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GEVREY REGULARITY AND TIME DECAY OF THE FRACTIONAL DEBYE-HÜCKEL SYSTEM IN FOURIER-BESOV SPACES

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Abstract. In this paper we mainly study existence and regularity of mild solutions to the parabolic-elliptic system of drift-diffusion type with small initial data in Fourier-Besov spaces. To be more detailed, we will explain that global-in-time mild solutions are well-posed and Gevrey regular by means of multilinear singular integrals and Fourier localization argument. Furthermore, we can get time decay rate estimate of mild solutions in Fourier-Besov spaces.

1. Introduction

In this paper, we study existence and regularity of mild solutions for the initial value problem of the following drift-diffusion system arising from the theory of electrolytes:

$$\begin{cases} \partial_t v + (-\Delta)^\beta v = -\nabla\cdot(v\nabla\phi) & \text{in} \quad \mathbb{R}^n\times(0,\infty), \\ \partial_t w + (-\Delta)^\beta w = \nabla\cdot(w\nabla\phi) & \text{in} \quad \mathbb{R}^n\times(0,\infty), \\ \Delta\phi = v - w & \text{in} \quad \mathbb{R}^n\times(0,\infty), \\ v(x,0)=v_0(x), w(x,0)=w_0(x) & \text{in} \quad \mathbb{R}^n, \end{cases} \tag{1}$$

where $v=v(x,t)$ and $w=w(x,t)$ denote densities of the electron and the hole in electrolytes, $\phi=\phi(x,t)$ denotes the electric potential, $v_0(x)$ and $w_0(x)$ are initial datum. Note that in this paper we assume that $n>1$.

The system (1) can be rewritten as the differential-integral Fokker-Planck system through the famous Duhamel principle:

$$\begin{cases} v = e^{-t(-\Delta)^\beta}v_0 - \int_0^t e^{-(t-s)(-\Delta)^\beta}\nabla\cdot\left[v\nabla(-\Delta)^{-1}(w-v)\right]ds, \\ w = e^{-t(-\Delta)^\beta}w_0 + \int_0^t e^{-(t-s)(-\Delta)^\beta}\nabla\cdot\left[w\nabla(-\Delta)^{-1}(w-v)\right]ds, \end{cases} \tag{2}$$

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where $e^{-t(-\Delta)^\beta} = \mathcal{F}^{-1}(e^{-t|\xi|^{2\beta}}\mathcal{F})$ is the heat flow operator, $\mathcal{F}$ is the Fourier transform and $\mathcal{F}^{-1}$ is the inverse Fourier transform. Any solution that satisfies system (2) is called a mild solution of the system (1).

It should be noted that the system (1) is scaling invariant, which means that if (v, w) solves (1) with initial data (v_0, w_0), so does (v_λ, w_λ) with initial data $(v_{0\lambda}, w_{0\lambda})$, where

$$(v_\lambda(x,t), w_\lambda(x,t)) := (\lambda^{2\beta} v(\lambda x, \lambda^{2\beta} t), \lambda^{2\beta} w(\lambda x, \lambda^{2\beta} t)),$$
$$(v_{0\lambda}(x), w_{0\lambda}(x)) := (\lambda^{2\beta} v_0(\lambda x), \lambda^{2\beta} w_0(\lambda x)).$$

In the long run, our seniors have done a lot of profound researches on the system (1) in many ways, see [1,2,13,14,18,20]. Mathematical study of the system (1) originated in 1980s, when attention was mainly focused on boundary values problems and scholars obtained some results related to global existence, uniqueness of classical solutions and asymptotic stability of stationary solutions, see [8, 9, 17, 19]. After the 1990s, some scholars, such as Biler and Hebisch in [3], began to study global and local existence of mild solutions. Subsequently, more and more scholars carried out deep exploration and researches into it and its related topic.

The first goal of this paper is to show existence of solutions of the system (1). In the case of $\beta = 2$, some previous scholars have come to the following results. Significantly, Karch in [10] solved the proof of existence and uniqueness of solutions of the system (1) for initial data in the Besov spaces $\dot{B}^s_{p,\infty}$ with the condition of $-1 < s < 0$ and $p = \frac{n}{s+2}$. Kurokiba and Ogawa in [11] obtained the same result in L^p space. J. Zhao, Q. Liu and S. Cui in [22] proved that solutions for the famous Debye-Hückel system with low regularity initial data in Besov spaces $\dot{B}^s_{p,r}(\mathbb{R}^n)$ for $-\frac{3}{2} < s \leq -2 + \frac{n}{2}$, $p = \frac{n}{s+2}$ exist. More similar studies can be consulted in [15, 16].

The second goal of this paper is to show analyticity of mild solutions to the system (1) by means of Gevrey regularity. In 1989, Foias and Temam created this method and used it for the first time to studying analyticity of the Navier-Stokes equations with space periodicity boundary condition, see [6, 7]. After that, many authors have fully exploited the advantages and potential of this method, and extended it to various functional spaces and equations. For example, Andrew B. Ferrari and Edriss S. Titi in [5] studied the regularity of solutions to a large class of analytic nonlinear parabolic equations on the two-dimensional sphere, I. Chueshov and M. Polat in [4] studied the Gevrey regularity of the global attractor of the dynamical system generated by the generalized Benjamin-Bona-Mahony equation with periodic boundary conditions. Recently, J. Zhao further proved that global-in-time mild solutions to system (1) are Gevrey regular for all $1 < p < 2n$ and $1 \leq r \leq \infty$, see [21].

After introducing the work done by our predecessors, let's make a summary of what this paper is going to prove. Now we will present the first two results of this paper, which can be used to explain existence of mild solutions and Gevrey regularity of the mild solution obtained in the first result.

Theorem 1.1. *Let* $p > \frac{2n}{2n-1}$, $\frac{1}{p} + \frac{1}{p'} = 1$, $1 \le r \le \infty$, *and let* $(v_0, w_0) \in F\dot{B}_{p,r}^{-2\beta+\frac{n}{p'}}(\mathbb{R}^n)$. *When* $\frac{1}{2} < \beta < \frac{1}{2} + \min\{\frac{n}{p'}, \frac{n}{2}\}$, *there exists a constant* $\varepsilon > 0$ *such that if* $\|(v_0, w_0)\|_{F\dot{B}_{p,r}^{-2\beta+\frac{n}{p'}}} \le \varepsilon$, *then the system* (1) *admits a global-in-time mild solution* $(v, w) \in \mathcal{X}_{p,r}$, *where*

$$\mathcal{X}_{p,r} = \widetilde{L}^\infty(0,\infty; F\dot{B}_{p,r}^{-2\beta+\frac{n}{p'}}(\mathbb{R}^n)) \bigcap \widetilde{L}^1(0,\infty; F\dot{B}_{p,r}^{\frac{n}{p'}}(\mathbb{R}^n)).$$

Moreover, if $r < \infty$, *we have* $(v, w) \in C([0,\infty), F\dot{B}_{p,r}^{-2\beta+\frac{n}{p'}}(\mathbb{R}^n))$.

Theorem 1.2. *Under the assumptions of Theorem 1.1, the global-in-time mild solution* $(v, w) \in \mathcal{X}_{p,r}$ *obtained in Theorem 1.1 satisfies*

$$(e^{\sqrt{t}(-\Delta)^{\frac{\beta}{2}}} v, e^{\sqrt{t}(-\Delta)^{\frac{\beta}{2}}} w) \in \mathcal{X}_{p,r},$$

where $e^{t(-\Delta)^{\frac{\beta}{2}}} = \mathcal{F}^{-1}(e^{t|\xi|^\beta}\mathcal{F})$.

In Theorem 1.2, we have proved analyticity of mild solutions, so that we can further obtain the time decay estimates of mild solutions.

