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SUFFICIENT CONDITIONS AND RADII PROBLEMS FOR A STARLIKE CLASS INVOLVING A DIFFERENTIAL INEQUALITY

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Abstract. Let $\mathcal{A}_n$ be the class of analytic functions $f(z)$ of the form $f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k$, $n \in \mathbb{N}$ defined on the open unit disk $\mathbb{D}$, and let

$$\Omega_n := \left\{ f \in \mathcal{A}_n : \left| zf'(z) - f(z) \right| < \frac{1}{2},\ z \in \mathbb{D} \right\}.$$

In this paper, we make use of differential subordination technique to obtain sufficient conditions for the class Ω_n. Writing $\Omega := \Omega_1$, we obtain inclusion properties of Ω with respect to functions which map $\mathbb{D}$ onto certain parabolic regions and as a consequence, establish a relation connecting the parabolic starlike class $\mathcal{S}_P$ and the uniformly starlike UST. Various radius problems for the class Ω are considered and the sharpness of the radii estimates is obtained analytically besides graphical illustrations.

1. Introduction

Let $\mathbb{C}$ be the set of complex numbers and let $\mathcal{H} := \mathcal{H}(\mathbb{D})$ be the totality of functions $f(z)$ that are analytic in the open unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. For $a \in \mathbb{C}$ and $n \in \mathbb{N} := \{1, 2, 3, \ldots\}$, we define the function classes $\mathcal{H}_n(a)$ and $\mathcal{A}_n$ as follows:

$$\mathcal{H}_n(a) := \left\{ f \in \mathcal{H} : f(z) = a + \sum_{k=n}^{\infty} a_k z^k,\ a_k \in \mathbb{C} \right\}$$

and

$$\mathcal{A}_n := \left\{ f \in \mathcal{H} : f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k,\ a_k \in \mathbb{C} \right\}.$$

In particular, we write $\mathcal{A} := \mathcal{A}_1$. For $0 \le \alpha < 1$, let $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$ be the subclasses of $\mathcal{A}$ which consist of functions that are, respectively, starlike and

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convex of order α. Analytically,

$$\mathcal{S}^*(\alpha) := \left\{ f : \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \right\} \text{ and } \mathcal{C}(\alpha) := \left\{ f : \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \right\}.$$

Further, $\mathcal{S}^* := \mathcal{S}^*(0)$ and $\mathcal{C} := \mathcal{C}(0)$ are the well-known classes of starlike and convex functions in $\mathbb{D}$. We note that $f \in \mathcal{C}$ if and only if $zf' \in \mathcal{S}^*$ and $\mathcal{C} \subsetneq \mathcal{S}^* \subsetneq \mathcal{S}$, where $\mathcal{S}$ is the collection of all functions $f \in \mathcal{A}$ that are univalent in $\mathbb{D}$. For further details related to these classes, we refer to the monograph of Duren [5]. For $f, g \in \mathcal{H}$, we say that $f(z)$ is *subordinate* to $g(z)$, written $f \prec g$, if there exists an analytic function $w(z)$ in $\mathbb{D}$ satisfying $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$. If $f \prec g$, then $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$. Further, if the function $g(z)$ happens to be univalent, then $f \prec g$ if and only if

$$f(0) = g(0) \text{ and } f(\mathbb{D}) \subset g(\mathbb{D}).$$

Let $\Psi : \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$ be a complex-valued analytic function and let $h : \mathbb{D} \to \mathbb{C}$ be univalent. If $p \in \mathcal{H}$ satisfies the first-order *differential subordination*

$$\Psi\left(p(z), zp'(z); z\right) \prec h(z), \qquad z \in \mathbb{D}, \tag{1}$$

then $p(z)$ is called a solution of the differential subordination (1). If $q : \mathbb{D} \to \mathbb{C}$ is univalent and $p \prec q$ for all $p(z)$ satisfying (1), then $q(z)$ is said to be a dominant of (1). A dominant $\tilde{q}(z)$ that satisfies $\tilde{q} \prec q$ for all dominants $q(z)$ of (1) is called the best dominant of (1). We note that the best dominant is unique up to a rotation of $\mathbb{D}$. For more insight into various forms of differential subordinations, we refer to the monograph of Miller and Mocanu [14] (see also Bulboacă [3]). For two functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in \mathcal{A}$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in \mathcal{A}$, the *Hadamard product* (or convolution) of f and g, denoted by $f * g$, is defined as the analytic function

$$h(z) = (f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k, \qquad z \in \mathbb{D}.$$

Under the operation of Hadamard product, the function $\ell(z) = z/(1-z) = z + \sum_{k=2}^{\infty} z^k$ mapping $\mathbb{D}$ onto the half-plane $\operatorname{Re}(w) > -1/2$ plays the role of identity element. That is, for any function $f \in \mathcal{A}$,

$$(f * \ell)(z) = f(z) = (\ell * f)(z).$$

Recently, Peng and Zhong [17] introduced a function class Ω involving a differential inequality and given by

$$\Omega := \left\{ f \in \mathcal{A} : |zf'(z) - f(z)| < \frac{1}{2},\ z \in \mathbb{D} \right\}. \tag{2}$$

The authors in [17] have shown that Ω is a subclass of the class of starlike functions $\mathcal{S}^*$, and hence established that the members of Ω are univalent in $\mathbb{D}$. Besides discussing several geometric properties of the members of Ω, Peng and Zhong [17] proved that the radius of convexity for Ω is $1/2$, and that Ω is closed under the operation of Hadamard product, i.e., if $f_1, f_2 \in \Omega$, then

$f_1 * f_2 \in \Omega$. Later, Obradović and Peng [15] considered the class Ω and used basic techniques to obtain two sufficient conditions for functions $f \in \mathcal{A}$ to be in the class Ω. Very recently, Peng and Obradović [16] discussed a few estimates on the logarithmic coefficients and the inverse function coefficients for the functions in Ω. Apart from many other important results, Peng and Obradović [16] proved that if $f \in \Omega$, then the Libera operator

$$L\left(f(z)\right) = \frac{2}{z}\int_0^z f(\zeta)d\zeta$$

is also in Ω.

