lambda}(\mathbb{R}^n)$. We need to prove

$$\|\chi_B T(f_1, f_2)\|_{L^{q(\cdot)}} \leq C|B|^{\lambda} \|\chi_B\|_{L^{q(\cdot)}} \prod_{i=1}^{2} \|f_i\|_{\dot{B}^{p_i(\cdot),\lambda_i}}. \tag{14}$$

The Minkowski inequality implies that

$$\begin{aligned} \|\chi_B T(f_1, f_2)\|_{L^{q(\cdot)}} \leq\ & \|\chi_B T(f_1\chi_{2B}, f_2\chi_{2B})\|_{L^{q(\cdot)}} \\ & + \|\chi_B T(f_1\chi_{(2B)^c}, f_2\chi_{2B})\|_{L^{q(\cdot)}} \\ & + \|\chi_B T(f_1\chi_{2B}, f_2\chi_{(2B)^c})\|_{L^{q(\cdot)}} \\ & + \|\chi_B T(f_1\chi_{(2B)^c}, f_2\chi_{(2B)^c})\|_{L^{q(\cdot)}} \\ =:\ & U_1 + U_2 + U_3 + U_4. \end{aligned} \tag{15}$$

For U_1, by Lemmas 2.5, 2.2 and 2.4, we get

$$\begin{aligned} U_1 &\leq C \prod_{i=1}^{2} \|f_i\chi_{2B}\|_{L^{p_i(\cdot)}} \\ &\leq C \prod_{i=1}^{2} |2B|^{\lambda_i} \|\chi_{2B}\|_{L^{p_i(\cdot)}} \|f_i\|_{\dot{B}^{p_i(\cdot),\lambda_i}} \\ &\leq C \prod_{i=1}^{2} |2B|^{\lambda_i} \|\chi_B\|_{L^{p_i(\cdot)}} \|f_i\|_{\dot{B}^{p_i(\cdot),\lambda_i}} \\ &\leq C|B|^{\lambda} \|\chi_B\|_{L^{q(\cdot)}} \prod_{i=1}^{2} \|f_i\|_{\dot{B}^{p_i(\cdot),\lambda_i}}. \end{aligned} \tag{16}$$

For U_2, noting that $|(x-y_1,x-y_2)|^{2n} \sim |x-y_1|^{2n} \sim |2^kB|^2$ for $x \in B$ and $y_1 \in (2^kB)^c$, we use Hölder's inequality, Lemma 2.1 and obtain

$$
\begin{aligned}
(17)\quad & |T(f_1\chi_{(2B)^c}, f_2\chi_{2B})(x)| \\
&\leq C\int_{2B}|f_2(y_2)|dy_2\int_{(2B)^c}\frac{|f_1(y_1)|}{|x-y_1|^{2n}}dy_1 \\
&\leq C\int_{2B}|f_2(y_2)|dy_2\sum_{k=1}^{\infty}\int_{2^{k+1}B\backslash 2^kB}\frac{|f_1(y_1)|}{|x-y_1|^{2n}}dy_1 \\
&\leq C\|f_2\chi_{2B}\|_{L^{p_2(\cdot)}}\|\chi_{2B}\|_{L^{p_2'(\cdot)}}\sum_{k=1}^{\infty}|2^kB|^{-2}\|f_1\chi_{2^{k+1}B}\|_{L^{p_1(\cdot)}}\|\chi_{2^{k+1}B}\|_{L^{p_1'(\cdot)}} \\
&\leq C|2B|^{\lambda_2}\|\chi_{2B}\|_{L^{p_2(\cdot)}}\|f_2\|_{\dot{B}^{p_2(\cdot),\lambda_2}}\|\chi_{2B}\|_{L^{p_2'(\cdot)}} \\
&\quad\times\sum_{k=1}^{\infty}|2^kB|^{-2}|2^{k+1}B|^{\lambda_1}\|\chi_{2^{k+1}B}\|_{L^{p_1(\cdot)}}\|f_1\|_{\dot{B}^{p_1(\cdot),\lambda_1}}\|\chi_{2^{k+1}B}\|_{L^{p_1'(\cdot)}} \\
&\leq C|B|^{\lambda_1+\lambda_2}\prod_{i=1}^{2}\|f_i\|_{\dot{B}^{p_i(\cdot),\lambda_i}}\sum_{k=1}^{\infty}2^{kn(\lambda_1-1)} \\
&\leq C|B|^{\lambda}\prod_{i=1}^{2}\|f_i\|_{\dot{B}^{p_i(\cdot),\lambda_i}}.
\end{aligned}
$$

Thus, we have

$$U_2 \leq C|B|^{\lambda}\|\chi_B\|_{L^{q(\cdot)}}\prod_{i=1}^{2}\|f_i\|_{\dot{B}^{p_i(\cdot),\lambda_i}}. \tag{18}$$

Similar to the estimates for U_2, we get

$$U_3 \leq C|B|^{\lambda}\|\chi_B\|_{L^{q(\cdot)}}\prod_{i=1}^{2}\|f_i\|_{\dot{B}^{p_i(\cdot),\lambda_i}}. \tag{19}$$

For U_4, noting that $|(x-y_1,x-y_2)|^{2n} \sim |x-y_1|^n \cdot |x-y_2|^n \sim |2^{k_1}B| \cdot |2^{k_2}B|$ for $x \in B$, $y_1 \in (2^kB)^c$and $y_2 \in (2^kB)^c$, by Hölder's inequality and Lemma 2.1, we deduce

$$
\begin{aligned}
(20)\quad & |T(f_1\chi_{(2B)^c}, f_2\chi_{(2B)^c})(x)| \\
&\leq C\int_{(2B)^c}\frac{|f_1(y_1)|}{|x-y_1|^{n}}dy_1\int_{(2B)^c}\frac{|f_2(y_2)|}{|x-y_2|^{n}}dy_2 \\
&\leq C\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\prod_{i=1}^{2}\int_{2^{k_i+1}B\backslash 2^{k_i}B}\frac{|f_i(y_i)|}{|x-y_i|^{n}}dy_i \\
&\leq C\prod_{i=1}^{2}\sum_{k_i=1}^{\infty}|2^{k_i}B|^{-1}\int_{2^{k_i+1}B}|f_i(y_i)|dy_i
\end{aligned}
$$

$$\leq C\prod_{i=1}^{2}\sum_{k_i=1}^{\infty}|2^{k_i}B|^{-1}\|f_i\chi_{2^{k_i+1}B}\|_{L^{p_i(\cdot)}}\|\chi_{2^{k_i+1}B}\|_{L^{p_i'(\cdot)}}$$
$$\leq C|B|^{\lambda_1+\lambda_2}\prod_{i=1}^{2}\|f_i\|_{\dot{B}^{p_i(\cdot),\lambda_i}}\sum_{k_i=1}^{\infty}2^{k_i n\lambda_i}$$
$$\leq C|B|^{\lambda}\prod_{i=1}^{2}\|f_i\|_{\dot{B}^{p_i(\cdot),\lambda_i}}.$$

Consequently, we derive the estimate

$$U_4 \leq C|B|^{\lambda}\|\chi_B\|_{L^{q(\cdot)}}\prod_{i=1}^{2}\|f_i\|_{\dot{B}^{p_i(\cdot),\lambda_i}}. \tag{21}$$

This completes our proof.