Theorem 1.3. *Under the assumptions of Theorem 1.1, for any* $\sigma > 0$, *the global-in-time mild solution* $(v, w) \in \mathcal{X}_{p,r}$ *and* $(e^{\sqrt{t}(-\Delta)^{\frac{\beta}{2}}} v, e^{\sqrt{t}(-\Delta)^{\frac{\beta}{2}}} w) \in \mathcal{X}_{p,r}$ *achieved from Theorem 1.2 satisfies the following time decay estimate*:

$$\|(\Lambda^\sigma v(t), \Lambda^\sigma w(t))\|_{F\dot{B}_{p,r}^{-2\beta+\frac{n}{p'}}} \le C_{\beta,\sigma} t^{-\frac{\sigma}{2\beta}} \|(v_0, w_0)\|_{F\dot{B}_{p,r}^{-2\beta+\frac{n}{p'}}},$$

where $C_{\beta,\sigma}$ *is a constant depended on* β *and* σ .

The overall structure of the article is as follows: In Section 2, we review the Littlewood-Paley dyadic decomposition theory and the definition of Fourier-Besov spaces. In Section 3 and Section 4, we prove Theorem 1.1 and Theorem 1.2 respectively by the standard fixed point argument. In Section 4, we prove Theorem 1.3.

2. Preliminaries

First of all, let's introduce some of the notations mentioned in the paper. For two constants A and B, if there is a finite constant C whose value of each line may vary such that $A \le CB$, we denote it as $A \lesssim B$. For a quasi-Banach space X and for any $0 < T \le \infty$, we use standard notation $L^p(0,T;X)$ to denote the quasi-Banach space of Bochner measurable functions f from $(0,T)$ to X endowed with the norm

$$\|f\|_{L^p_T X} := \begin{cases} (\int_0^T \|f(\cdot,t)\|_X^p \, dt)^{\frac{1}{p}} & \text{if } 1 \le p < \infty, \\ \sup_{0 \le t \le T} \|f(\cdot,t)\|_X & \text{if } p = \infty. \end{cases}$$

Especially, if $T=\infty$, we still use $\|f\|_{L^p_T X}$ rather than $\|f\|_{L^p_\infty X}$. Given two quasi-Banach spaces X and Y, the product of these two spaces, $X\times Y$, will be endowed with the usual norm $\|(u,v)\|:=\|u\|_X+\|v\|_Y$. If $X=Y$, we use $\|(u,v)\|_X$ instead of $\|(u,v)\|_{X\times X}$. Now we introduce some basic knowledge on Littlewood-Paley theory and Fourier-Besov spaces.

Let $\varphi\in C_c^\infty(\mathbb{R}^n)$ be a radial positive function such that

$$supp\ \varphi\subset\{\xi\in\mathbb{R}^n:\frac{3}{4}\le|\xi|\le\frac{8}{3}\},\quad \sum_{j\in Z}\varphi(2^{-j}\xi)=1 \text{ for any } \xi\neq 0.$$

We denote by $\mathcal{F}$ the Fourier transform and $\mathcal{F}^{-1}$ the inverse Fourier transform. Define the frequency localization operators as follows:

$$\triangle_j u=\varphi_j(D)u=\mathcal{F}^{-1}\varphi_j(\xi)\mathcal{F}u;\quad S_j u=\psi_j(D)u=\mathcal{F}^{-1}\psi_j(\xi)\mathcal{F}u,$$

here $\varphi_j(\xi)=\varphi(2^{-j}\xi)$ and $\psi_j=\sum_{k\le j-1}\varphi_j$.

By Bony's decomposition we can split the product uv into three parts:

$$uv=T_u v+T_v u+R(u,v),$$

with

$$T_u v=\sum_j S_{j-1}u\Delta_j v,\quad R(u,v)=\sum_j\Delta_j u\widetilde{\Delta}_j v,\quad \widetilde{\Delta}_j v=\Delta_{j-1}v+\Delta_j v+\Delta_{j+1}v.$$

Let us now define the Fourier-Besov spaces as follows.

Definition 2.1. For $s\in\mathbb{R}$, $p,r\in[1,\infty]$, we define the Fourier-Besov space $F\dot{B}^s_{p,r}$ as

$$F\dot{B}^s_{p,r}=\{f\in\mathscr{S}'/\mathbb{P}:\|f\|_{F\dot{B}^s_{p,r}}=\Big(\sum_{j\in\mathbb{Z}}2^{jsr}\|\widehat{\Delta_j f}\|^r_{L^p}\Big)^{1/r}<\infty\}.$$

Here the norm changes normally when $p=\infty$ or $r=\infty$, and $\mathbb{P}$ is the set of all polynomials.

Definition 2.2. For $0<T\le\infty$, $s\in\mathbb{R}$, $1\le p,r,\rho\le\infty$, we set (with the usual convention if $r=\infty$)

$$\|f\|_{\widetilde{L}^\rho_T(F\dot{B}^s_{p,r})}:=(\sum_{j\in Z}2^{jsr}\|\widehat{\Delta_j f}\|^r_{L^\rho(0,T;L^p)})^{\frac{1}{r}}.$$

We then define the space $\widetilde{L}^\rho(0,T;F\dot{B}^s_{p,r}(\mathbb{R}^n))$ as the set of temperate distributions f over $(0,T)\times\mathbb{R}^n$ such that $\lim_{j\to-\infty}S_j f=0$ in $S'((0,T)\times\mathbb{R}^n)$ and $\|f\|_{\widetilde{L}^\rho_T(F\dot{B}^s_{p,r})}<\infty$.

Lemma 2.3. *Let f be a smooth function on $\mathbb{R}^n\setminus\{0\}$ which is homogeneous of degree m. The operator $f(D)$ is continuous from $F\dot{B}^s_{p,r}(\mathbb{R}^n)$ to $F\dot{B}^{s-m}_{p,r}(\mathbb{R}^n)$.*

Proof. Set $u \in F\dot{B}^s_{p,r}(\mathbb{R}^n)$. Then

$$\begin{aligned}
\|f(D)u\|_{F\dot{B}^{s-m}_{p,r}(\mathbb{R}^n)} &= \Big(\sum_{j\in\mathbb{Z}} 2^{j(s-m)r}\|\Delta_j\widehat{f(D)}u\|^r_{L^p}\Big)^{1/r} \\
&= \Big(\sum_{j\in\mathbb{Z}} 2^{j(s-m)r}\|\varphi_j(\xi)f(\xi)\hat{u}\|^r_{L^p}\Big)^{1/r} \\
&= \Big(\sum_{j\in\mathbb{Z}} 2^{j(s-m)r}\|\varphi_j(\xi)|\xi|^m f(\xi/|\xi|)\hat{u}\|^r_{L^p}\Big)^{1/r} \\
&\le C\Big(\sum_{j\in\mathbb{Z}} 2^{j(s-m)r}2^{jmr}\|\varphi_j(\xi)\hat{u}\|^r_{L^p}\Big)^{1/r} \\
&= C\|u\|_{F\dot{B}^s_{p,r}(\mathbb{R}^n)}.
\end{aligned}$$

□

Proposition 2.4 ([12])**.** *Let $\mathcal{X}$ be a Banach space with norm $\|\cdot\|_{\mathcal{X}}$, and $\boldsymbol{B}:\mathcal{X}\times\mathcal{X}\to\mathcal{X}$ be a bilinear bounded operator. Assume that for any $u_1,u_2\in\mathcal{X}$, we have $\|\boldsymbol{B}(u_1,u_2)\|_{\mathcal{X}}\le C_0\|u_1\|_{\mathcal{X}}\|u_2\|_{\mathcal{X}}$. Then for any $y\in\mathcal{X}$ such that $\|y\|_{\mathcal{X}}\le\varepsilon<\frac{1}{4C_0}$, the equation $u=y+\boldsymbol{B}(u,u)$ has a solution u in $\mathcal{X}$. Moreover, this solution is the only one such that $\|u\|_{\mathcal{X}}\le 2\varepsilon$, and depends continuously on y in the following sense: if $\|\widetilde{y}\|_{\mathcal{X}}\le\varepsilon, \widetilde{u}=\widetilde{y}+\boldsymbol{B}(\widetilde{u},\widetilde{u})$ and $\|\widetilde{u}\|_{\mathcal{X}}\le 2\varepsilon$, then $\|u-\widetilde{u}\|_{\mathcal{X}}\le\frac{1}{1-4\varepsilon C_0}\|y-\widetilde{y}\|_{\mathcal{X}}$.*