In this paper, we consider the class

$$\Omega_n := \left\{ f \in \mathcal{A}_n : |zf'(z) - f(z)| < \frac{1}{2},\ z \in \mathbb{D} \right\},$$

which is, in some sense, a natural generalization of $\Omega := \Omega_1$, and discuss several interesting and characteristic properties of the members of Ω_n. More explicitly, in Section 2, we use the techniques of differential subordination to establish sufficient conditions for $f \in \mathcal{A}_n$ to be in the class Ω_n. The sufficient conditions derived by Obradović and Peng [15] are obtained as special cases. Moreover, we use these results to construct functions of the form

$$f(z) = \int_0^1\int_0^1 \mathcal{J}(s,t,z)dsdt, \qquad z \in \mathbb{D}, \tag{3}$$

and obtain conditions on the analytic kernel-function $\mathcal{J}(z)$ so that the function $f(z)$ given by (3) is a member of Ω_n. In Section 3, inclusion relations between the parabolic starlike class $\mathcal{S}_P$, the uniformly starlike class UST and the class Ω given by (2) are studied, and as a consequence a remarkable result connecting $\mathcal{S}_P$ and UST is derived. Section 4 discusses several radius problems for Ω and, in particular, the parabolic radius for Ω is found to be $2/3$. Furthermore, certain geometrically defined subclasses (e.g., $\mathcal{S}_e^*$, $\mathcal{S}_C^*$, $\mathcal{S}_{\leftmoon}^*$ etc.) of $\mathcal{S}^*$ are also considered in Section 4 and the corresponding radii problems for Ω are settled. Most importantly, the sharpness of radius estimates proved analytically is illustrated graphically as well. In the end, we pose a problem on the Hadamard product of members of Ω.

2. Sufficient conditions for Ω_n

To prove our results, we use the following lemma related to first-order differential subordination.

Lemma 2.1 ([14, Theorem 3.1b, p. 71])**.** *Let $h(z)$ be a convex function in $\mathbb{D}$ with $h(0) = a$, $\gamma \neq 0$ and $\mathrm{Re}\gamma \geq 0$. If $p(z) \in \mathcal{H}_n(a)$ and*

$$p(z) + \frac{zp'(z)}{\gamma} \prec h(z),$$

then

$$p(z) \prec q(z) \prec h(z),$$

where

$$q(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(\xi)\, \xi^{\gamma/n-1} d\xi.$$

The function $q(z)$ is convex and is the best dominant.

Theorem 2.2. *Let $n \in \mathbb{N}$ and $\gamma \geq 1$. If $f \in \mathcal{A}_n$ satisfies the inequality*

$$\left| zf''(z) + (\gamma - 1)\left(f'(z) - \frac{f(z)}{z}\right)\right| < \frac{n+\gamma}{2}, \tag{4}$$

then $f \in \Omega_n$. The result is sharp for the function $\mathcal{L}_{n,\mu}(z)$ given by

$$\mathcal{L}_{n,\mu}(z) := z + \frac{\mu}{2n} z^{n+1}, \qquad |\mu| = 1. \tag{5}$$

Proof. In terms of subordination, the inequality (4) can be rewritten as

$$zf''(z) + (\gamma - 1)\left(f'(z) - \frac{f(z)}{z}\right) \prec \left(\frac{n+\gamma}{2}\right) z, \quad z \in \mathbb{D}. \tag{6}$$

Setting

$$p(z) = f'(z) - \frac{f(z)}{z} = \sum_{k=n}^{\infty} (ka_{k+1}) z^k \in \mathcal{H}_n(0),$$

the subordination (6) takes the form

$$\gamma p(z) + zp'(z) \prec \left(\frac{n+\gamma}{2}\right) z =: h(z).$$

It can be easily seen that $h(z)$ is convex in $\mathbb{D}$ and $h(0) = 0 = p(0)$. Therefore, Lemma 2.1 is applicable and hence, we have

$$p(z) \prec \frac{1}{nz^{\gamma/n}} \int_0^z \left(\left(\frac{n+\gamma}{2}\right)\xi\right) \xi^{\gamma/n-1} d\xi = \frac{z}{2}.$$

This further implies that

$$\left| f'(z) - \frac{f(z)}{z} \right| < \frac{1}{2}. \tag{7}$$

Now, making use of (7) and the fact that $f(0) = 0$, we obtain

$$|zf'(z) - f(z)| = |z| \left| f'(z) - \frac{f(z)}{z} \right| < \frac{1}{2}.$$

This proves that $f \in \Omega_n$. For the function $\mathcal{L}_{n,\mu}(z)$ defined in (5),

$$\left| z\mathcal{L}''_{n,\mu}(z) + (\gamma - 1)\left(\mathcal{L}'_{n,\mu}(z) - \frac{\mathcal{L}_{n,\mu}(z)}{z}\right)\right| = \left| \left(\frac{n+\gamma}{2}\right) \mu z^n \right| < \frac{n+\gamma}{2},$$

proving that $\mathcal{L}_{n,\mu}(z)$ satisfies the condition (4), and hence is a member of Ω_n. Further, for $z \in \mathbb{D}$, the function $\mathcal{L}_{n,\mu}(z)$ satisfies

$$\left|z\mathcal{L}'_{n,\mu}(z) - \mathcal{L}_{n,\mu}(z)\right| = \left|\frac{z^{n+1}}{2}\right| < \frac{1}{2}.$$

This shows that the result is sharp for the function $\mathcal{L}_{n,\mu} \in \Omega_n$ given by (5) and completes the proof. □

Fixing $n = 1$, and then taking $\gamma = 1$ and $\gamma = 2$ in Theorem 2.2, respectively, we obtain the following sufficient conditions established by Obradović and Peng [15].