4. Proof of Theorem 2.9

Without loss of generality, we may assume that $m = 2$. Let f_1, f_2 be functions in $\dot{B}^{p_1(\cdot),\gamma_1}(\mathbb{R}^n)$ and $\dot{B}^{p_2(\cdot),\gamma_2}(\mathbb{R}^n)$, respectively. In the case $m = 2$, denote $[\vec{b}, T]$ by $[b_1, b_2, T]$, we need to prove

$$\begin{aligned}&\|\chi_B[b_1,b_2,T](f_1,f_2)\|_{L^{q(\cdot)}}\\&\leq C|B|^{\lambda}\|\chi_B\|_{L^{q(\cdot)}}\prod_{i=1}^{2}(\|b_i\|_{\mathrm{CBMO}^{s_i(\cdot),\eta_i}}\|f_i\|_{\dot{B}^{p_i(\cdot),\gamma_i}}).\end{aligned} \tag{22}$$

Applying the Minkowski inequality, we obtain

$$\begin{aligned}&\|\chi_B[b_1,b_2,T](f_1,f_2)\|_{L^{q(\cdot)}}\\&\leq \|\chi_B(b_1-(b_1)_B)(b_2-(b_2)_B)T(f_1,f_2)\|_{L^{q(\cdot)}}\\&\quad+\|\chi_B(b_1-(b_1)_B)T(f_1,(b_2-(b_2)_B)f_2)\|_{L^{q(\cdot)}}\\&\quad+\|\chi_B(b_2-(b_2)_B)T((b_1-(b_1)_B)f_1,f_2)\|_{L^{q(\cdot)}}\\&\quad+\|\chi_B T((b_1-(b_1)_B)f_1,(b_2-(b_2)_B)f_2)\|_{L^{q(\cdot)}}\\&=: V_1+V_2+V_3+V_4.\end{aligned} \tag{23}$$

For V_1, we have

$$\begin{aligned}V_1 &\leq \|\chi_B(b_1-(b_1)_B)(b_2-(b_2)_B)T(f_1\chi_{2B},f_2\chi_{2B})\|_{L^{q(\cdot)}}\\&\quad+\|\chi_B(b_1-(b_1)_B)(b_2-(b_2)_B)T(f_1\chi_{2B},f_2\chi_{(2B)^c})\|_{L^{q(\cdot)}}\\&\quad+\|\chi_B(b_1-(b_1)_B)(b_2-(b_2)_B)T(f_1\chi_{(2B)^c},f_2\chi_{2B})\|_{L^{q(\cdot)}}\\&\quad+\|\chi_B(b_1-(b_1)_B)(b_2-(b_2)_B)T(f_1\chi_{(2B)^c},f_2\chi_{(2B)^c})\|_{L^{q(\cdot)}}\\&=: V_{11}+V_{12}+V_{13}+V_{14}.\end{aligned} \tag{24}$$

For V_{11}, set $\frac{1}{p(\cdot)} = \frac{1}{p(\cdot)} + \frac{1}{p_2(\cdot)}$, then $\frac{1}{q(\cdot)} = \frac{1}{p_1(\cdot)} + \prod_{i=1}^{2} \frac{1}{s_i(\cdot)}$. Using Hölder's inequality, Lemmas 2.5, 2.2 and 2.4, we obtain

(25)

$$\begin{aligned}
V_{11} &\le \|\chi_B T(f_1\chi_{2B}, f_2\chi_{2B})\|_{L^{p(\cdot)}} \prod_{i=1}^{2} \|(b_i - (b_i)_B)\chi_B\|_{L^{s_i(\cdot)}} \\
&\le C \prod_{i=1}^{2} \|f_i\chi_{2B}\|_{L^{p_i(\cdot)}} \prod_{i=1}^{2} (\|b_i\|_{\mathrm{CBMO}^{s_i(\cdot),\eta_i}} |B|^{\eta_i} \|\chi_B\|_{L^{s_i(\cdot)}}) \\
&\le C \prod_{i=1}^{2} (\|f_i\|_{\dot{B}^{p_i(\cdot),\gamma_i}} |2B|^{\gamma_i} \|\chi_{2B}\|_{L^{p_i(\cdot)}}) \prod_{i=1}^{2} (\|b_i\|_{\mathrm{CBMO}^{s_i(\cdot),\eta_i}} |B|^{\eta_i} \|\chi_B\|_{L^{s_i(\cdot)}}) \\
&\le C \prod_{i=1}^{2} (\|f_i\|_{\dot{B}^{p_i(\cdot),\gamma_i}} |B|^{\gamma_i} \|\chi_B\|_{L^{p_i(\cdot)}}) \prod_{i=1}^{2} (\|b_i\|_{\mathrm{CBMO}^{s_i(\cdot),\eta_i}} |B|^{\eta_i} \|\chi_B\|_{L^{s_i(\cdot)}}) \\
&\le C|B|^{\lambda} \|\chi_B\|_{L^{q(\cdot)}} \prod_{i=1}^{2} (\|b_i\|_{\mathrm{CBMO}^{s_i(\cdot),\eta_i}} \|f_i\|_{\dot{B}^{p_i(\cdot),\gamma_i}}).
\end{aligned}$$

For V_{12}, noting that $|(x-y_1, x-y_2)|^{2n} \sim |x-y_2|^{2n} \sim |2^kB|^2$ for $x \in B$ and $y_2 \in (2^kB)^c$, by Hölder's inequality and Lemma 2.1, we get

(26)

$$\begin{aligned}
&|T(f_1\chi_{2B}, f_2\chi_{(2B)^c})(x)| \\
&\le C \int_{2B} |f_1(y_1)| dy_1 \int_{(2B)^c} \frac{|f_2(y_2)|}{|x-y_2|^{2n}} dy_2 \\
&\le C \int_{2B} |f_1(y_1)| dy_1 \sum_{k=1}^{\infty} \int_{2^{k+1}B \setminus 2^kB} \frac{|f_2(y_2)|}{|x-y_2|^{2n}} dy_2 \\
&\le C \|f_1\chi_{2B}\|_{L^{p_1(\cdot)}} \|\chi_{2B}\|_{L^{p_1'(\cdot)}} \sum_{k=1}^{\infty} |2^kB|^{-2} \|f_2\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}} \|\chi_{2^{k+1}B}\|_{L^{p_2'(\cdot)}} \\
&\le C|2B|^{\gamma_1} \|\chi_{2B}\|_{L^{p_1(\cdot)}} \|f_1\|_{\dot{B}^{p_1(\cdot),\gamma_1}} \|\chi_{2B}\|_{L^{p_1'(\cdot)}} \\
&\quad \times \sum_{k=1}^{\infty} |2^kB|^{-2} |2^{k+1}B|^{\gamma_2} \|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}} \|f_2\|_{\dot{B}^{p_2(\cdot),\gamma_2}} \|\chi_{2^{k+1}B}\|_{L^{p_2'(\cdot)}} \\
&\le C|B|^{\gamma_1+\gamma_2} \prod_{i=1}^{2} \|f_i\|_{\dot{B}^{p_i(\cdot),\gamma_i}} \sum_{k=1}^{\infty} 2^{kn(\gamma_2-1)} \\
&\le C|B|^{\gamma_1+\gamma_2} \prod_{i=1}^{2} \|f_i\|_{\dot{B}^{p_i(\cdot),\gamma_i}}.
\end{aligned}$$