3. Existence of solutions

In this section we will prove Theorem 1.1. We have learned from Section 1 that the integral form of the system (1) is as follows

$$\begin{cases} v = e^{-t(-\Delta)^\beta}v_0 - \int_0^t e^{-(t-s)(-\Delta)^\beta}\nabla\cdot[v\nabla(-\Delta)^{-1}(w-v)]\,ds, \\ w = e^{-t(-\Delta)^\beta}w_0 + \int_0^t e^{-(t-s)(-\Delta)^\beta}\nabla\cdot[w\nabla(-\Delta)^{-1}(w-v)]\,ds. \end{cases} \tag{3}$$

Lemma 3.1. *Let $v_0\in F\dot{B}^{-2\beta+\frac{n}{p'}}_{p,r}$ for $p>\frac{2n}{2n-1}$ and $1\le r\le+\infty$. Then there holds $e^{-(-\Delta)^\beta t}v_0\in\mathcal{X}_{p,r}$ and $\|e^{-(-\Delta)^\beta t}v_0\|_{\mathcal{X}_{p,r}}\lesssim\|v_0\|_{F\dot{B}^{-2\beta+\frac{n}{p'}}_{p,r}}$.*

Proof. Firstly, we need to prove that

$$(\sum_{j\in Z}2^{j(-2\beta+\frac{n}{p'})r}\|\Delta_j(\widehat{e^{-(-\Delta)^\beta t}v_0})\|^r_{L^\infty_t L^p})^{\frac{1}{r}}\lesssim\|v_0\|_{F\dot{B}^{-2\beta+\frac{n}{p'}}_{p,r}}.$$

In fact,

$$\begin{aligned}
&\|\Delta_j(\widehat{e^{-(-\Delta)^\beta t}v_0})\|_{L^p} = \|\varphi_j e^{-t|\xi|^{2\beta}}\widehat{v_0}(\xi)\|_{L^p}\lesssim\|\varphi_j\widehat{v_0}(\xi)\|_{L^p}, \\
&\|\Delta_j(\widehat{e^{-(-\Delta)^\beta t}v_0})\|_{L^\infty_t L^p}\lesssim\|\widehat{\Delta_j v_0}\|_{L^p}. \qquad (4)
\end{aligned}$$

Then multiplying (4) by $2^{-2\beta+\frac{n}{p'}}$ and taking l^r-norm, we get what we want. Secondly, we need to prove that

$$(\sum_{j\in Z} 2^{j\frac{n}{p'}r}\|\Delta_j(\widehat{e^{-(-\Delta)^\beta t}v_0})\|^r_{L^1_tL^p})^{\frac{1}{r}} \lesssim \|v_0\|_{F\dot{B}^{-2\beta+\frac{n}{p'}}_{p,r}}.$$

Due to $\|\Delta_j(\widehat{e^{-(-\Delta)^\beta t}v_0})\|_{L^p} \lesssim e^{-t2^{2\beta j}}\|\varphi_j\widehat{v_0}(\xi)\|_{L^p}$,

$$\begin{aligned}(5)\quad \|\Delta_j(\widehat{e^{-(-\Delta)^\beta t}v_0})\|_{L^1_tL^p} &\lesssim \int_0^T e^{-t2^{2\beta j}}\,dt\cdot\|\varphi_j\widehat{v_0}(\xi)\|_{L^p}\\ &= \frac{1-e^{-T2^{2\beta j}}}{2^{2\beta j}}\|\varphi_j\widehat{v_0}(\xi)\|_{L^p} \lesssim 2^{-2\beta j}\|\varphi_j\widehat{v_0}(\xi)\|_{L^p}.\end{aligned}$$

Similarly, multiplying (5) by $2^{\frac{n}{p'}}$ and taking l^r-norm, we acquire the results described in the lemma. □

Lemma 3.2. *Let $v, w \in \mathcal{X}_{p,r}$ and $\Delta\phi = w - v$. Thus $\|\mathscr{B}(v,\phi)\|_{\mathcal{X}_{p,r}} \lesssim \|(v,w)\|^2_{\mathcal{X}_{p,r}}$.*

Proof. For convenience, we use $\mathscr{B}(v,\phi)$ to represent the nonliner term, that is

$$\mathscr{B}(v,\phi) = \int_0^t e^{-(t-s)(-\Delta)^\beta}\nabla\cdot(v\nabla\phi)\,ds.$$

Applying Δ_j to $\mathscr{B}(v,\phi)$, then doing Fourier transform on it and taking the L^p-norm, by using Minkowski's inequality, we find that

$$\begin{aligned}\|\Delta_j\widehat{\mathscr{B}(v,\phi)}\|_{L^p} &\lesssim \int_0^t \|\Delta_j e^{-(t-s)\widehat{(-\Delta)^\beta}}\nabla(v\nabla\phi)\|_{L^p}\,ds\\ &\lesssim \int_0^t e^{-(t-s)2^{2\beta j}}2^j\|\Delta_j\widehat{(v\nabla\phi)}\|_{L^p}\,ds.\end{aligned}$$

According to Bony's paraproduct decomposition, we find that

$$v\nabla\phi = \sum_{j'\in Z} S_{j'-1}v\Delta_{j'}\nabla\phi + \sum_{j'\in Z}\Delta_{j'}vS_{j'-1}\nabla\phi + \sum_{j'\in Z}\Delta_{j'}v\widetilde{\Delta_{j'}}\nabla\phi.$$

Then we have

$$\begin{aligned}\|\Delta_j\widehat{\mathscr{B}(v,\phi)}\|_{L^p} \lesssim\ & \int_0^t e^{-(t-s)2^{2\beta j}}2^j\|\mathcal{F}(\Delta_j\sum_{j'\in Z}(S_{j'-1}v\Delta_{j'}\nabla\phi))\|_{L^p}\,ds\\ &+\int_0^t e^{-(t-s)2^{2\beta j}}2^j\|\mathcal{F}(\Delta_j\sum_{j'\in Z}(\Delta_{j'}vS_{j'-1}\nabla\phi))\|_{L^p}\,ds\\ &+\int_0^t e^{-(t-s)2^{2\beta j}}2^j\|\mathcal{F}(\Delta_j\sum_{j'\in Z}(\Delta_{j'}v\widetilde{\Delta_{j'}}\nabla\phi))\|_{L^p}\,ds\\ :=\ & I_1+I_2+I_3.\end{aligned}$$

Based on the properties of support sets, we have

$$I_1 \lesssim \int_0^t e^{-(t-s)2^{2\beta j}} 2^j \sum_{|j-j'|\le 4} \|\mathcal{F}(\Delta_j(S_{j'-1}v\Delta_{j'}\nabla\phi))\|_{L^p}\, ds.$$

As for the term $\|\mathcal{F}(\Delta_j(S_{j'-1}v\Delta_{j'}\nabla\phi))\|_{L^p}$, applying Hölder's inequality and Young's inequality,

$$\begin{aligned}
&\sum_{|j-j'|\le 4} \|\mathcal{F}(\Delta_j(S_{j'-1}v\Delta_{j'}\nabla\phi))\|_{L^p}\\
=&\sum_{|j-j'|\le 4} \|\varphi_j\cdot(\widehat{S_{j'-1}v})*(\widehat{\Delta_{j'}\nabla\phi})\|_{L^p}\\
\lesssim&\sum_{|j-j'|\le 4} \|(\widehat{S_{j'-1}v})*(\widehat{\Delta_{j'}\nabla\phi})\|_{L^p}\\
\lesssim&\sum_{|j-j'|\le 4} \|\widehat{S_{j'-1}v}\|_{L^1}\cdot\|\widehat{\Delta_{j'}\nabla\phi}\|_{L^p}\\
\lesssim&\sum_{|j-j'|\le 4}\sum_{k\leqslant j'-2} 2^{\frac{kn}{p'}}\|\widehat{\Delta_k v}\|_{L^p}\cdot\|\widehat{\Delta_{j'}\nabla\phi}\|_{L^p}\\
\lesssim&\sum_{|j-j'|\le 4} 2^{j'(-1-\frac{n}{p'})}\sum_{k\leqslant j'-2} 2^{2\beta k}2^{k(-2\beta+\frac{n}{p'})}\|\widehat{\Delta_k v}\|_{L^p}2^{j'(1+\frac{n}{p'})}\|\widehat{\Delta_{j'}\nabla\phi}\|_{L^p}.
\end{aligned}$$