Corollary 2.3. *If $f \in \mathcal{A}$ satisfies $|zf''(z)| < 1$, then $f \in \Omega$. The number 1 is best possible.*

Corollary 2.4. *Let $f \in \mathcal{A}$. If*

$$\left|z^2 f''(z) + zf'(z) - f(z)\right| < \frac{3}{2},$$

then $f \in \Omega$. The number $3/2$ is best possible.

The following theorem also provides sufficient conditions for Ω_n and is a direct application of Theorem 2.2.

Theorem 2.5. *Let $\gamma \geq 1$, $n \in \mathbb{N}$, and let*

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \in \mathcal{A}_n, \quad z \in \mathbb{D}.$$

If

$$\sum_{k=n+1}^{\infty} (k-1)(k+\gamma-1)|a_k| \leq \frac{n+\gamma}{2}, \tag{8}$$

then $f \in \Omega_n$. Equality holds for the function $\mathcal{L}_{n,\mu} \in \Omega_n$ given by (5).

Proof. Suppose that (8) holds, then for $z \in \mathbb{D}$,

$$\begin{aligned}\left|zf''(z) + (\gamma-1)\left(f'(z) - \frac{f(z)}{z}\right)\right| &= \left|\sum_{k=n+1}^{\infty} (k-1)(k+\gamma-1)\, a_k z^{k-1}\right| \\ &< \sum_{k=n+1}^{\infty} (k-1)(k+\gamma-1)|a_k| \\ &\leq \frac{n+\gamma}{2},\end{aligned}$$

and the desired result follows from Theorem 2.2. It is easy to verify that the function $\mathcal{L}_{n,\mu}(z)$ given by (5) satisfies

$$\sum_{k=n+1}^{\infty} (k-1)[k+\gamma-1]|a_k| = \frac{n+\gamma}{2}.$$

□

As afore, Theorem 2.5 yields:

Corollary 2.6. *Let* $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in \mathcal{A}$. *If* $\sum_{k=2}^{\infty} k(k-1)|a_k| \leq 1$, *then* $f \in \Omega$. *The result is sharp.*

Corollary 2.7. *Let the function* $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in \mathcal{A}$ *satisfies the inequality*

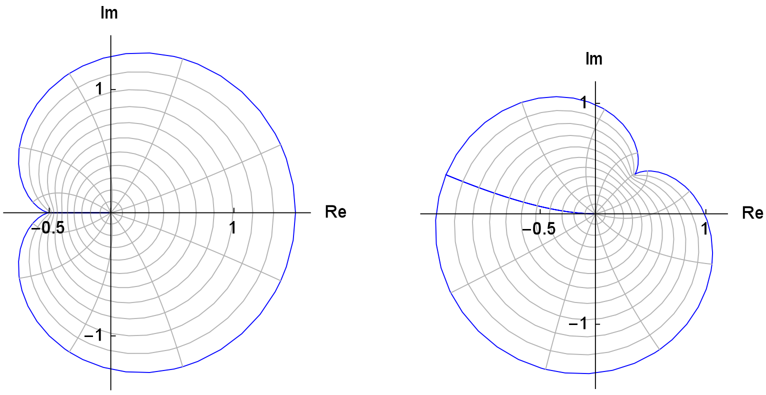
$$\sum_{k=2}^{\infty}(k^2-1)|a_k| \leq \frac{3}{2}.$$

Then $f \in \Omega$ *and the result is sharp.*

Remark 2.8. In both the results, Corollary 2.6 and Corollary 2.7, the equality holds for the function $\mathcal{L}_\mu(z) := \mathcal{L}_{1,\mu}(z) \in \Omega$ given by

$$\mathcal{L}_\mu(z) := z + \frac{\mu}{2}z^2, \qquad |\mu| = 1. \tag{9}$$

Figure 1 shows the region $\mathcal{L}_\mu(\mathbb{D})$ for different values of μ.



(A) $\mu = 1$ (B) $\mu = (-1+i)/\sqrt{2}$

FIGURE 1. The region $\mathcal{L}_\mu(\mathbb{D})$.

We now use Theorem 2.2 to construct functions involving double integrals that are members of the function class Ω_n.

Theorem 2.9. *Let* $\gamma \geq 1$, $n \in \mathbb{N}$, *and let* $\mathcal{J}(z)$ *be analytic in* $\mathbb{D}$ *such that*

$$|\mathcal{J}(z)| \leq \frac{n+\gamma}{2}. \tag{10}$$

Then the function

$$f(z) = z + z^{n+1}\int_0^1\int_0^1 \mathcal{J}(stz)s^{n-1}t^{n+\gamma-1}dsdt \tag{11}$$

belongs to the class Ω_n*. Moreover, if the equality holds in* (10)*, then* (11) *is the function* $\mathcal{L}_{n,\mu} \in \Omega_n$ *given by* (5)*.*

Proof. Let us consider the function $f \in \mathcal{A}_n$ satisfying the second-order differential equation

$$zf''(z) + (\gamma - 1)\left(f'(z) - \frac{f(z)}{z}\right) = z^n \mathcal{J}(z). \tag{12}$$