Thus, from Hölder's inequality and Lemma 2.4, it follows that

(27)

$$\begin{aligned}
V_{12} &\le C|B|^{\gamma_1+\gamma_2}\prod_{i=1}^{2}\|f_i\|_{\dot{B}^{p_i(\cdot),\gamma_i}}\|(b_1-(b_1)_B)(b_2-(b_2)_B)\chi_B\|_{L^{q(\cdot)}}\\
&\le C|B|^{\gamma_1+\gamma_2}\prod_{i=1}^{2}\|f_i\|_{\dot{B}^{p_i(\cdot),\gamma_i}}\|\chi_B\|_{L^{p(\cdot)}}\prod_{i=1}^{2}\|(b_i-(b_i)_B)\chi_B\|_{L^{s_i(\cdot)}}\\
&\le C|B|^{\gamma_1+\gamma_2}\prod_{i=1}^{2}\|f_i\|_{\dot{B}^{p_i(\cdot),\gamma_i}}\|\chi_B\|_{L^{p(\cdot)}}\prod_{i=1}^{2}(\|b_i\|_{\mathrm{CBMO}^{s_i(\cdot),\eta_i}}|B|^{\eta_i}\|\chi_B\|_{L^{s_i(\cdot)}})\\
&\le C|B|^{\lambda}\|\chi_B\|_{L^{q(\cdot)}}\prod_{i=1}^{2}(\|b_i\|_{\mathrm{CBMO}^{s_i(\cdot),\eta_i}}\|f_i\|_{\dot{B}^{p_i(\cdot),\gamma_i}}).
\end{aligned}$$

Similarly, we have

$$V_{13} \le C|B|^{\lambda}\|\chi_B\|_{L^{q(\cdot)}}\prod_{i=1}^{2}(\|b_i\|_{\mathrm{CBMO}^{s_i(\cdot),\eta_i}}\|f_i\|_{\dot{B}^{p_i(\cdot),\gamma_i}}). \tag{28}$$

For V_{14}, noting that $|(x-y_1,x-y_2)|^{2n}\sim|x-y_1|^n\cdot|x-y_2|^n\sim|2^{k_1}B|\cdot|2^{k_2}B|$ for $x\in B$, $y_1\in(2^kB)^c$and $y_2\in(2^kB)^c$, by Hölder's inequality and Lemma 2.1, we obtain

(29)

$$\begin{aligned}
&|T(f_1\chi_{(2B)^c},f_2\chi_{(2B)^c})(x)|\\
&\le C\int_{(2B)^c}\frac{|f_1(y_1)|}{|x-y_1|^n}dy_1\int_{(2B)^c}\frac{|f_2(y_2)|}{|x-y_2|^n}dy_2\\
&\le C\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\prod_{i=1}^{2}\int_{2^{k_i+1}B\setminus 2^{k_i}B}\frac{|f_i(y_i)|}{|x-y_i|^n}dy_i\\
&\le C\prod_{i=1}^{2}\sum_{k_i=1}^{\infty}|2^{k_i}B|^{-1}\int_{2^{k_i+1}B}|f_i(y_i)|dy_i\\
&\le C\prod_{i=1}^{2}\sum_{k_i=1}^{\infty}|2^{k_i}B|^{-1}\|f_i\chi_{2^{k_i+1}B}\|_{L^{p_i(\cdot)}}\|\chi_{2^{k_i+1}B}\|_{L^{p_i'(\cdot)}}\\
&\le C|B|^{\gamma_1+\gamma_2}\prod_{i=1}^{2}\|f_i\|_{\dot{B}^{p_i(\cdot),\gamma_i}}\sum_{k_i=1}^{\infty}2^{k_in\gamma_i}\\
&\le C|B|^{\gamma_1+\gamma_2}\prod_{i=1}^{2}\|f_i\|_{\dot{B}^{p_i(\cdot),\gamma_i}}.
\end{aligned}$$

Thus, similar to the estimates for V_{12}, we deduce that

$$V_{14} \leq C|B|^{\lambda}\|\chi_B\|_{L^{q(\cdot)}}\prod_{i=1}^{2}(\|b_i\|_{\mathrm{CBMO}^{s_i(\cdot),\eta_i}}\|f_i\|_{\dot{B}^{p_i(\cdot),\gamma_i}}). \tag{30}$$

For V_2, we have the following decomposition:

$$\begin{aligned} V_2 \leq\ & \|\chi_B(b_1-(b_1)_B)T(f_1\chi_{2B},(b_2-(b_2)_B)f_2\chi_{2B})\|_{L^{q(\cdot)}} \\ &+ \|\chi_B(b_1-(b_1)_B)T(f_1\chi_{(2B)^c},(b_2-(b_2)_B)f_2\chi_{2B})\|_{L^{q(\cdot)}} \\ &+ \|\chi_B(b_1-(b_1)_B)T(f_1\chi_{2B},(b_2-(b_2)_B)f_2\chi_{(2B)^c})\|_{L^{q(\cdot)}} \\ &+ \|\chi_B(b_1-(b_1)_B)T(f_1\chi_{(2B)^c},(b_2-(b_2)_B)f_2\chi_{(2B)^c})\|_{L^{q(\cdot)}} \\ =:\ & V_{21}+V_{22}+V_{23}+V_{24}. \end{aligned} \tag{31}$$

For V_{21}, set $\frac{1}{r(\cdot)}=\frac{1}{p_2(\cdot)}+\frac{1}{s_2(\cdot)}$, $\frac{1}{q_1(\cdot)}=\frac{1}{p_1(\cdot)}+\frac{1}{r(\cdot)}$, then $\frac{1}{q(\cdot)}=\frac{1}{s_1(\cdot)}+\frac{1}{q_1(\cdot)}$. By Lemmas 2.5, 2.3 and 2.1, we have

(32)

$$\begin{aligned} V_{21} &\leq C\|(b_1-(b_1)_B)\chi_B\|_{L^{s_1(\cdot)}}\|\chi_B T(f_1\chi_{2B},(b_2-(b_2)_B)f_2\chi_{2B})\|_{L^{q_1(\cdot)}} \\ &\leq C\|(b_1-(b_1)_B)\chi_B\|_{L^{s_1(\cdot)}}\|f_1\chi_{2B}\|_{L^{p_1(\cdot)}}\|(b_2-(b_2)_B)f_2\chi_{2B}\|_{L^{r(\cdot)}} \\ &\leq C\|(b_1-(b_1)_B)\chi_B\|_{L^{s_1(\cdot)}}\|f_1\chi_{2B}\|_{L^{p_1(\cdot)}}\|(b_2-(b_2)_B)\chi_{2B}\|_{L^{s_2(\cdot)}}\|f_2\chi_{2B}\|_{L^{p_2(\cdot)}} \\ &\leq C\|(b_1-(b_1)_B)\chi_B\|_{L^{s_1(\cdot)}}\|f_1\chi_{2B}\|_{L^{p_1(\cdot)}} \\ &\quad\times\left\{\|(b_2-(b_2)_{2B})\chi_{2B}\|_{L^{s_2(\cdot)}}+|(b_2)_{2B}-(b_2)_B|\|\chi_{2B}\|_{L^{s_2(\cdot)}}\right\}\|f_2\chi_{2B}\|_{L^{p_2(\cdot)}} \\ &\leq C\|(b_1-(b_1)_B)\chi_B\|_{L^{s_1(\cdot)}}\|f_1\chi_{2B}\|_{L^{p_1(\cdot)}}\|f_2\chi_{2B}\|_{L^{p_2(\cdot)}} \\ &\quad\times\left\{\|(b_2-(b_2)_{2B})\chi_{2B}\|_{L^{s_2(\cdot)}}+\frac{1}{|B|}\|(b_2-(b_2)_{2B})\chi_{2B}\|_{L^{s_2(\cdot)}}\|\chi_{2B}\|_{L^{s_2'(\cdot)}}\|\chi_{2B}\|_{L^{s_2(\cdot)}}\right\} \\ &\leq C\|(b_1-(b_1)_B)\chi_B\|_{L^{s_1(\cdot)}}\|f_1\chi_{2B}\|_{L^{p_1(\cdot)}}\|(b_2-(b_2)_{2B})\chi_{2B}\|_{L^{s_2(\cdot)}}\|f_2\chi_{2B}\|_{L^{p_2(\cdot)}} \\ &\leq C|B|^{\lambda}\|\chi_B\|_{L^{q(\cdot)}}\prod_{i=1}^{2}(\|b_i\|_{\mathrm{CBMO}^{s_i(\cdot),\eta_i}}\|f_i\|_{\dot{B}^{p_i(\cdot),\gamma_i}}). \end{aligned}$$