In conclusion,

$$\begin{aligned}
I_1 \lesssim& \int_0^t [e^{-(t-s)2^{2\beta j}} 2^j \sum_{|j-j'|\le 4} 2^{j'(-1-\frac{n}{p'})}\\
&\times \sum_{k\leqslant j'-2} 2^{2\beta k}2^{k(-2\beta+\frac{n}{p'})}\|\widehat{\Delta_k v}\|_{L^p}2^{j'(1+\frac{n}{p'})}\|\widehat{\Delta_{j'}\nabla\phi}\|_{L^p}]\, ds.
\end{aligned}$$

We can estimate I_2 in the same way:

$$I_2 \lesssim \int_0^t [e^{-(t-s)2^{2\beta j}} 2^j \sum_{|j-j'|\le 4} \|\widehat{\Delta_{j'}v}\|_{L^p} \sum_{k\leqslant j'-2} 2^{(2\beta-1)k}2^{k(1-2\beta+\frac{n}{p'})}\|\widehat{\Delta_k\nabla\phi}\|_{L^p}]\, ds.$$

Next, it's clear that there exists a constant N_0 such that

$$I_3 \lesssim \int_0^t [e^{-(t-s)2^{2\beta j}} 2^j \sum_{j'\geqslant j-N_0} \|\mathcal{F}(\Delta_j(\Delta_{j'}v\widetilde{\Delta_{j'}}\nabla\phi))\|_{L^p}]\, ds.$$

We divide the estimate of the term $\|\Delta_j(\widehat{\Delta_{j'}v\widetilde{\Delta_{j'}}}\nabla\phi)\|_{L^p}$ into two steps. When $1\le p\le 2$, utilizing Hölder's inequality and Young's inequality, one has

$$\begin{aligned}
&\sum_{j'\geqslant j-N_0} \|\mathcal{F}(\Delta_j(\Delta_{j'}v\widetilde{\Delta_{j'}}\nabla\phi))\|_{L^p}\\
=&\sum_{j'\geqslant j-N_0} \|\varphi_j\cdot(\widehat{\Delta_{j'}v})*(\widehat{\widetilde{\Delta_{j'}}\nabla\phi})\|_{L^p}
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \sum_{j'\geqslant j-N_0} 2^{\frac{nj}{p'}} \|\widehat{(\Delta_{j'}v)} * \widehat{(\widetilde{\Delta_{j'}}\nabla\phi)}\|_{L^{\frac{pp'}{p'-p}}} \\
&\lesssim \sum_{j'\geqslant j-N_0} 2^{\frac{nj}{p'}} \|\widehat{\Delta_{j'}v}\|_{L^p} \cdot \|\widehat{\widetilde{\Delta_{j'}}\nabla\phi}\|_{L^p} \\
&\lesssim 2^{\frac{nj}{p'}} \sum_{j'\geqslant j-N_0} 2^{j'(2\beta-1-\frac{2n}{p'})} 2^{j'(-2\beta+\frac{n}{p'})} \|\widehat{\Delta_{j'}v}\|_{L^p} 2^{j'(1+\frac{n}{p'})} \|\widehat{\widetilde{\Delta_{j'}}\nabla\phi}\|_{L^p}.
\end{aligned}
$$

When $p>2$, we can still come to a similar conclusion that

$$
\begin{aligned}
&\sum_{j'\geqslant j-N_0} \|\varphi_j \cdot \widehat{(\Delta_{j'}v)} * \widehat{(\widetilde{\Delta_{j'}}\nabla\phi)}\|_{L^p} \\
&\lesssim \sum_{j'\geqslant j-N_0} 2^{\frac{nj}{p}} \|\widehat{(\Delta_{j'}v)} * \widehat{(\widetilde{\Delta_{j'}}\nabla\phi)}\|_{L^\infty} \\
&\lesssim 2^{\frac{nj}{p}} \sum_{j'\geqslant j-N_0} \|\widehat{\Delta_{j'}v}\|_{L^{p'}} \|\widehat{\widetilde{\Delta_{j'}}\nabla\phi}\|_{L^p} \\
&\lesssim 2^{nj(1-\frac{1}{p'})} \sum_{j'\geqslant j-N_0} 2^{nj'(\frac{1}{p'}-\frac{1}{p})} \|\widehat{\Delta_{j'}v}\|_{L^p} \|\widehat{\widetilde{\Delta_{j'}}\nabla\phi}\|_{L^p} \\
&\lesssim 2^{nj(1-\frac{1}{p'})} \sum_{j'\geqslant j-N_0} 2^{j'(2\beta-1-n)} 2^{j'(-2\beta+\frac{n}{p'})} \|\widehat{\Delta_{j'}v}\|_{L^p} 2^{j'(1+\frac{n}{p'})} \|\widehat{\widetilde{\Delta_{j'}}\nabla\phi}\|_{L^p}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
I_3 \lesssim &\int_0^t \Big[e^{-(t-s)2^{2\beta j}} 2^{j(1+\frac{n}{p'})} \sum_{j'\geqslant j-N_0} 2^{j'(2\beta-1-\frac{2n}{p'})} 2^{j'(-2\beta+\frac{n}{p'})} \\
&\qquad \|\widehat{\Delta_{j'}v}\|_{L^p} 2^{j'(1+\frac{n}{p'})} \|\widehat{\widetilde{\Delta_{j'}}\nabla\phi}\|_{L^p} \Big] ds \\
&+ \int_0^t \Big[e^{-(t-s)2^{2\beta j}} 2^{j(1+n-\frac{n}{p'})} \sum_{j'\geqslant j-N_0} 2^{j'(2\beta-1-n)} 2^{j'(-2\beta+\frac{n}{p'})} \\
&\qquad \|\widehat{\Delta_{j'}v}\|_{L^p} 2^{j'(1+\frac{n}{p'})} \|\widehat{\widetilde{\Delta_{j'}}\nabla\phi}\|_{L^p} \Big] ds.
\end{aligned}
$$

Taking the L^∞ norm of $\|\Delta_j \widehat{\mathscr{B}(v,\phi)}\|_{L^p}$ in time, we can see that

$$
\begin{aligned}
(6)\quad & \|\Delta_j \widehat{\mathscr{B}(v,\phi)}\|_{L^\infty_t L^p} \\
&\lesssim 2^{j(2\beta-\frac{n}{p'})} \|v\|_{\widetilde{L}^\infty_t F\dot{B}^{-2\beta+\frac{n}{p'}}_{p,r}} 2^{j(1+\frac{n}{p'})} \|\widehat{\Delta_j\nabla\phi}\|_{L^1_t L^p} \\
&\quad + 2^{j(2\beta-\frac{n}{p'})} \|\nabla\phi\|_{\widetilde{L}^\infty_t F\dot{B}^{1-2\beta+\frac{n}{p'}}_{p,r}} 2^{j\frac{n}{p'}} \|\widehat{\Delta_j v}\|_{L^1_t L^p} \\
&\quad + 2^{j(1+\frac{n}{p'})} \|v\|_{\widetilde{L}^\infty_t F\dot{B}^{-2\beta+\frac{n}{p'}}_{p,r}} \sum_{j'\geqslant j-N_0} 2^{j'(2\beta-1-\frac{2n}{p'})} 2^{j'(1+\frac{n}{p'})} \|\widehat{\widetilde{\Delta_{j'}}\nabla\phi}\|_{L^1_t L^p}
\end{aligned}
$$

$$+ 2^{j(1+n-\frac{n}{p'})}\|v\|_{\widetilde{L}^\infty_t F\dot{B}^{-2\beta+\frac{n}{p'}}_{p,r}} \sum_{j'\geqslant j-N_0} 2^{j'(2\beta-1-n)}2^{j'(1+\frac{n}{p'})}\|\widehat{\widetilde{\Delta_{j'}}\nabla\phi}\|_{L^1_tL^p}.$$