From (10) and (12), we have

$$\left| zf''(z) + (\gamma - 1)\left(f'(z) - \frac{f(z)}{z}\right) \right| < \frac{n+\gamma}{2}.$$

In view of Theorem 2.2, we conclude that the solution of the differential equation (12) must lie Ω_n. We show that the solution of (12) is the function defined in (11). Writing

$$q(z) = f'(z) - \frac{f(z)}{z},$$

the equation (12) reduces to the form

$$z^{1-\gamma}\big(z^\gamma q(z)\big)' = z^n \mathcal{J}(z).$$

On solving the above equation, we obtain

$$q(z) = z^n \int_0^1 \mathcal{J}(tz) t^{n+\gamma-1} dt,$$

or equivalently,

$$f'(z) - \frac{f(z)}{z} = z^n \int_0^1 \mathcal{J}(tz) t^{n+\gamma-1} dt. \tag{13}$$

The differential equation (13) can be rewritten as

$$z\left(\frac{f(z)}{z} - 1\right)' = z^n \int_0^1 \mathcal{J}(tz) t^{n+\gamma-1} dt,$$

and whose solution can be easily verified to be the function given by (11). Now if the equality holds in (10), then $\mathcal{J}(z) = \mu(n+\gamma)/2$ for some $\mu \in \mathbb{C}$ satisfying $|\mu| = 1$. Substituting this value of $\mathcal{J}(z)$ in (11) and doing some basic analysis, we obtain the function $\mathcal{L}_{n,\mu}(z)$ which is a member of Ω_n. This completes the proof. □

Corollary 2.10. *Let* $\mathcal{J} \in \mathcal{H}$ *such that* $|\mathcal{J}(z)| \leq 1$*. Then the function*

$$f(z) = z + z^2 \int_0^1\int_0^1 \mathcal{J}(stz) t ds dt$$

is a member of Ω *defined in* (2)*.*

From the problems discussed so far, it was observed that the function $\mathcal{L}_{n,\mu}(z)$ given by (5) played the role of an extremal function in Ω_n. We now claim that the function $\mathcal{L}_{n,\mu}(z)$ is indeed an extreme point of Ω_n for each $n \in \mathbb{N}$ and for each $\mu \in \mathbb{C}$ satisfying $|\mu| = 1$. To establish this fact, we will use the following generalized version of Theorem 3.14 of Peng and Zhong [17].

Lemma 2.11. *Let $\phi(z)$ be analytic in $\mathbb{D}$ and satisfies $|\phi(z)| \leq 1$ for each $z \in \mathbb{D}$. Then the function $f \in \mathcal{A}_n$ is an extreme point of Ω_n if and only if*

$$f(z) = z + \frac{z^n}{2n} \int_0^z \phi(\xi)\, d\xi,$$

and

$$\int_0^{2\pi} \log\left[1 - |\phi(e^{i\theta})|\right] d\theta = -\infty.$$

Since $\mathcal{L}_{n,\mu}(z)$ given by (5) can be written as

$$\mathcal{L}_{n,\mu}(z) = z + \frac{\mu}{2n} z^{n+1} = z + \frac{z^n}{2n} \int_0^z \phi(\xi)\, d\xi,$$

where $\phi(z) = \mu z$ satisfies $|\phi(z)| < 1$ and

$$\int_0^{2\pi} \log\left[1 - |\phi(e^{i\theta})|\right] d\theta = \int_0^{2\pi} \log[0] d\theta = -\infty,$$

the desired claim follows from Lemma 2.11.

3. Inclusion properties of Ω

The following necessary condition for a function $f \in \Omega$ was established by Peng and Zhong [17].

Lemma 3.1 ([17, Corollary 3.12])**.** *If $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ is in Ω, then*

$$|a_k| \leq \frac{1}{2(k-1)}, \quad k \geq 2. \tag{14}$$

We now show that (14) is also sufficient for some special kind of functions.

Theorem 3.2. *The function $f_k(z) = z + a_k z^k,\ k \geq 2$ is in Ω if and only if (14) holds.*

Proof. The necessary part easily follows from Lemma 3.1. Now suppose (14) holds, then

$$|z f_k'(z) - f_k(z)| = \left|(k-1) a_k z^k\right| < (k-1)|a_k| \leq \frac{1}{2}.$$

Hence, $f_k \in \Omega$. □

In view of the condition (14), it is easy to verify that the Koebe function $k(z) = z/(1-z)^2 = z + \sum_{k=2}^{\infty} kz^k \in \mathcal{S}^*$ is not a member of Ω, and hence the inclusion $\Omega \subset \mathcal{S}^*$ is proper. We now try to seek the relationship of Ω with other proper subclasses of $\mathcal{S}^*$, e.g., $\mathcal{S}_P$ and UST. To establish our results, the theory of coefficient bounds in the respective function classes is utilized.

Definition (Parabolic Starlike Functions - $\mathcal{S}_P$)**.** A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}_P \subset \mathcal{S}^*$ if and only if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \left|\frac{zf'(z)}{f(z)} - 1\right|, \quad z \in \mathbb{D}.$$

These functions were introduced by Rønning [19] and later studied and generalized by many authors (see [2, 9, 10]). Geometrically, $f \in \mathcal{S}_P$ if and only if all the values taken by the expression $zf'(z)/f(z)$ lie in the parabolic region

$$\mathcal{R}_P := \left\{w = u + iv \in \mathbb{C} : v^2 < 2u - 1\right\}. \tag{15}$$

Lemma 3.3 ([19, Theorem 3])**.** *The function $f_k(z) = z + a_k z^k$ is in $\mathcal{S}_P$ if and only if*

$$|a_k| \leq \frac{1}{(2k-1)}, \quad k \geq 2.$$

In view of Lemma 3.3, the function $\mathcal{L}_\mu(z) = z + \mu z^2/2 \in \Omega$ clearly shows that $\Omega \not\subset \mathcal{S}_P$. For the other direction, we give the following result.