For V_{22}, in view of $p_2'(\cdot) < s_2(\cdot)$, we set $\frac{1}{\phi(\cdot)}=\frac{1}{p_2'(\cdot)}-\frac{1}{s_2(\cdot)}$. Noting that $|(x-y_1,x-y_2)|^{2n}\sim|x-y_1|^{2n}\sim|2^kB|^2$ for $x\in B$ and $y_1\in(2^kB)^c$, we apply Hölder's inequality, Lemmas 2.4 and 2.1 to get the estimate

(33)

$$\begin{aligned} &|T(f_1\chi_{(2B)^c},(b_2-(b_2)_B)f_2\chi_{2B})(x)| \\ &\leq C\int_{2B}|b_2(y_2)-(b_2)_B||f_2(y_2)|dy_2\int_{(2B)^c}\frac{|f_1(y_1)|}{|x-y_1|^{2n}}dy_1 \\ &\leq C\int_{2B}|b_2(y_2)-(b_2)_B||f_2(y_2)|dy_2\sum_{k=1}^{\infty}\int_{2^{k+1}B\backslash 2^kB}\frac{|f_1(y_1)|}{|x-y_1|^{2n}}dy_1 \end{aligned}$$

$$\leq C\|(b_2-(b_2)_B)\chi_{2B}\|_{L^{s_2(\cdot)}}\|f_2\chi_{2B}\|_{L^{p_2(\cdot)}}\|\chi_{2B}\|_{L^{\phi(\cdot)}}$$
$$\times\sum_{k=1}^{\infty}|2^kB|^{-2}\|f_1\chi_{2^{k+1}B}\|_{L^{p_1(\cdot)}}\|\chi_{2^{k+1}B}\|_{L^{p_1'(\cdot)}}$$
$$\leq C\Big\{\|(b_2-(b_2)_{2B})\chi_{2B}\|_{L^{s_2(\cdot)}}+|(b_2)_{2B}-(b_2)_B|\|\chi_{2B}\|_{L^{s_2(\cdot)}}\Big\}\|f_2\chi_{2B}\|_{L^{p_2(\cdot)}}\|\chi_{2B}\|_{L^{\phi(\cdot)}}$$
$$\times\sum_{k=1}^{\infty}|2^kB|^{-2}\|f_1\chi_{2^{k+1}B}\|_{L^{p_1(\cdot)}}\|\chi_{2^{k+1}B}\|_{L^{p_1'(\cdot)}}$$
$$\leq C|B|^{\eta_2+\gamma_1+\gamma_2}\prod_{i=1}^{2}\|f_i\|_{\dot{B}^{p_i(\cdot),\gamma_i}}\|b_2\|_{\mathrm{CBMO}^{s_2(\cdot),\eta_2}}\sum_{k=1}^{\infty}2^{kn(\gamma_1-1)}$$
$$\leq C|B|^{\eta_2+\gamma_1+\gamma_2}\|b_2\|_{\mathrm{CBMO}^{s_2(\cdot),\eta_2}}\prod_{i=1}^{2}\|f_i\|_{\dot{B}^{p_i(\cdot),\gamma_i}}.$$

Therefore, recalling that $\frac{1}{q(\cdot)}=\frac{1}{s_1(\cdot)}+\frac{1}{q_1(\cdot)}$, by Lemmas 2.3 and 2.4, we have

(34)
$$V_{22}\leq C|B|^{\eta_2+\gamma_1+\gamma_2}\|b_2\|_{\mathrm{CBMO}^{s_2(\cdot),\eta_2}}\|(b_1-(b_1)_B)\chi_B\|_{L^{q(\cdot)}}\prod_{i=1}^{2}\|f_i\|_{\dot{B}^{p_i(\cdot),\gamma_i}}$$
$$\leq C|B|^{\eta_2+\gamma_1+\gamma_2}\|b_2\|_{\mathrm{CBMO}^{s_2(\cdot),\eta_2}}\|(b_1-(b_1)_B)\chi_B\|_{L^{s_1(\cdot)}}\|\chi_B\|_{L^{q_1(\cdot)}}\prod_{i=1}^{2}\|f_i\|_{\dot{B}^{p_i(\cdot),\gamma_i}}$$
$$\leq C|B|^{\lambda}\|\chi_B\|_{L^{q(\cdot)}}\prod_{i=1}^{2}(\|b_i\|_{\mathrm{CBMO}^{s_i(\cdot),\eta_i}}\|f_i\|_{\dot{B}^{p_i(\cdot),\gamma_i}}).$$

For V_{23}, noting the fact that (see [13, Page 516])
$$|(b_2)_{2^{k+1}B}-(b_2)_B|\leq C\|b_2\|_{\mathrm{CBMO}^{s_2(\cdot),\eta_2}}(k+1)|2^{k+1}B|^{\eta_2}.$$
By Hölder's inequality, Lemmas 2.1 and 2.4, we have