Multiplying (6) by $2^{j(-2\beta+\frac{n}{p'})}$, and taking l^r-norm with index j, by using Minkowski's inequality and Young's inequality, we can conclude that

$$\begin{aligned}&\|\mathscr{B}(v,\phi)\|_{\widetilde{L}^\infty_t F\dot{B}^{-2\beta+\frac{n}{p'}}_{p,r}}\\ \lesssim\ &\|v\|_{\widetilde{L}^\infty_t F\dot{B}^{-2\beta+\frac{n}{p'}}_{p,r}}\|\nabla\phi\|_{\widetilde{L}^1_t F\dot{B}^{1+\frac{n}{p'}}_{p,r}} + \|\nabla\phi\|_{\widetilde{L}^\infty_t F\dot{B}^{1-2\beta+\frac{n}{p'}}_{p,r}}\|v\|_{\widetilde{L}^1_t F\dot{B}^{\frac{n}{p'}}_{p,r}}.\end{aligned}$$

Due to $\Delta\phi = w - v$, quoting Lemma 2.3, we find that

$$\|\nabla\phi\|_{\widetilde{L}^1_t F\dot{B}^{1+\frac{n}{p'}}_{p,r}} \lesssim \|(v,w)\|_{\widetilde{L}^1_t F\dot{B}^{\frac{n}{p'}}_{p,r}}$$

and

$$\|\nabla\phi\|_{\widetilde{L}^\infty_t F\dot{B}^{1-2\beta+\frac{n}{p'}}_{p,r}} \lesssim \|(v,w)\|_{\widetilde{L}^\infty_t F\dot{B}^{-2\beta+\frac{n}{p'}}_{p,r}}.$$

Therefore we have

$$\|\mathscr{B}(v,\phi)\|_{\widetilde{L}^\infty_t F\dot{B}^{-2\beta+\frac{n}{p'}}_{p,r}} \lesssim \|(v,w)\|^2_{\mathcal{X}_{p,r}}. \tag{7}$$

When taking the L^1-norm in time to $\|\Delta_j\widehat{\mathscr{B}(v,\phi)}\|_{L^p}$, we will obtain similar results that

$$\begin{aligned}&\|\Delta_j\widehat{\mathscr{B}(v,\phi)}\|_{L^1_tL^p} \qquad (8)\\ \lesssim\ &2^{-j\frac{n}{p'}}\|v\|_{\widetilde{L}^\infty_t F\dot{B}^{-2\beta+\frac{n}{p'}}_{p,r}} 2^{j(1+\frac{n}{p'})}\|\widehat{\Delta_j\nabla\phi}\|_{L^1_tL^p}\\ &+ 2^{-j\frac{n}{p'}}\|\nabla\phi\|_{\widetilde{L}^\infty_t F\dot{B}^{1-2\beta+\frac{n}{p'}}_{p,r}} 2^{j\frac{n}{p'}}\|\widehat{\Delta_j v}\|_{L^1_tL^p}\\ &+2^{j(-2\beta+1+\frac{n}{p'})}\|v\|_{\widetilde{L}^\infty_t F\dot{B}^{-2\beta+\frac{n}{p'}}_{p,r}} \sum_{j'\geqslant j-N_0} 2^{j'(2\beta-1-\frac{2n}{p'})}2^{j'(1+\frac{n}{p'})}\|\widehat{\widetilde{\Delta_{j'}}\nabla\phi}\|_{L^1_tL^p}\\ &+2^{(-2\beta+1+n-\frac{n}{p'})}\|v\|_{\widetilde{L}^\infty_t F\dot{B}^{-2\beta+\frac{n}{p'}}_{p,r}} \sum_{j'\geqslant j-N_0} 2^{j'(2\beta-1-n)}2^{j'(1+\frac{n}{p'})}\|\widehat{\widetilde{\Delta_{j'}}\nabla\phi}\|_{L^1_tL^p}.\end{aligned}$$

Multiplying (8) by $2^{j\frac{n}{p'}}$ and taking l^r-norm, we can get

$$\begin{aligned}&\|\mathscr{B}(v,\phi)\|_{\widetilde{L}^1_t F\dot{B}^{\frac{n}{p'}}_{p,r}} \qquad (9)\\ \lesssim\ &\|v\|_{\widetilde{L}^\infty_t F\dot{B}^{-2\beta+\frac{n}{p'}}_{p,r}}\|\nabla\phi\|_{\widetilde{L}^1_t F\dot{B}^{1+\frac{n}{p'}}_{p,r}} + \|\nabla\phi\|_{\widetilde{L}^\infty_t F\dot{B}^{1-2\beta+\frac{n}{p'}}_{p,r}}\|v\|_{\widetilde{L}^1_t F\dot{B}^{\frac{n}{p'}}_{p,r}}\\ \lesssim\ &\|(v,w)\|^2_{\mathcal{X}_{p,r}}.\end{aligned}$$

Through (5), (7) and (9), we can get

$$\|\mathscr{B}(v,\phi)_{\mathcal{X}_{p,r}} \lesssim \|(v,w)\|^2_{\mathcal{X}_{p,r}}.$$

Thus we have completed the proof of Lemma 3.2.

From the above results, we can prove Theorem 1.1 according to the fixed point theorem. □

4. Gevrey regularity of mild solutions

Setting $V(t)=e^{\sqrt{t}(-\Delta)^{\frac{\beta}{2}}}v(t), W(t)=e^{\sqrt{t}(-\Delta)^{\frac{\beta}{2}}}w(t)$, and $\Phi(t)=e^{\sqrt{t}(-\Delta)^{\frac{\beta}{2}}}\phi(t)$. In addition, $\Phi(t) = W(t) - V(t)$, so (V(t),W(t)) satisfies the following integral system:

$$\begin{cases} V(t) = e^{\sqrt{t}(-\Delta)^{\frac{\beta}{2}}-t(-\Delta)^{\beta}}v_0 - \int_0^t e^{[(\sqrt{t}-\sqrt{s})(-\Delta)^{\frac{\beta}{2}}-(t-s)(-\Delta)^{\beta}]} \\ \qquad\times\nabla e^{\sqrt{s}(-\Delta)^{\frac{\beta}{2}}}(e^{-\sqrt{s}(-\Delta)^{\frac{\beta}{2}}}V(s)e^{-\sqrt{s}(-\Delta)^{\frac{\beta}{2}}}\nabla\Phi(s))\,ds \\ W(t) = e^{\sqrt{t}(-\Delta)^{\frac{\beta}{2}}-t(-\Delta)^{\beta}}w_0 - \int_0^t e^{[(\sqrt{t}-\sqrt{s})(-\Delta)^{\frac{\beta}{2}}-(t-s)(-\Delta)^{\beta}]} \\ \qquad\times\nabla e^{\sqrt{s}(-\Delta)^{\frac{\beta}{2}}}(e^{-\sqrt{s}(-\Delta)^{\frac{\beta}{2}}}W(s)e^{-\sqrt{s}(-\Delta)^{\frac{\beta}{2}}}\nabla\Phi(s))\,ds. \end{cases} \tag{10}$$

Significantly, in this section $\mathscr{B}(V,\Phi)$ represents new nonlinear terms, that is

$$\mathscr{B}(V,\Phi) = \int_0^t e^{[(\sqrt{t}-\sqrt{s})(-\Delta)^{\frac{\beta}{2}}-(t-s)(-\Delta)^{\beta}]}\nabla e^{\sqrt{s}(-\Delta)^{\frac{\beta}{2}}}(e^{-\sqrt{s}(-\Delta)^{\frac{\beta}{2}}}V(s)e^{-\sqrt{s}(-\Delta)^{\frac{\beta}{2}}}\nabla\Phi(s))\,ds.$$