Theorem 3.4. *If $f_k(z) = z + a_k z^k$ belongs to $\mathcal{S}_P$, then $f_k \in \Omega$ for every $k \geq 2$.*

Proof. Let $f_k(z) = z + a_k z^k \in \mathcal{S}_P$. Then from Lemma 3.3

$$|a_k| \leq \frac{1}{(2k-1)}, \quad k \geq 2.$$

Since,

$$\frac{1}{(2k-1)} = \frac{1}{2(k-1)+1} < \frac{1}{2(k-1)}, \quad \forall\ k \geq 2,$$

the desired result follows from Theorem 3.2. □

Definition (Uniformly Starlike Functions - UST)**.** A function $f \in \mathcal{A}$ is said to be in the class $UST \subset \mathcal{S}^*$ if and only if

$$\operatorname{Re}\left(\frac{f(z) - f(\xi)}{(z-\xi)f'(z)}\right) > 0 \tag{16}$$

for every pair $(\xi, z) \in \mathbb{D} \times \mathbb{D}$.

This class UST was introduced by Goodman [8]. These functions have the property that for every circular arc γ contained in $\mathbb{D}$ with center ζ also in $\mathbb{D}$, the arc $f(\gamma)$ is starlike with respect to $f(\zeta)$.

Lemma 3.5 ([13, Theorem 5])**.** *If*

$$|a_k| \leq \sqrt{\frac{k+1}{2k^3}}, \quad k \geq 2,$$

then the function $f_k(z) = z + a_k z^k$ *is in* UST.

Using Lemma 3.5, we prove the following theorem.

Theorem 3.6. *Let* $f_k(z) = z + a_k z^k$ *be in* Ω. *Then, for all* $k \geq 3$, $f_k(z)$ *is in* UST.

Proof. Given $f_k(z) = z + a_k z^k \in \Omega$, we have from Theorem 3.2 that

$$|a_k| \leq \frac{1}{2(k-1)}, \quad k \geq 2.$$

Since,

$$\frac{1}{2(k-1)} \leq \sqrt{\frac{k+1}{2k^3}} \quad \text{for all} \quad k \geq 3,$$

it follows from Lemma 3.5 that $f_k(z) = z + a_k z^k$ is in UST for all $k \geq 3$. $\square$

A Remarkable Result. We know that $\mathcal{S}_P \not\subset UST$ and $UST \not\subset \mathcal{S}_P$, see Ali and Ravichandran [1, p. 21] and Rønning [18, p. 125]. In view of Theorem 3.4 and Theorem 3.6, we remark the following important result which is not available in the literature. This remark gives an inclusion type relation between the members of $\mathcal{S}_P$ and UST.

Remark 3.7. If $f_k(z) = z + a_k z^k$ is in $\mathcal{S}_P$, then $f_k \in UST$ for all $k \geq 3$.

4. Radii problems for Ω

Let $\mathcal{F}$ and $\mathcal{G}$ be two function families in $\mathcal{A}$. Then the $\mathcal{F}$-radius for $\mathcal{G}$ is the largest number $\rho \in (0,1)$ such that $r^{-1}f(rz) \in \mathcal{F}$ for all $f \in \mathcal{G}$, where $0 < r \leq \rho$. Moreover, the number ρ is said to be *sharp* if there exists a function $f_0 \in \mathcal{G}$ such that $r^{-1}f_0(rz) \notin \mathcal{F}$ whenever $r > \rho$ and such a function f_0 is called *extremal function*. The problem of finding the number "ρ" is called a *radius problem* in geometric function theory. In the sequel, by

$$\mathscr{R}_{\mathcal{F}}(\mathcal{G}) = \rho,$$

we mean that ρ is the $\mathcal{F}$-radius for $\mathcal{G}$. For instance, see Duren [5],

$$\mathscr{R}_{\mathcal{S}^*}(\mathcal{S}) = \tanh(\pi/4) \quad \text{and} \quad \mathscr{R}_{\mathcal{C}}(\mathcal{S}) = \mathscr{R}_{\mathcal{C}}(\mathcal{S}^*) = 2 - \sqrt{3}.$$

For a comprehensive list of classical radii results in univalent function theory, we refer to Goodman [7, Chapter 13].

For the function class Ω defined in (2), Peng and Zhong [17, Theorem 3.4] proved that $\mathscr{R}_{\mathcal{C}}(\Omega) = 1/2$ and the estimate is sharp. In this section, we will solve some more radii problems for Ω. Before proceeding, we require a result to be proved with the help of the following lemma.

Lemma 4.1 ([17, Theorem 3.1])**.** *If $f \in \Omega$, then*

$$|z| - \frac{1}{2}|z|^2 \leq |f(z)| \leq |z| + \frac{1}{2}|z|^2, \tag{17}$$

and

$$1 - |z| \leq |f'(z)| \leq 1 + |z|.$$

Further, for each $0 \neq z \in \mathbb{D}$, the equality occurs in both the estimates if and only if

$$f(z) = \mathcal{L}_\mu(z) = z + \frac{\mu}{2}z^2 \quad \textit{with} \quad |\mu| = 1.$$

We now prove the instrumental lemma.