(35)
$$|T(f_1\chi_{2B},(b_2-(b_2)_B)f_2\chi_{(2B)^c})(x)|$$
$$\leq C\int_{2B}|f_1(y_1)|dy_1\sum_{k=1}^{\infty}|2^kB|^{-2}\int_{2^{k+1}B}|b_2(y_2)-(b_2)_B||f_2(y_2)|dy_2$$
$$\leq C\|f_1\chi_{2B}\|_{L^{p_1(\cdot)}}\|\chi_{2B}\|_{L^{p_1'(\cdot)}}$$
$$\times\sum_{k=1}^{\infty}|2^kB|^{-2}\|(b_2-(b_2)_B)\chi_{2^{k+1}B}\|_{L^{s_2(\cdot)}}\|f_2\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}}\|\chi_{2^{k+1}B}\|_{L^{\phi(\cdot)}}$$
$$\leq C\|f_1\chi_{2B}\|_{L^{p_1(\cdot)}}\|\chi_{2B}\|_{L^{p_1'(\cdot)}}\sum_{k=1}^{\infty}|2^kB|^{-2}\Big\{\|(b_2-(b_2)_{2^{k+1}B})\chi_{2^{k+1}B}\|_{L^{s_2(\cdot)}}$$
$$+|(b_2)_{2^{k+1}B}-(b_2)_B|\|\chi_{2^{k+1}B}\|_{L^{s_2(\cdot)}}\Big\}\|f_2\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}}\|\chi_{2^{k+1}B}\|_{L^{\phi(\cdot)}}$$

$$\leq C|B|^{\eta_2+\gamma_1+\gamma_2}\prod_{i=1}^{2}\|f_i\|_{\dot{B}^{p_i(\cdot),\gamma_i}}\|b_2\|_{\mathrm{CBMO}^{s_2(\cdot),\eta_2}}\sum_{k=1}^{\infty}k2^{kn(\eta_2+\gamma_2-1)}$$
$$\leq C|B|^{\eta_2+\gamma_1+\gamma_2}\prod_{i=1}^{2}\|f_i\|_{\dot{B}^{p_i(\cdot),\gamma_i}}\|b_2\|_{\mathrm{CBMO}^{s_2(\cdot),\eta_2}}.$$

Similar to the estimates for V_{22}, we get

$$V_{23}\leq C|B|^{\lambda}\|\chi_B\|_{L^{q(\cdot)}}\prod_{i=1}^{2}(\|b_i\|_{\mathrm{CBMO}^{s_i(\cdot),\eta_i}}\|f_i\|_{\dot{B}^{p_i(\cdot),\gamma_i}}). \tag{36}$$

For V_{24}, by Hölder's inequality, Lemmas 2.1 and 2.4, we derive

(37)

$$\begin{aligned}
&|T(f_1\chi_{(2B)^c},(b_2-(b_2)_B)f_2\chi_{(2B)^c})(x)|\\
&\leq C\sum_{k_1=1}^{\infty}|2^{k_1}B|^{-1}\int_{2^{k_1+1}B}|f_1(y_1)|dy_1\\
&\quad\times\sum_{k_2=1}^{\infty}|2^{k_2}B|^{-1}\int_{2^{k_2+1}B}|b_2(y_2)-(b_2)_B||f_2(y_2)|dy_2\\
&\leq C\sum_{k_1=1}^{\infty}|2^{k_1}B|^{-1}\|f_1\chi_{2^{k_1+1}B}\|_{L^{p_1(\cdot)}}\|\chi_{2^{k_1+1}B}\|_{L^{p_1'(\cdot)}}\\
&\quad\times\sum_{k_2=1}^{\infty}|2^{k_2}B|^{-1}\|(b_2-(b_2)_B)\chi_{2^{k_2+1}B}\|_{L^{s_2(\cdot)}}\|f_2\chi_{2^{k_2+1}B}\|_{L^{p_2(\cdot)}}\|\chi_{2^{k_2+1}B}\|_{L^{\phi(\cdot)}}\\
&\leq C\sum_{k_1=1}^{\infty}|2^{k_1}B|^{-1}\|f_1\chi_{2^{k_1+1}B}\|_{L^{p_1(\cdot)}}\|\chi_{2^{k_1+1}B}\|_{L^{p_1'(\cdot)}}\\
&\quad\times\sum_{k_2=1}^{\infty}|2^{k_2}B|^{-1}\Big\{\|(b_2-(b_2)_{2^{k_2+1}B})\chi_{2^{k_2+1}B}\|_{L^{s_2(\cdot)}}\\
&\quad+|(b_2)_{2^{k_2+1}B}-(b_2)_B|\|\chi_{2^{k_2+1}B}\|_{L^{s_2(\cdot)}}\Big\}\|f_2\chi_{2^{k_2+1}B}\|_{L^{p_2(\cdot)}}\|\chi_{2^{k_2+1}B}\|_{L^{\phi(\cdot)}}\\
&\leq C|B|^{\eta_2+\gamma_1+\gamma_2}\prod_{i=1}^{2}\|f_i\|_{\dot{B}^{p_i(\cdot),\gamma_i}}\|b_2\|_{\mathrm{CBMO}^{s_2(\cdot),\eta_2}}.
\end{aligned}$$

Thus, we have

$$V_{24}\leq C|B|^{\lambda}\|\chi_B\|_{L^{q(\cdot)}}\prod_{i=1}^{2}(\|b_i\|_{\mathrm{CBMO}^{s_i(\cdot),\eta_i}}\|f_i\|_{\dot{B}^{p_i(\cdot),\gamma_i}}). \tag{38}$$

For V_4, we decompose

$$V_4\leq\|\chi_B T((b_1-(b_1)_B)f_1\chi_{2B},(b_2-(b_2)_B)f_2\chi_{2B})\|_{L^{q(\cdot)}} \tag{39}$$

$$
\begin{aligned}
&+ \|\chi_B T((b_1-(b_1)_B)f_1\chi_{2B},(b_2-(b_2)_B)f_2\chi_{(2B)^c})\|_{L^{q(\cdot)}}\\
&+ \|\chi_B T((b_1-(b_1)_B)f_1\chi_{(2B)^c},(b_2-(b_2)_B)f_2\chi_{2B})\|_{L^{q(\cdot)}}\\
&+ \|\chi_B T((b_1-(b_1)_B)f_1\chi_{(2B)^c},(b_2-(b_2)_B)f_2\chi_{(2B)^c})\|_{L^{q(\cdot)}}\\
=:&\ V_{41}+V_{42}+V_{43}+V_{44}.
\end{aligned}
$$

For V_{41}, by Hölder's inequality Lemmas 2.5, 2.4 and 2.1, we get

(40)

$$
\begin{aligned}
V_{41} &\le C\prod_{i=1}^{2}\|f_i\chi_{2B}\|_{L^{p_i(\cdot)}}\|(b_i-(b_i)_B)\chi_{2B})\|_{L^{s_i(\cdot)}}\\
&\le C\prod_{i=1}^{2}\|f_i\chi_{2B}\|_{L^{p_i(\cdot)}}\Big\{\|(b_i-(b_i)_{2B})\chi_{2B}\|_{L^{s_i(\cdot)}}+|(b_i)_{2B}-(b_i)_B|\|\chi_{2B}\|_{L^{s_i(\cdot)}}\Big\}\\
&\le C|B|^{\lambda}\|\chi_B\|_{L^{q(\cdot)}}\prod_{i=1}^{2}(\|b_i\|_{\mathrm{CBMO}^{s_i(\cdot),\eta_i}}\|f_i\|_{\dot{B}^{p_i(\cdot),\gamma_i}}).
\end{aligned}
$$