Lemma 4.1. *Let $v_0 \in F\dot{B}_{p,r}^{-2\beta+\frac{n}{p'}}$ for $p > \frac{2n}{2n-1}$ and $1 \le r \le +\infty$. Then there holds $e^{\sqrt{t}(-\Delta)^{\frac{\beta}{2}}-t(-\Delta)^{\beta}}v_0 \in \mathcal{X}_{p,r}$ and $\|e^{\sqrt{t}(-\Delta)^{\frac{\beta}{2}}-t(-\Delta)^{\beta}}v_0\|_{\mathcal{X}_{p,r}} \lesssim \|v_0\|_{F\dot{B}_{p,r}^{-2\beta+\frac{n}{p'}}}$.*

Proof. We first find

$$\begin{aligned} \|\mathcal{F}(\Delta_j e^{\sqrt{t}(-\Delta)^{\frac{\beta}{2}}-t(-\Delta)^{\beta}}v_0)\|_{L^p} &= \|\varphi_j e^{\sqrt{t}|\xi|^{\beta}-t|\xi|^{2\beta}}\widehat{v_0}(\xi)\|_{L^p} \\ &= \|\varphi_j e^{-(\sqrt{t}|\xi|^{\beta}-\frac{1}{2})^2+\frac{1}{4}}\widehat{v_0}(\xi)\|_{L^p} \\ &\lesssim \|\varphi_j\widehat{v_0}(\xi)\|_{L^p} = \|\widehat{\Delta_j v_0}\|_{L^p}. \end{aligned}$$

Hence $\|\mathcal{F}(\Delta_j e^{\sqrt{t}(-\Delta)^{\frac{\beta}{2}}-t(-\Delta)^{\beta}}v_0)\|_{L_t^{\infty}L^p} \lesssim \|\widehat{\Delta_j v_0}\|_{L^p}$, multiplying by $2^{j(-2\beta+\frac{n}{p'})}$ and taking l^r-norm,

$$\|e^{\sqrt{t}(-\Delta)^{\frac{\beta}{2}}-t(-\Delta)^{\beta}}v_0\|_{\widetilde{L}_t^{\infty}F\dot{B}_{p,r}^{-2\beta+\frac{n}{p'}}} \lesssim \|v_0\|_{F\dot{B}_{p,r}^{-2\beta+\frac{n}{p'}}}.$$

On the other hand,

$$\begin{aligned} \|\mathcal{F}(\Delta_j e^{\sqrt{t}(-\Delta)^{\frac{\beta}{2}}-t(-\Delta)^{\beta}}v_0)\|_{L^p} &= \|e^{\sqrt{t}|\xi|^{\beta}-\frac{t}{2}|\xi|^{2\beta}}\varphi_j e^{-\frac{t}{2}|\xi|^{2\beta}}\widehat{v_0}(\xi)\|_{L^p} \\ &\lesssim \|\varphi_j e^{-\frac{t}{2}|\xi|^{2\beta}}\widehat{v_0}(\xi)\|_{L^p} \lesssim e^{-2^{2\beta j-1}t}\|\varphi_j\widehat{v_0}(\xi)\|_{L^p}. \end{aligned}$$

Now taking L^1-norm with respect on s over $[0,T)$, multiplying $2^{j\frac{n}{p'}}$ and then taking l^r-norm, one has

$$\|e^{\sqrt{t}(-\Delta)^{\frac{\beta}{2}}-t(-\Delta)^{\beta}}v_0\|_{\widetilde{L}_t^1F\dot{B}_{p,r}^{\frac{n}{p'}}} \lesssim \|v_0\|_{F\dot{B}_{p,r}^{-2\beta+\frac{n}{p'}}}.$$

From the above we can see that Lemma 4.1 holds. □

Lemma 4.2. *Let* $V, W \in \mathcal{X}_{p,r}$ *and* $\Delta\Phi = W - V$. *Then*

$$\|\mathscr{B}(V,\Phi)\|_{\mathcal{X}_{p,r}} \lesssim \|(V,W)\|^2_{\mathcal{X}_{p,r}}.$$

Proof. We first have

$$\|\Delta_j \widehat{\mathscr{B}(V,\Phi)}\|_{L^p} = \| \int_0^t \mathcal{F}(\Delta_j e^{[(\sqrt{t}-\sqrt{s})(-\Delta)^{\frac{\beta}{2}}-(t-s)(-\Delta)^{\beta}]}$$

$$\nabla e^{\sqrt{s}(-\Delta)^{\frac{\beta}{2}}}(e^{-\sqrt{s}(-\Delta)^{\frac{\beta}{2}}}V(s)e^{-\sqrt{s}(-\Delta)^{\frac{\beta}{2}}}\nabla\Phi(s)))\,ds\|_{L^p}. \tag{11}$$

Due to the equation

$$\begin{aligned}&(\sqrt{t}-\sqrt{s})(-\Delta)^{\frac{\beta}{2}}-(t-s)(-\Delta)^{\beta}\\ =\ & -(\sqrt{t-s}+\sqrt{s}-\sqrt{t})(-\Delta)^{\frac{\beta}{2}}+(\sqrt{t-s}\cdot(-\Delta)^{\frac{\beta}{2}}\\ &-\frac{(t-s)}{2}(-\Delta)^{\beta})-\frac{(t-s)}{2}(-\Delta)^{\beta},\end{aligned}$$

we have

(12) Equation (11)

$$\begin{aligned}&\lesssim \|\int_0^t e^{-\frac{t-s}{2}|\xi|^{2\beta}}\xi\cdot\mathcal{F}(\Delta_j e^{\sqrt{s}(-\Delta)^{\frac{\beta}{2}}}(e^{-\sqrt{s}(-\Delta)^{\frac{\beta}{2}}}Ve^{-\sqrt{s}(-\Delta)^{\frac{\beta}{2}}}\nabla\Phi))\,ds\|_{L^p}\\ &\lesssim \int_0^t e^{-(t-s)2^{2\beta j}}2^j\|\mathcal{F}(\Delta_j e^{\sqrt{s}(-\Delta)^{\frac{\beta}{2}}}(e^{-\sqrt{s}(-\Delta)^{\frac{\beta}{2}}}Ve^{-\sqrt{s}(-\Delta)^{\frac{\beta}{2}}}\nabla\Phi))\|_{L^p}\,ds.\end{aligned}$$

Using Bony's paraproduct decomposition, we can get

$$\begin{aligned}e^{-\sqrt{s}(-\Delta)^{\frac{\beta}{2}}}Ve^{-\sqrt{s}(-\Delta)^{\frac{\beta}{2}}}\nabla\Phi = &\sum_{j'\in Z} e^{-\sqrt{s}(-\Delta)^{\frac{\beta}{2}}}S_{j'-1}Ve^{-\sqrt{s}(-\Delta)^{\frac{\beta}{2}}}\Delta_{j'}\nabla\Phi\\ &+\sum_{j'\in Z} e^{-\sqrt{s}(-\Delta)^{\frac{\beta}{2}}}\Delta_{j'}Ve^{-\sqrt{s}(-\Delta)^{\frac{\beta}{2}}}S_{j'-1}\nabla\Phi\\ &+\sum_{j'\in Z} e^{-\sqrt{s}(-\Delta)^{\frac{\beta}{2}}}\Delta_{j'}Ve^{-\sqrt{s}(-\Delta)^{\frac{\beta}{2}}}\widetilde{\Delta_{j'}}\nabla\Phi.\end{aligned} \tag{13}$$