Lemma 4.2. *Let $f \in \Omega$. Then for $|z| = r < 1$, we have the sharp estimate*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{r}{2-r}.$$

Proof. Since $f \in \Omega$, we have

$$|zf'(z) - f(z)| < \frac{1}{2}.$$

This can be equivalently written in the form

$$zf'(z) - f(z) = \frac{1}{2}z^2\phi(z),$$

where $\phi \in \mathcal{H}$ and $|\phi(z)| \leq 1$. This further implies that

$$|zf'(z) - f(z)| \leq \frac{1}{2}|z|^2. \tag{18}$$

Inequality (18) along with (17) yields

$$\left|\frac{zf'(z)}{f(z)} - 1\right| = \frac{1}{|f(z)|}\,|zf'(z) - f(z)| \leq \frac{\frac{1}{2}|z|^2}{|z| - \frac{1}{2}|z|^2} = \frac{r}{2-r}.$$

The sharpness of the estimate follows from Lemma 4.1. □

For brevity, we write

$$\mathcal{Q}_f(z) := \frac{zf'(z)}{f(z)}, \quad f \in \mathcal{A}. \tag{19}$$

Theorem 4.3. *For the function class Ω given by (2), we have*

(i) $\mathscr{R}_{\mathcal{S}^*(\alpha)}(\Omega) = \rho_\alpha := 2(1-\alpha)/(2-\alpha)$, *where $0 \leq \alpha < 1$,*
(ii) $\mathscr{R}_{\mathcal{S}_P}(\Omega) = 2/3$.

Both estimates are sharp.

Proof. (i) To prove this part, we will use the fact that

$$|w(z) - 1| < 1 - \alpha \implies \mathrm{Re}\,(w(z)) > \alpha, \quad z \in \mathbb{D}.$$

Let $f \in \Omega$ and $|z| = r < 1$. Then from Lemma 4.2,

$$|\mathcal{Q}_f(z) - 1| \le \frac{r}{2-r} < 1 - \alpha,$$

provided $r < (1-\alpha)(2-r)$, or $r < \rho_\alpha$. Thus, $f(z)$ is starlike of order α in $|z| < \rho_\alpha$. For sharpness, consider the function $\mathcal{L}(z) := \mathcal{L}_1(z) \in \Omega$ given by

$$\mathcal{L}(z) := z + \frac{z^2}{2}. \tag{20}$$

At the point $z_0 = -2(1-\alpha)/(2-\alpha)$ lying on the circle $|z| = \rho_\alpha$, the function $\mathcal{L}(z)$ satisfies

$$\mathcal{Q}_{\mathcal{L}}(z_0) = \frac{z_0\mathcal{L}'(z_0)}{\mathcal{L}(z_0)} = \frac{1+z_0}{1+z_0/2} = \alpha.$$

This shows that the radius estimate ρ_α is sharp.

(ii) In this part we will make use of the following lemma proved by Shanmugam and Ravichandran [21].

Lemma 4.4 ([21, Lemma 1])**.** *For* $3/4 < a < 3$*, let*

$$r_a = \begin{cases} a - 1/2, & 3/4 < a \le 3/2, \\ \sqrt{2a-2}, & 3/2 \le a < 3. \end{cases}$$

If $|\mathcal{Q}_f(z) - a| < r_a$ *for all* $z \in \mathbb{D}$*, then* $f \in \mathcal{S}_P$*.*

Now, let $f \in \Omega$ and $|z| = r < 1$. Then Lemma 4.2 gives

$$|\mathcal{Q}_f(z) - 1| \le r/(2-r),$$

so that $|\mathcal{Q}_f(z) - 1| < 1/2$ if $r/(2-r) < 1/2$, or equivalently, if $r < 2/3$. Therefore, from Lemma 4.4, it follows that $f \in \mathcal{S}_P$ for $|z| < 2/3$. Since the function $\mathcal{L} \in \Omega$ given by (20) satisfies $\mathrm{Re}(\mathcal{Q}_{\mathcal{L}}(z)) = 1/2$ for $z = -2/3$, the sharpness of the estimate follows from the fact that for each $f \in \mathcal{S}_P$, $\mathrm{Re}(\mathcal{Q}_f(z)) > 1/2$ (cf. Rønning [19]). Figure 2 shows that the domain $\mathcal{Q}_{\mathcal{L}}(|z| < 2/3)$ completely lies inside the parabolic region $\mathcal{R}_P$ given by (15) with their boundaries touching at $u = 1/2$. □

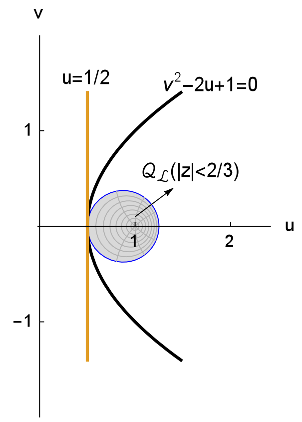


FIGURE 2. Sharpness of $\mathcal{S}_P$-radius for Ω.

For $\mathcal{Q}_f(z)$ defined in (19), Ma and Minda [11] introduced a general method of constructing the function classes $\mathcal{S}^*(\varphi) \subset \mathcal{S}^*$ as

$$\mathcal{S}^*(\varphi) := \{f \in \mathcal{A} : \mathcal{Q}_f(z) \prec \varphi(z)\},$$

where the analytic function $\varphi : \mathbb{D} \to \mathbb{C}$ satisfies (i) $\varphi(z)$ is univalent with positive real part, (ii) $\varphi(z)$ maps $\mathbb{D}$ onto a region that is starlike with respect to $\varphi(0) = 1$, (iii) $\varphi(\mathbb{D})$ is symmetric about the real axis, and (iv) $\varphi'(0) > 0$. Using this unified approach, several classes of starlike functions with nice geometric properties have been introduced and studied in the recent past. In the sequel, we mention some of the important Ma-Minda type classes along with the prerequisite results and then solve the corresponding radii problems for Ω in Theorem 4.10.