For V_{42}, set $\frac{1}{\phi_i(\cdot)}=\frac{1}{p_i'(\cdot)}-\frac{1}{s_i(\cdot)}$, $i=1,2$. By Hölder's inequality, Lemmas 2.4 and 2.1, we deduce

$$
\begin{aligned}
(41)\quad & |T((b_1-(b_1)_B)f_1\chi_{2B},(b_2-(b_2)_B)f_2\chi_{(2B)^c})(x)|\\
&\le C\int_{2B}|b_1(y_1)-(b_1)_B||f_1(y_1)|dy_1\\
&\quad\times\sum_{k=1}^{\infty}|2^kB|^{-2}\int_{2^{k+1}B}|b_2(y_2)-(b_2)_B||f_2(y_2)|dy_2\\
&\le C\|(b_1-(b_1)_B)\chi_{2B}\|_{L^{s_1(\cdot)}}\|f_1\chi_{2B}\|_{L^{p_1(\cdot)}}\|\chi_{2B}\|_{L^{\phi_1(\cdot)}}\\
&\quad\times\sum_{k=1}^{\infty}|2^kB|^{-2}\|(b_2-(b_2)_B)\chi_{2^{k+1}B}\|_{L^{s_2(\cdot)}}\|f_2\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}}\|\chi_{2^{k+1}B}\|_{L^{\phi_2(\cdot)}}\\
&\le C\|(b_1-(b_1)_B)\chi_{2B}\|_{L^{s_1(\cdot)}}\|f_1\chi_{2B}\|_{L^{p_1(\cdot)}}\|\chi_{2B}\|_{L^{\phi_1(\cdot)}}\\
&\quad\times\sum_{k=1}^{\infty}|2^kB|^{-2}\Big\{\|(b_2-(b_2)_{2^{k+1}B})\chi_{2^{k+1}B}\|_{L^{s_2(\cdot)}}\\
&\quad+|(b_2)_{2^{k+1}B}-(b_2)_B|\|\chi_{2^{k+1}B}\|_{L^{s_2(\cdot)}}\Big\}\|f_2\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}}\|\chi_{2^{k+1}B}\|_{L^{\phi_2(\cdot)}}\\
&\le C|B|^{\lambda}\prod_{i=1}^{2}(\|f_i\|_{\dot{B}^{p_i(\cdot),\gamma_i}}\|b_i\|_{\mathrm{CBMO}^{s_i(\cdot),\eta_i}})\sum_{k=1}^{\infty}k2^{kn(\eta_2+\gamma_2-1)}\\
&\le C|B|^{\lambda}\prod_{i=1}^{2}(\|f_i\|_{\dot{B}^{p_i(\cdot),\gamma_i}}\|b_i\|_{\mathrm{CBMO}^{s_i(\cdot),\eta_i}}).
\end{aligned}
$$

This implies that

$$V_{42} \leq C|B|^{\lambda}\|\chi_B\|_{L^{q(\cdot)}}\prod_{i=1}^{2}(\|b_i\|_{\mathrm{CBMO}^{s_i(\cdot),\eta_i}}\|f_i\|_{\dot{B}^{p_i(\cdot),\gamma_i}}). \tag{42}$$

Similar to the estimates for V_{42}, we get

$$V_{43} \leq C|B|^{\lambda}\|\chi_B\|_{L^{q(\cdot)}}\prod_{i=1}^{2}(\|b_i\|_{\mathrm{CBMO}^{s_i(\cdot),\eta_i}}\|f_i\|_{\dot{B}^{p_i(\cdot),\gamma_i}}). \tag{43}$$

For V_{44}, in view of $\eta_i+\gamma_i<0$, by Hölder's inequality and Lemma 2.4, we have (44)

$$\begin{aligned}
&|T((b_1-(b_1)_B)f_1\chi_{(2B)^c},(b_2-(b_2)_B)f_2\chi_{(2B)^c})(x)|\\
&\leq C\prod_{i=1}^{2}\bigg(\sum_{k_i=1}^{\infty}|2^{k_i}B|^{-1}\int_{2^{k_i+1}B}|b_i(y_i)-(b_i)_B||f_i(y_i)|dy_i\bigg)\\
&\leq C\prod_{i=1}^{2}\bigg(\sum_{k_i=1}^{\infty}|2^{k_i}B|^{-1}\|(b_i-(b_i)_B)\chi_{2^{k_i+1}B}\|_{L^{s_i(\cdot)}}\|f_i\chi_{2^{k_i+1}B}\|_{L^{p_i(\cdot)}}\|\chi_{2^{k_i+1}B}\|_{L^{\phi_i(\cdot)}}\bigg)\\
&\leq C|B|^{\lambda}\prod_{i=1}^{2}\|b_i\|_{\mathrm{CBMO}^{s_i(\cdot),\eta_i}}\|f_i\|_{\dot{B}^{p_i(\cdot),\gamma_i}}\bigg(\sum_{k_i=1}^{\infty}k_i2^{k_in(\eta_i+\gamma_i)}\bigg)\\
&\leq C|B|^{\lambda}\prod_{i=1}^{2}\|b_i\|_{\mathrm{CBMO}^{s_i(\cdot),\eta_i}}\|f_i\|_{\dot{B}^{p_i(\cdot),\gamma_i}}.
\end{aligned}$$

Thus we get

$$V_{44} \leq C|B|^{\lambda}\|\chi_B\|_{L^{q(\cdot)}}\prod_{i=1}^{2}(\|b_i\|_{\mathrm{CBMO}^{s_i(\cdot),\eta_i}}\|f_i\|_{\dot{B}^{p_i(\cdot),\gamma_i}}). \tag{45}$$

The proof of Theorem 2.9 is complete.

5. Proof of Theorem 2.10

Without loss of generality, we may assume that $m=2$. For $f\in\dot{B}^{p(\cdot),v}(\mathbb{R}^n)$, we write

$$f(x)=f(x)\chi_{2B}+f(x)(1-\chi_{2B})=f_1(x)+f_2(x).$$

We need to prove

$$\|\chi_BT_{\vec{b}}(f)\|_{L^{q(\cdot)}}\leq C|B|^{\lambda}\|\chi_B\|_{L^{q(\cdot)}}\|f\|_{\dot{B}^{p(\cdot),v}}\prod_{i=1}^{2}\|b_i\|_{\mathrm{CBMO}^{s_i(\cdot),u_i}}. \tag{46}$$

Applying the Minkowski inequality, we get

$$\begin{aligned}\|\chi_BT_{\vec{b}}(f)\|_{L^{q(\cdot)}}&\leq\|\chi_B(b_1-(b_1)_B)(b_2-(b_2)_B)Tf\|_{L^{q(\cdot)}}\\
&\quad+\|\chi_B(b_1-(b_1)_B)T((b_2-(b_2)_B)f)\|_{L^{q(\cdot)}}\end{aligned} \tag{47}$$

$$
\begin{aligned}
&+ \|\chi_B(b_2-(b_2)_B)T((b_1-(b_1)_B)f)\|_{L^{q(\cdot)}} \\
&+ \|\chi_B T((b_1-(b_1)_B)(b_2-(b_2)_B)f)\|_{L^{q(\cdot)}} \\
=:\ & W_1+W_2+W_3+W_4.
\end{aligned}
$$

For W_1, we have

$$
\begin{aligned}
W_1 &\le \|\chi_B(b_1-(b_1)_B)(b_2-(b_2)_B)Tf_1\|_{L^{q(\cdot)}} \\
&\quad + \|\chi_B(b_1-(b_1)_B)(b_2-(b_2)_B)Tf_2\|_{L^{q(\cdot)}} \\
&=: W_{11}+W_{12}.
\end{aligned} \tag{48}
$$