Substituting (13) into (12) and using the properties of support sets, we have

(14) $\|\Delta_j \widehat{\mathscr{B}(V,\Phi)}\|_{L^p}$

$$\begin{aligned}&\lesssim \int_0^t e^{-(t-s)2^{2\beta j}}2^j\sum_{|j-j'|\le 4}\|\mathcal{F}(\Delta_j e^{\sqrt{s}(-\Delta)^{\frac{\beta}{2}}}(e^{-\sqrt{s}(-\Delta)^{\frac{\beta}{2}}}S_{j'-1}\\ &\qquad Ve^{-\sqrt{s}(-\Delta)^{\frac{\beta}{2}}}\Delta_{j'}\nabla\Phi))\|_{L^p}\,ds\\ &+\int_0^t e^{-(t-s)2^{2\beta j}}2^j\sum_{|j-j'|\le 4}\|\mathcal{F}(\Delta_j e^{\sqrt{s}(-\Delta)^{\frac{\beta}{2}}}(e^{-\sqrt{s}(-\Delta)^{\frac{\beta}{2}}}\Delta_{j'}\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad\qquad\qquad V e^{-\sqrt{s}(-\Delta)^{\frac{\beta}{2}}} S_{j'-1}\nabla\Phi))\|_{L^p}\, ds\\
&+\int_0^t e^{-(t-s)2^{2\beta j}} 2^j \sum_{j'\geq j-N_0} \|\mathcal{F}(\Delta_j e^{\sqrt{s}(-\Delta)^{\frac{\beta}{2}}}(e^{-\sqrt{s}(-\Delta)^{\frac{\beta}{2}}}\Delta_{j'}\\
&\qquad\qquad\qquad\qquad V e^{-\sqrt{s}(-\Delta)^{\frac{\beta}{2}}} \widetilde{\Delta_{j'}}\nabla\Phi))\|_{L^p}\, ds\\
&:= I_1 + I_2 + I_3.
\end{aligned}
$$

As for the term $\|\mathcal{F}(\Delta_j e^{\sqrt{s}(-\Delta)^{\frac{\beta}{2}}}(e^{-\sqrt{s}(-\Delta)^{\frac{\beta}{2}}} S_{j'-1} V e^{-\sqrt{s}(-\Delta)^{\frac{\beta}{2}}}\Delta_{j'}\nabla\Phi))\|_{L^p}$, applying Hölder's inequality and Young's inequality, one has

$$
\begin{aligned}
&\sum_{|j-j'|\leq 4} \|\mathcal{F}(\Delta_j e^{\sqrt{s}(-\Delta)^{\frac{\beta}{2}}}(e^{-\sqrt{s}(-\Delta)^{\frac{\beta}{2}}} S_{j'-1} V e^{-\sqrt{s}(-\Delta)^{\frac{\beta}{2}}}\Delta_{j'}\nabla\Phi))\|_{L^p}\\
&= \sum_{|j-j'|\leq 4} \|\varphi_j e^{\sqrt{s}|\xi|^\beta}(e^{-\sqrt{s}|\xi|^\beta}\widehat{S_{j'-1}V}(\xi) * e^{-\sqrt{s}|\xi|^\beta}\widehat{\Delta_{j'}\nabla\Phi}(\xi))\|_{L^p}\\
&= \sum_{|j-j'|\leq 4} \|\varphi_j e^{\sqrt{s}|\xi|^\beta}\int_{R^n} e^{-\sqrt{s}|\xi-\eta|^\beta}(\widehat{S_{j'-1}V})(\xi-\eta) e^{-\sqrt{s}|\eta|^\beta}(\widehat{\Delta_{j'}\nabla\Phi})(\eta)\, d\eta\|_{L^p}\\
&\lesssim \sum_{|j-j'|\leq 4} \|\varphi_j \int_{R^n} e^{\sqrt{s}(|\xi|^\beta-|\xi-\eta|^\beta-|\eta|^\beta)}(\widehat{S_{j'-1}V})(\xi-\eta)(\widehat{\Delta_{j'}\nabla\Phi})(\eta)\, d\eta\|_{L^p}.
\end{aligned}
$$

Since $e^{\sqrt{s}(|\xi|^\beta-|\xi-\eta|^\beta-|\eta|^\beta)}$ is uniformly bounded when $\beta\in[0,1]$,

$$
\begin{aligned}
&\text{the formula above}\\
&\lesssim \sum_{|j-j'|\leq 4} \|\int_{R^n} |(\widehat{S_{j'-1}V})(\xi-\eta)|\cdot|(\widehat{\Delta_{j'}\nabla\Phi})(\eta)|\, d\eta\|_{L^p}\\
&= \sum_{|j-j'|\leq 4} \||(\widehat{S_{j'-1}V})| * |(\widehat{\Delta_{j'}\nabla\Phi})|\|_{L^p}\\
&\lesssim \sum_{|j-j'|\leq 4} 2^{j'(-1-\frac{n}{p'})} \sum_{k\leqslant j'-2} 2^{2\beta k} 2^{k(-2\beta+\frac{n}{p'})}\|\widehat{\Delta_k v}\|_{L^p} 2^{j'(1+\frac{n}{p'})}\|\widehat{\Delta_{j'}\nabla\phi}\|_{L^p}.
\end{aligned}
$$

In conclusion,

$$
\begin{aligned}
I_1 \lesssim &\int_0^t [e^{-(t-s)2^{2\beta j}} 2^j \sum_{|j-j'|\leq 4} 2^{j'(-1-\frac{n}{p'})}\\
&\times \sum_{k\leqslant j'-2} 2^{2\beta k} 2^{k(-2\beta+\frac{n}{p'})}\|\widehat{\Delta_k v}\|_{L^p} 2^{j'(1+\frac{n}{p'})}\|\widehat{\Delta_{j'}\nabla\phi}\|_{L^p}]\, ds.
\end{aligned}
$$

In the same way, we can get:

$$
I_2 \lesssim \int_0^t [e^{-(t-s)2^{2\beta j}} 2^j \sum_{|j-j'|\leq 4} \|\widehat{\Delta_{j'}v}\|_{L^p} \sum_{k\leqslant j'-2} 2^{(2\beta-1)k} 2^{k(1-2\beta+\frac{n}{p'})}\|\widehat{\Delta_k\nabla\phi}\|_{L^p}]\, ds.
$$

When it comes to I_3, for the term

$$\|\mathcal{F}(\Delta_j e^{\sqrt{s}(-\Delta)^{\frac{\beta}{2}}}(e^{-\sqrt{s}(-\Delta)^{\frac{\beta}{2}}}\Delta_{j'}Ve^{-\sqrt{s}(-\Delta)^{\frac{\beta}{2}}}\widetilde{\Delta_{j'}}\nabla\Phi))\|_{L^p},$$

we have

$$\begin{aligned}
&\sum_{j'\geq j-N_0}\|\mathcal{F}(\Delta_j e^{\sqrt{s}(-\Delta)^{\frac{\beta}{2}}}(e^{-\sqrt{s}(-\Delta)^{\frac{\beta}{2}}}\Delta_{j'}Ve^{-\sqrt{s}(-\Delta)^{\frac{\beta}{2}}}\widetilde{\Delta_{j'}}\nabla\Phi))\|_{L^p}\\
=&\sum_{j'\geq j-N_0}\|\varphi_j e^{\sqrt{s}|\xi|^\beta}\int_{R^n}e^{-\sqrt{s}|\xi-\eta|^\beta}(\widehat{\Delta_{j'}V})(\xi-\eta)e^{-\sqrt{s}|\eta|^\beta}(\widehat{\widetilde{\Delta_{j'}}\nabla\Phi})(\eta)\,d\eta\|_{L^p}\\
=&\sum_{j'\geq j-N_0}\|\varphi_j\int_{R^n}e^{\sqrt{s}(|\xi|^\beta-|\eta|^\beta-|\xi-\eta|^\beta)}(\widehat{\Delta_{j'}V})(\xi-\eta)(\widehat{\widetilde{\Delta_{j'}}\nabla\Phi})(\eta)\,d\eta\|_{L^p}\\
\lesssim&\sum_{j'\geq j-N_0}\|\varphi_j\int_{R^n}|(\widehat{\Delta_{j'}V})(\xi-\eta)|\cdot|(\widehat{\widetilde{\Delta_{j'}}\nabla\Phi})(\eta)|\,d\eta\|_{L^p}\\
=&\sum_{j'\geq j-N_0}\|\varphi_j|(\widehat{\Delta_{j'}V})|*|(\widehat{\widetilde{\Delta_{j'}}\nabla\Phi})|\|_{L^p}.
\end{aligned}$$