(1) $\mathcal{S}^*_e$: Mendiratta et al. [12] introduced the class $\mathcal{S}^*_e$ associated with the exponential function e^z defined by

$$\mathcal{S}^*_e := \{f \in \mathcal{A} : \mathcal{Q}_f(z) \prec e^z\}.$$

The authors in [12] verified that $f \in \mathcal{S}^*_e$ if and only if $\mathcal{Q}_f(z)$, $z \in \mathbb{D}$, lies in the region

$$\mathcal{R}_e := \{w \in \mathbb{C} : |\log w| < 1\},$$

and established the following result.

Lemma 4.5 ([12, Lemma 2.2]). *For $1/e < a < e$, let r_a be given by*

$$r_a = \begin{cases} a - 1/e, & 1/e < a \le (e + e^{-1})/2, \\ e - a, & (e + e^{-1})/2 \le a < e. \end{cases}$$

Then $\{w \in \mathbb{C} : |w - a| < r_a\} \subset \mathcal{R}_e$.

(2) $\mathcal{S}^*_C$: K. Sharma et al. [22] introduced and discussed the class

$$\mathcal{S}^*_C := \left\{ f \in \mathcal{A} : \mathcal{Q}_f(z) \prec 1 + 4z/3 + 2z^2/3 \right\}.$$

A function $f \in \mathcal{A}$ is in the class $\mathcal{S}^*_C$ if and only if $\mathcal{Q}_f(z)$ lies in the region bounded by the cardiod $\left(9u^2 - 18u + 9v^2 + 5\right)^2 - 16\left(9u^2 - 6u + 9v^2 + 1\right) = 0$. Let $\mathcal{R}_C$ denotes the region bounded by this cardioid.

Lemma 4.6 ([22, Lemma 2.5])**.** *For $1/3 < a < 3$, let r_a be given by*

$$r_a = \begin{cases} (3a-1)/3, & 1/3 < a \le 5/3, \\ 3 - a, & 5/3 \le a < 3. \end{cases}$$

Then $\{w \in \mathbb{C} : |w - a| < r_a\} \subset \mathcal{R}_C$.

(3) $\mathcal{S}^*_{☾}$: P. Sharma et al. [23] considered the class

$$\mathcal{S}^*_{☾} := \left\{ f \in \mathcal{A} : \mathcal{Q}_f(z) \prec z + \sqrt{1 + z^2} \right\}.$$

The function $\varphi_{☾}(z) = z + \sqrt{1+z^2}$ maps $\mathbb{D}$ onto the crescent shaped region

$$\mathcal{R}_{☾} := \left\{ w : |w^2 - 1| < 2|w|,\ \mathrm{Re}(w) > 0 \right\}.$$

Lemma 4.7 ([23, p. 1892])**.** *The disk $\{w : |w - a| < r\}$ lies inside the region $\mathcal{R}_{☾}$ if and only if $|a - \sqrt{2}| \le 1 - r$.*

(4) $\mathcal{S}^*_S$: Cho et al. [4] introduced another important class

$$\mathcal{S}^*_S := \{ f \in \mathcal{A} : \mathcal{Q}_f(z) \prec 1 + \sin z \}.$$

The function $\varphi_S(z) = 1 + \sin z$ maps $\mathbb{D}$ onto the interior of an eight-shaped curve.

Lemma 4.8 ([4, Lemma 3.3])**.** *Let $1 - \sin 1 \le a \le 1 + \sin 1$ and $r_a = \sin 1 - |a - 1|$. Then $\{w \in \mathbb{C} : |w - a| < r_a\} \subset \mathcal{R}_S$, where $\mathcal{R}_S := \varphi_S(\mathbb{D})$.*

(5) $\mathcal{S}^*_{SG}$: Very recently, Goel and Kumar [6] introduced

$$\mathcal{S}^*_{SG} := \left\{ f \in \mathcal{A} : \mathcal{Q}_f(z) \prec 2/(1 + e^{-z}) \right\}.$$

The function $\varphi_{SG}(z) = 2/(1 + e^{-z})$ is the modified sigmoid function which maps $\mathbb{D}$ onto the region $\mathcal{R}_{SG} := \{w \in \mathbb{C} : |\log(w/(2-w))| < 1\}$.

Lemma 4.9 ([6, Lemma 2.2])**.** *Let $2/(1+e) < a < 2e/(1+e)$. If*

$$r_a = \frac{e-1}{e+1} - |a - 1|,$$

then $\{w \in \mathbb{C} : |w - a| < r_a\} \subset \mathcal{R}_{SG}$.

We now go for the main theorem.

Theorem 4.10. *For the function class Ω, we have the following radii results*:

(1) $\mathscr{R}_{\mathcal{S}^*_e}(\Omega) = \rho_e := 1 - 1/(2e-1) \approx 0.7746$.
(2) $\mathscr{R}_{\mathcal{S}^*_C}(\Omega) = 4/5$.

(3) $\mathscr{R}_{\mathcal{S}^*_{\mathfrak{C}}}(\Omega) = \rho_{\mathfrak{C}} := 4/(4+\sqrt{2}) \approx 0.738796$.
(4) $\mathscr{R}_{\mathcal{S}^*_S}(\Omega) = \rho_S := 2\sin 1/(1+\sin 1) \approx 0.913912$.
(5) $\mathscr{R}_{\mathcal{S}^*_{SG}}(\Omega) = (e-1)/e \approx 0.632121$.