For W_{11}, using the Hölder inequality and the $L^{p(\cdot)}$-boundedness of T (see [7, Corollary 2.5]), Lemmas 2.2 and 2.4, we obtain

$$
\begin{aligned}
W_{11} &\le \|\chi_B Tf_1\|_{L^{p(\cdot)}}\prod_{i=1}^{2}\|(b_i-(b_i)_B)\chi_B\|_{L^{s_i(\cdot)}} \\
&\le C\|f_1\|_{L^{p(\cdot)}}\prod_{i=1}^{2}(\|b_i\|_{\mathrm{CBMO}^{s_i(\cdot),u_i}}|B|^{u_i}\|\chi_B\|_{L^{s_i(\cdot)}}) \\
&\le C|2B|^{v}\|\chi_{2B}\|_{L^{p(\cdot)}}\|f\|_{\dot{B}^{p(\cdot),v}}\prod_{i=1}^{2}(\|b_i\|_{\mathrm{CBMO}^{s_i(\cdot),u_i}}|B|^{u_i}\|\chi_B\|_{L^{s_i(\cdot)}}) \\
&\le C|B|^{\lambda}\|\chi_B\|_{L^{q(\cdot)}}\|f\|_{\dot{B}^{p(\cdot),v}}\prod_{i=1}^{2}\|b_i\|_{\mathrm{CBMO}^{s_i(\cdot),u_i}}.
\end{aligned} \tag{49}
$$

For W_{12}, in view of $v<0$, by Hölder's inequality and Lemma 2.1, we get

$$
\begin{aligned}
|Tf_2(x)| &\le \sum_{k=1}^{\infty}|2^kB|^{-1}\int_{2^{k+1}B\setminus 2^kB}|f(y)|dy \\
&\le C\sum_{k=1}^{\infty}|2^kB|^{-1}\|f\chi_{2^{k+1}B}\|_{L^{p(\cdot)}}\|\chi_{2^{k+1}B}\|_{L^{p'(\cdot)}} \\
&\le C\sum_{k=1}^{\infty}|2^kB|^{-1}|2^{k+1}B|^{v}\|\chi_{2^{k+1}B}\|_{L^{p(\cdot)}}\|f\|_{\dot{B}^{p(\cdot),v}}\|\chi_{2^{k+1}B}\|_{L^{p'(\cdot)}} \\
&\le C|B|^{v}\|f\|_{\dot{B}^{p(\cdot),v}}\sum_{k=1}^{\infty}2^{knv} \\
&\le C|B|^{v}\|f\|_{\dot{B}^{p(\cdot),v}}.
\end{aligned} \tag{50}
$$

Thus, similar to the estimates for W_{11}, we obtain

$$
\begin{aligned}
W_{12} &\le C|B|^{v}\|f\|_{\dot{B}^{p(\cdot),v}}\|\chi_B(b_1-(b_1)_B)(b_2-(b_2)_B)\|_{L^{q(\cdot)}} \\
&\le C|B|^{v}\|f\|_{\dot{B}^{p(\cdot),v}}\|\chi_B\|_{L^{p(\cdot)}}\prod_{i=1}^{2}\|(b_i-(b_i)_B)\chi_B\|_{L^{s_i(\cdot)}}
\end{aligned} \tag{51}
$$

$$\le C|B|^{\lambda}\|\chi_B\|_{L^{q(\cdot)}}\|f\|_{\dot{B}^{p(\cdot),v}}\prod_{i=1}^{2}\|b_i\|_{\mathrm{CBMO}^{s_i(\cdot),u_i}}.$$

For W_2, we have

$$\begin{aligned}W_2 &\le \|\chi_B(b_1-(b_1)_B)T((b_2-(b_2)_B)f_1)\|_{L^{q(\cdot)}}\\ &\quad+\|\chi_B(b_1-(b_1)_B)T((b_2-(b_2)_B)f_2)\|_{L^{q(\cdot)}}\\ &=: W_{21}+W_{22}.\end{aligned}\tag{52}$$

For W_{21}, set $\frac{1}{q_1(\cdot)}=\frac{1}{s_2(\cdot)}+\frac{1}{p(\cdot)}$, then $\frac{1}{q(\cdot)}=\frac{1}{q_1(\cdot)}+\frac{1}{s_1(\cdot)}$, by Hölder's inequality, the $L^{q_1(\cdot)}$-boundedness of T, Lemmas 2.3, 2.1 and 2.4, we obtain

$$\begin{aligned}W_{21} &\le C\|(b_1-(b_1)_B)\chi_B\|_{L^{s_1(\cdot)}}\|\chi_B T((b_2-(b_2)_B)f_1)\|_{L^{q_1(\cdot)}}\\ &\le C\|(b_1-(b_1)_B)\chi_B\|_{L^{s_1(\cdot)}}\|(b_2-(b_2)_B)f_1\|_{L^{q_1(\cdot)}}\\ &\le C\|(b_1-(b_1)_B)\chi_B\|_{L^{s_1(\cdot)}}\|(b_2-(b_2)_B)\chi_{2B}\|_{L^{s_2(\cdot)}}\|f_1\|_{L^{p(\cdot)}}\\ &\le C|B|^{u_1}\|\chi_B\|_{L^{s_1(\cdot)}}\|b_1\|_{\mathrm{CBMO}^{s_1(\cdot),u_1}}|2B|^{v}\|\chi_{2B}\|_{L^{p(\cdot)}}\|f\|_{\dot{B}^{p(\cdot),v}}\\ &\quad\times\left\{\|(b_2-(b_2)_{2B})\chi_{2B}\|_{L^{s_2(\cdot)}}+|(b_2)_{2B}-(b_2)_B|\|\chi_{2B}\|_{L^{s_2(\cdot)}}\right\}\\ &\le C|B|^{\lambda}\|\chi_B\|_{L^{q(\cdot)}}\|f\|_{\dot{B}^{p(\cdot),v}}\prod_{i=1}^{2}\|b_i\|_{\mathrm{CBMO}^{s_i(\cdot),u_i}}.\end{aligned}\tag{53}$$

For W_{22}, set $\frac{1}{\varphi(\cdot)}=\frac{1}{p'(\cdot)}-\frac{1}{s_2(\cdot)}$, by Hölder's inequality and Lemma 2.4, we have

(54)

$$\begin{aligned}&|T((b_2-(b_2)_B)f_2)(x)|\\ &\le \sum_{k=1}^{\infty}|2^kB|^{-1}\int_{2^{k+1}B\setminus 2^kB}|b_2(y)-(b_2)_B||f(y)|dy\\ &\le C\sum_{k=1}^{\infty}|2^kB|^{-1}\|(b_2-(b_2)_B)\chi_{2^{k+1}B}\|_{L^{s_2(\cdot)}}\|f\chi_{2^{k+1}B}\|_{L^{p(\cdot)}}\|\chi_{2^{k+1}B}\|_{L^{\varphi(\cdot)}}\\ &\le C\sum_{k=1}^{\infty}|2^kB|^{-1}|2^{k+1}B|^{v}\|\chi_{2^{k+1}B}\|_{L^{p(\cdot)}}\|f\|_{\dot{B}^{p(\cdot),v}}\|\chi_{2^{k+1}B}\|_{L^{\varphi(\cdot)}}\\ &\quad\times\left\{\|(b_2-(b_2)_{2^{k+1}B})\chi_{2^{k+1}B}\|_{L^{s_2(\cdot)}}+|(b_2)_{2^{k+1}B}-(b_2)_B|\|\chi_{2^{k+1}B}\|_{L^{s_2(\cdot)}}\right\}\\ &\le C\|f\|_{\dot{B}^{p(\cdot),v}}\sum_{k=1}^{\infty}|2^kB|^{-1}|2^{k+1}B|^{v}\|\chi_{2^{k+1}B}\|_{L^{p(\cdot)}}\|\chi_{2^{k+1}B}\|_{L^{\varphi(\cdot)}}\\ &\quad\times k|2^{k+1}B|^{u_2}\|b_2\|_{\mathrm{CBMO}^{s_2(\cdot),u_2}}\|\chi_{2^{k+1}B}\|_{L^{s_2(\cdot)}}\\ &\le C|B|^{u_2+v}\|b_2\|_{\mathrm{CBMO}^{s_2(\cdot),u_2}}\|f\|_{\dot{B}^{p(\cdot),v}}\sum_{k=1}^{\infty}k2^{kn(u_2+v)}\end{aligned}$$

$$\leq C|B|^{u_2+v}\|b_2\|_{\mathrm{CBMO}^{s_2(\cdot),u_2}}\|f\|_{\dot{B}^{p(\cdot),v}}.$$