According to the results that have been proved in Section 3, we can get that when $1\leq p\leq 2$,

$$\begin{aligned}
I_3\lesssim\int_0^t\Big[&e^{-(t-s)2^{2\beta j}}2^{j(1+\frac{n}{p'})}\sum_{j'\geqslant j-N_0}2^{j'(2\beta-1-\frac{2n}{p'})}2^{j'(-2\beta+\frac{n}{p'})}\\
&\|\widehat{\Delta_{j'}v}\|_{L^p}2^{j'(1+\frac{n}{p'})}\|\widehat{\widetilde{\Delta_{j'}}\nabla\phi}\|_{L^p}\Big]\,ds.
\end{aligned}$$

When $p>2$,

$$\begin{aligned}
I_3\lesssim\int_0^t\Big[&e^{-(t-s)2^{2\beta j}}2^{j(1+n-\frac{n}{p'})}\sum_{j'\geqslant j-N_0}2^{j'(2\beta-1-n)}2^{j'(-2\beta+\frac{n}{p'})}\\
&\|\widehat{\Delta_{j'}v}\|_{L^p}2^{j'(1+\frac{n}{p'})}\|\widehat{\widetilde{\Delta_{j'}}\nabla\phi}\|_{L^p}\Big]\,ds.
\end{aligned}$$

By the above derivation, we have

$$\begin{aligned}
&\|\Delta_j\widehat{\mathscr{B}(V,\Phi)}\|_{L^p}\\
\lesssim&\int_0^t\Big[e^{-(t-s)2^{2\beta j}}2^j\sum_{|j-j'|\leq 4}2^{j'(-1-\frac{n}{p'})}\sum_{k\leqslant j'-2}2^{2\beta k}2^{k(-2\beta+\frac{n}{p'})}\\
&\qquad\|\widehat{\Delta_k v}\|_{L^p}2^{j'(1+\frac{n}{p'})}\|\widehat{\Delta_{j'}\nabla\phi}\|_{L^p}\Big]\,ds\\
&+\int_0^t\Big[e^{-(t-s)2^{2\beta j}}2^j\sum_{|j-j'|\leq 4}\|\widehat{\Delta_{j'}v}\|_{L^p}\sum_{k\leqslant j'-2}2^{(2\beta-1)k}2^{k(1-2\beta+\frac{n}{p'})}\|\widehat{\Delta_k\nabla\phi}\|_{L^p}\Big]\,ds\\
&+\int_0^t\Big[e^{-(t-s)2^{2\beta j}}2^{j(1+\frac{n}{p'})}\sum_{j'\geqslant j-N_0}2^{j'(2\beta-1-\frac{2n}{p'})}2^{j'(-2\beta+\frac{n}{p'})}
\end{aligned}$$

$$\|\widehat{\Delta_{j'} v}\|_{L^p} 2^{j'(1+\frac{n}{p'})} \|\widehat{\widetilde{\Delta_{j'}}\nabla\phi}\|_{L^p}\Big] ds$$
$$+\int_0^t \Big[e^{-(t-s)2^{2\beta j}} 2^{j(1+n-\frac{n}{p'})} \sum_{j'\geqslant j-N_0} 2^{j'(2\beta-1-n)} 2^{j'(-2\beta+\frac{n}{p'})}$$
$$\|\widehat{\Delta_{j'} v}\|_{L^p} 2^{j'(1+\frac{n}{p'})} \|\widehat{\widetilde{\Delta_{j'}}\nabla\phi}\|_{L^p}\Big] ds.$$

Since the remaining proof part is exactly the same as that corresponding to Section 3, we can already obtain the final result, that is,

$$\|\mathscr{B}(V,\Phi)\|_{\mathcal{X}_{p,r}} \lesssim \|(V,W)\|^2_{\mathcal{X}_{p,r}}.$$

Using the fixed point theorem again we can prove Theorem 1.2. □

5. Time decay of mild solutions

From the contents of the previous two sections, we can prove Theorem 1.3.

Lemma 5.1. *For any $\sigma > 0$, the system* (1.1) *exists global-in-time mild solution $(v,w) \in \mathcal{X}_{p,r}$ and $(e^{\sqrt{t}(-\Delta)^{\frac{\beta}{2}}} v, e^{\sqrt{t}(-\Delta)^{\frac{\beta}{2}}} w) \in \mathcal{X}_{p,r}$. Actually, they satisfy the following time decay estimate*:

$$\|(\Lambda^\sigma v(t), \Lambda^\sigma w(t))\|_{F\dot{B}_{p,r}^{-2\beta+\frac{n}{p'}}} \le C_{\beta,\sigma} t^{-\frac{\sigma}{2\beta}} \|(v_0,w_0)\|_{F\dot{B}_{p,r}^{-2\beta+\frac{n}{p'}}},$$

where $C_{\beta,\sigma}$ is a constant.

Proof. It's worth noting that we just need to prove

$$\|\Lambda^\sigma v(t)\|_{F\dot{B}_{p,r}^{-2\beta+\frac{n}{p'}}} \le C_{\beta,\sigma} t^{-\frac{\sigma}{2\beta}} \|v_0\|_{F\dot{B}_{p,r}^{-2\beta+\frac{n}{p'}}},$$

because we can prove $\|\Lambda^\sigma w(t)\|_{F\dot{B}_{p,r}^{-2\beta+\frac{n}{p'}}} \le C_{\beta,\sigma} t^{-\frac{\sigma}{2\beta}} \|w_0\|_{F\dot{B}_{p,r}^{-2\beta+\frac{n}{p'}}}$ in the same way. According to the above two formulas, the result described in the theorem can be obtained immediately.

$$\begin{aligned}\|\Lambda^\sigma v(t)\|_{F\dot{B}_{p,r}^{-2\beta+\frac{n}{p'}}} &= \|\Lambda^\sigma e^{-\sqrt{t}(-\Delta)^{\frac{\beta}{2}}} e^{\sqrt{t}(-\Delta)^{\frac{\beta}{2}}} v(t)\|_{F\dot{B}_{p,r}^{-2\beta+\frac{n}{p'}}}\\ &= (\sum_{j\in Z} 2^{j(-2\beta+\frac{n}{p'})r} \|\mathcal{F}(\Delta_j \Lambda^\sigma e^{-\sqrt{t}(-\Delta)^{\frac{\beta}{2}}} e^{\sqrt{t}(-\Delta)^{\frac{\beta}{2}}} v(t))\|^r_{L^p})^{\frac{1}{r}}\\ &= (\sum_{j\in Z} 2^{j(-2\beta+\frac{n}{p'})r} \||\xi|^\sigma e^{-\sqrt{t}|\xi|^\beta} \mathcal{F}(\Delta_j e^{\sqrt{t}(-\Delta)^{\frac{\beta}{2}}} v(t))\|^r_{L^p})^{\frac{1}{r}}.\end{aligned}$$

Suppose the function $f(\mu) = \mu^\sigma e^{-\sqrt{t}\mu^\beta}$, where $\mu \ge 0$, σ is a constant greater than 0. As can be seen from the derivation of the function, $f(\mu) \le f((\frac{\sigma}{\beta\sqrt{t}})^{\frac{1}{\beta}}) \le C_{\beta,\sigma} t^{-\frac{\sigma}{2\beta}}$. Therefore,

$$\|\Lambda^\sigma v(t)\|_{F\dot{B}_{p,r}^{-2\beta+\frac{n}{p'}}} \le C_{\beta,\sigma} t^{-\frac{\sigma}{2\beta}} (\sum_{j\in Z} 2^{j(-2\beta+\frac{n}{p'})r} \|\mathcal{F}(\Delta_j e^{\sqrt{t}\Lambda} v(t))\|^r_{L^p})^{\frac{1}{r}}$$

$$= C_{\beta,\sigma} t^{-\frac{\sigma}{2\beta}} \|e^{\sqrt{t}\Lambda} v\|_{F\dot{B}_{p,r}^{-2\beta+\frac{n}{p'}}}$$
$$\le C_{\beta,\sigma} t^{-\frac{\sigma}{2\beta}} \|v_0\|_{F\dot{B}_{p,r}^{-2\beta+\frac{n}{p'}}}.$$

Thus we have proved Lemma 5.1. □

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