All estimates are sharp.

Proof. (1) Let $f \in \Omega$. Then, for $|z| = r < 1$, Lemma 4.2 gives

$$|\mathcal{Q}_f(z) - 1| \leq r/(2-r), \tag{21}$$

which represents a disk centered at $(1,0)$ and radius $r/(2-r)$. Now it follows from Lemma 4.5 that the disk (21) is contained in the region $\mathcal{R}_e$ provided

$$r/(2-r) \leq 1 - 1/e, \quad \text{or} \quad r \leq \rho_e.$$

For sharpness of the radius estimate ρ_e, consider the function $\mathcal{L}(z)$ given by (20). It is easy to verify that $\mathcal{Q}_{\mathcal{L}}(z)$ takes the value $1/e \in \partial\mathcal{R}_e$ (the boundary of $\mathcal{R}_e$) at the point $z = -\rho_e$ lying on the boundary of $|z| < \rho_e$. This shows that the estimate is best possible.

(2) In view of Lemma 4.2 and Lemma 4.6, it follows that for any $f \in \Omega$, the disk (21) will lie inside the region $\mathcal{R}_C$ if and only if $r/(2-r) \leq 2/3$, or $r \leq 4/5$. Again, from Lemma 4.6, it is easy to verify that the largest disk with center at $(1,0)$ and lying completely inside $\mathcal{R}_C$ is $\{w : |w-1| < 2/3\}$. Clearly the left diametric end point of this disk is $1/3$. The sharpness of our result will follow if we can find at least one function $f \in \Omega$ and a point z_0 on the circle $|z| = 4/5$ such that the value of $\mathcal{Q}_f(z_0)$ is $1/3$. We see that one such function in Ω is $\mathcal{L}(z)$ given by (20) and the corresponding point is $z_0 = -4/5$. See Figure 3 for graphical illustration.

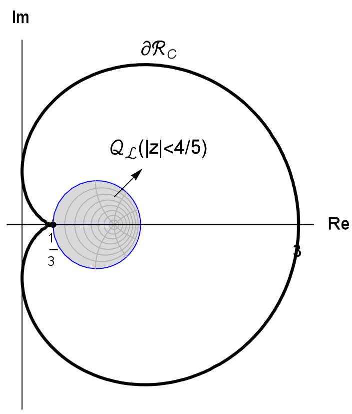


FIGURE 3. Sharpness of $\mathcal{S}^*_C$-radius for Ω.

(3) Let $f \in \Omega$. Then, in view of Lemma 4.7, the disk (21) lies inside the region $\mathcal{R}_{☾}$ if and only if

$$\left|1-\sqrt{2}\right| \leq 1 - r(2-r) \iff r \leq 4/(4+\sqrt{2}) = \rho_{☾} .$$

The result is sharp for the function $\mathcal{L} \in \Omega$ given by (20), as $\mathcal{Q}_{\mathcal{L}}(z)$ attains the value $\sqrt{2}-1 \in \partial\mathcal{R}_{☾}$ at the point $z = -\rho_{☾}$, see Figure 4.

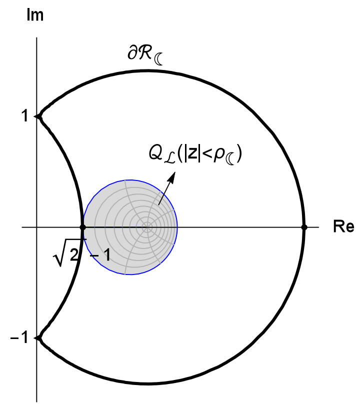


FIGURE 4. Sharpness of $\mathcal{S}^*_{☾}$-radius for Ω.

(4) It is obvious from Lemma 4.8 that the disk (21) will completely lie inside the region $\mathcal{R}_S$ provided $r/(2-r) \leq \sin 1$, which further gives $r \leq \rho_S$. Verification of sharpness of the estimate ρ_S for the function $\mathcal{L} \in \Omega$ is easy.

(5) Let $f \in \Omega$. Then Lemma 4.2 shows that the inequality (21) holds for $|z| = r < 1$. In view of Lemma 4.9, the disk (21) lies completely in the interior of the region $\mathcal{R}_{SG}$ if

$$\frac{r}{2-r} \leq \frac{e-1}{e+1}.$$

The above inequality on simplification yields $r \leq (e-1)/e$. For the function $\mathcal{L}(z)$ given by (20), we have

$$\mathcal{Q}_{\mathcal{L}}\left(\frac{1-e}{e}\right) = \frac{2}{1+e} \in \partial\mathcal{R}_{SG}.$$

Thus the estimate is best possible. This completes the proof of the theorem. $\square$

We finish this paper by presenting a justified open problem related to the convolution of members of Ω. This problem carries its significance due to the celebrated Pólya-Schoenberg Theorem [20].

Problem. Let $f_1, f_2 \in \Omega$. Then $f_1 * f_2 \in \mathcal{C}$, i.e., the convolution of two members of Ω is a convex function.

Justification. Consider the function $\mathcal{L}(z) = z + z^2/2 \in \Omega$ playing a central role in Ω, as far as its extremal behaviour is concerned. The convolution of $\mathcal{L}(z)$ with itself is the function

$$g(z) = (\mathcal{L} * \mathcal{L})(z) = z + \frac{z^2}{4}.$$

The function $g(z)$ satisfies

$$\sum_{n=2}^{\infty} n(n-1)|a_n| = 2\left(\frac{1}{4}\right) = \frac{1}{2}.$$

Applying the result of Kanas and Wisniowska [9, Corollary 3.2], it follows that $g \in \mathcal{C}$.

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