Thus, we get

(55)

$$\begin{aligned}
W_{22} &\leq C|B|^{u_2+v}\|b_2\|_{\mathrm{CBMO}^{s_2(\cdot),u_2}}\|f\|_{\dot{B}^{p(\cdot),v}}\|(b_1-(b_1)_B)\chi_B\|_{L^{q(\cdot)}}\\
&\leq C|B|^{u_2+v}\|b_2\|_{\mathrm{CBMO}^{s_2(\cdot),u_2}}\|f\|_{\dot{B}^{p(\cdot),v}}\|(b_1-(b_1)_B)\chi_B\|_{L^{s_1(\cdot)}}\|\chi_B\|_{L^{q_1(\cdot)}}\\
&\leq C|B|^{u_1+u_2+v}\|\chi_B\|_{L^{s_1(\cdot)}}\|\chi_B\|_{L^{q_1(\cdot)}}\|f\|_{\dot{B}^{p(\cdot),v}}\prod_{i=1}^{2}\|b_i\|_{\mathrm{CBMO}^{s_i(\cdot),u_i}}\\
&\leq C|B|^{\lambda}\|\chi_B\|_{L^{q(\cdot)}}\|f\|_{\dot{B}^{p(\cdot),v}}\prod_{i=1}^{2}\|b_i\|_{\mathrm{CBMO}^{s_i(\cdot),u_i}}.
\end{aligned}$$

Noting that W_3 is symmetric to W_2, therefore, we have

$$W_3 \leq C|B|^{\lambda}\|\chi_B\|_{L^{q(\cdot)}}\|f\|_{\dot{B}^{p(\cdot),v}}\prod_{i=1}^{2}\|b_i\|_{\mathrm{CBMO}^{s_i(\cdot),u_i}}. \tag{56}$$

For W_4, we have

$$\begin{aligned}
W_4 &\leq \|\chi_B T((b_1-(b_1)_B)(b_2-(b_2)_B)f_1)\|_{L^{q(\cdot)}}\\
&\quad + \|\chi_B T((b_1-(b_1)_B)(b_2-(b_2)_B)f_2)\|_{L^{q(\cdot)}}\\
&=: W_{41}+W_{42}.
\end{aligned} \tag{57}$$

For W_{41}, by the $L^{p(\cdot)}$-boundedness of T, Lemmas 2.3 and 2.4, we obtain

$$\begin{aligned}
W_{41} &\leq C\|(b_1-(b_1)_B)(b_2-(b_2)_B)f_1\|_{L^{q(\cdot)}}\\
&\leq C\|f_1\|_{L^{p(\cdot)}}\prod_{i=1}^{2}\|(b_i-(b_i)_B)\chi_{2B}\|_{L^{s_i(\cdot)}}\\
&\leq C|2B|^{v}\|\chi_{2B}\|_{L^{p(\cdot)}}\|f\|_{\dot{B}^{p(\cdot),v}}\\
&\quad\times\prod_{i=1}^{2}\left\{\|(b_i-(b_i)_{2B})\chi_{2B}\|_{L^{s_i(\cdot)}}+|(b_i)_{2B}-(b_i)_B|\|\chi_{2B}\|_{L^{s_i(\cdot)}}\right\}\\
&\leq C|2B|^{v}\|\chi_{2B}\|_{L^{p(\cdot)}}\|f\|_{\dot{B}^{p(\cdot),v}}\prod_{i=1}^{2}|2B|^{u_i}\|\chi_{2B}\|_{L^{s_i(\cdot)}}\|b_i\|_{\mathrm{CBMO}^{s_i(\cdot),u_i}}\\
&\leq C|B|^{\lambda}\|\chi_B\|_{L^{q(\cdot)}}\|f\|_{\dot{B}^{p(\cdot),v}}\prod_{i=1}^{2}\|b_i\|_{\mathrm{CBMO}^{s_i(\cdot),u_i}}.
\end{aligned} \tag{58}$$

For W_{42}, by Hölder's inequality, Lemmas 2.1 and 2.4, we deduce that

(59)

$$|T((b_1-(b_1)_B)(b_2-(b_2)_B)f_2)(x)|$$

$$\leq C\sum_{k=1}^{\infty}|2^kB|^{-1}\int_{2^{k+1}B\setminus 2^kB}|b_1(y)-(b_1)_B||b_2(y)-(b_2)_B||f(y)|dy$$
$$\leq C\sum_{k=1}^{\infty}|2^kB|^{-1}\prod_{i=1}^{2}\|(b_i-(b_i)_B)\chi_{2^{k+1}B}\|_{L^{s_i(\cdot)}}\|f\chi_{2^{k+1}B}\|_{L^{p(\cdot)}}\|\chi_{2^{k+1}B}\|_{L^{q'(\cdot)}}$$
$$\leq C\sum_{k=1}^{\infty}|2^kB|^{-1}|2^{k+1}B|^{v}\|\chi_{2^{k+1}B}\|_{L^{p(\cdot)}}\|f\|_{\dot{B}^{p(\cdot),v}}\|\chi_{2^{k+1}B}\|_{L^{q'(\cdot)}}$$
$$\times\prod_{i=1}^{2}\left\{\|(b_i-(b_i)_{2^{k+1}B})\chi_{2^{k+1}B}\|_{L^{s_i(\cdot)}}+|(b_i)_{2^{k+1}B}-(b_i)_B|\|\chi_{2^{k+1}B}\|_{L^{s_i(\cdot)}}\right\}$$
$$\leq C|B|^{\lambda}\|f\|_{\dot{B}^{p(\cdot),v}}\prod_{i=1}^{2}\|b_i\|_{\mathrm{CBMO}^{s_i(\cdot),u_i}}\sum_{k=1}^{\infty}k^2 2^{kn\lambda}$$
$$\leq C|B|^{\lambda}\|f\|_{\dot{B}^{p(\cdot),v}}\prod_{i=1}^{2}\|b_i\|_{\mathrm{CBMO}^{s_i(\cdot),u_i}}.$$

Thus, we arrive at the desired inequality

$$W_{42}\leq C|B|^{\lambda}\|\chi_B\|_{L^{q(\cdot)}}\|f\|_{\dot{B}^{p(\cdot),v}}\prod_{i=1}^{2}\|b_i\|_{\mathrm{CBMO}^{s_i(\cdot),u_i}}. \tag{60}$$

The proof of Theorem 2.10 is complete.

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