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ON THE DENOMINATORS OF ε-HARMONIC NUMBERS

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Abstract. Let H_n be the n-th harmonic number and let v_n be its denominator. Shiu proved that there are infinitely many positive integers n with $v_n = v_{n+1}$. Recently, Wu and Chen proved that the set of positive integers n with $v_n = v_{n+1}$ has density one. They also proved that the same result is true for the denominators of alternating harmonic numbers. In this paper, we prove that the result is true for the denominators of ε-harmonic numbers, where $\varepsilon = \{\varepsilon_i\}_{i=1}^{\infty}$ is a pure recurring sequence with $\varepsilon_i \in \{-1, 1\}$.

1. Introduction

For any positive integer n, let

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \frac{u_n}{v_n}, \quad (u_n, v_n) = 1, v_n > 0.$$

The number H_n is called *the n-th harmonic number*.

For any prime number p, let J_p be the set of positive integers n with $p \mid u_n$. Eswarathasan and Levine [2] conjectured that J_p is finite for any prime number p. Boyd [1] conjectured that $|J_p| = O(p^2(\log\log p)^{2+\varepsilon})$. For any set S of positive integers and any real number $x \geq 1$, let $S(x) = |S \cap [1, x]|$. Sanna [3] proved that

$$J_p(x) \leq 129p^{\frac{2}{3}}x^{0.765}.$$

This was improved by Wu and Chen [5] to

$$J_p(x) \leq 3x^{\frac{2}{3}+\frac{1}{25\log p}}. \tag{1.1}$$

Shiu [4] proved that there are infinitely many positive integers n with $v_n = v_{n+1}$. Recently, Wu and Chen [6] proved that the set of positive integers n with $v_n = v_{n+1}$ has density one. They also proved that the same result is true for the denominators of alternating harmonic numbers.

For any sequence $\varepsilon = \{\varepsilon_i\}_{i=1}^{\infty}$, if there is a positive integer s such that

$$\varepsilon_{i+s} = \varepsilon_i, \ i = 1, 2, 3, \ldots,$$

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then the sequence ε is called a pure recurring sequence and s is a period of the sequence.

For any positive integer n and any sequence ε, let

$$H_{n,\varepsilon} = \sum_{i=1}^{n} \frac{\varepsilon_i}{i} = \frac{a_n(\varepsilon)}{b_n(\varepsilon)}, \quad (a_n(\varepsilon), b_n(\varepsilon)) = 1, b_n(\varepsilon) > 0.$$

The number $H_{n,\varepsilon}$ is called *the n-th ε-harmonic number.*

In this paper, the following result is proved.

Theorem 1.1. *Let $\varepsilon = \{\varepsilon_i\}_{i=1}^{\infty}$ be a pure recurring sequence with $\varepsilon_i \in \{-1, 1\}$. Then the set of positive integers n with $b_n(\varepsilon) = b_{n+1}(\varepsilon)$ has density one.*

2. Proof of Theorem 1.1

Let $\varepsilon = \{\varepsilon_i\}_{i=1}^{\infty}$ be a pure recurring sequence with $\varepsilon_i \in \{-1, 1\}$ and s be a period of the sequence. For any prime number p, define each sequence $\delta_j = \{\delta_{j,i}\}_{i=1}^{\infty}$ $(j = 0, 1, \ldots)$ as follows:

$$\delta_{0,i} = \varepsilon_i, \ \delta_{j+1,i} = \delta_{j,pi}, \ j = 0, 1, \ldots. \tag{2.1}$$

It is clear that the sequences $\delta_1, \delta_2, \ldots$ have a period s and $\delta_{j,i} \in \{1, -1\}$.

For any prime number p and any positive integer m, let $S_{p,\varepsilon}$ be the set of positive integers n with $p \mid a_n(\varepsilon)$, let $I_{m,\varepsilon}$ be the set of positive integers n with $m \nmid b_n(\varepsilon)$.

Lemma 2.1. *Let $\varepsilon = \{\varepsilon_i\}_{i=1}^{\infty}$ be a pure recurring sequence with $\varepsilon_i \in \{-1, 1\}$ and s be a period of the sequence. Let p be a prime number and let x and y be two real numbers with $1 \le y < p$. Write*

$$S_{p,\varepsilon} \cap [x, x+y] = \{n_1 < n_2 < \cdots < n_l\}.$$

Then, for any integer $d \ge 1$, we have

$$|\{i : n_i - n_{i-1} = d\}| \le s(d-1).$$

Proof. If $l \le 1$, the result is obvious. We may assume that $l \ge 2$. For any positive integer d and any real number x, let

$$\begin{aligned}
f_d(x) &= (x+1)(x+2)\cdots(x+d), \\
g_{1,d}(x) &= \frac{f_d(x)}{x+1}\varepsilon_1 + \frac{f_d(x)}{x+2}\varepsilon_2 + \cdots + \frac{f_d(x)}{x+d}\varepsilon_d, \\
g_{2,d}(x) &= \frac{f_d(x)}{x+1}\varepsilon_2 + \frac{f_d(x)}{x+2}\varepsilon_3 + \cdots + \frac{f_d(x)}{x+d}\varepsilon_{d+1}, \\
&\ \ \vdots \\
g_{s,d}(x) &= \frac{f_d(x)}{x+1}\varepsilon_s + \frac{f_d(x)}{x+2}\varepsilon_{s+1} + \cdots + \frac{f_d(x)}{x+d}\varepsilon_{s+d-1}.
\end{aligned}$$

It is clear that $g_{j,d}(x)$ is an integer-valued polynomial. Since $g_{j,d}(-1) = (d-1)!$ or $g_{j,d}(-1) = -(d-1)!$, it follows that $g_{j,d}(x)$ is a nonzero polynomial of degree at most $d-1$ for every j. Let

$$\{n_1, n_2, \dots, n_l\} = \bigcup_{j=0}^{s-1} A_j, \;\; A_j = \{n_t \in S_{p,\varepsilon} \cap [x, x+y] : n_t \equiv j \pmod{s}\}.$$

For every integer $0 \le j \le s-1$, if there exists an integer $n_t \in A_j$ and $n_{t+1} - n_t = d$, then

$$\begin{aligned} g_{j+1,d}(n_t) &= (H_{n_{t+1},\varepsilon} - H_{n_t,\varepsilon}) f_d(n_t) \\ &= \left(\frac{a_{n_{t+1}}(\varepsilon)}{b_{n_{t+1}}(\varepsilon)} - \frac{a_{n_t}(\varepsilon)}{b_{n_t}(\varepsilon)} \right) f_d(n_t). \end{aligned}$$

Since

$$p \mid a_{n_{t+1}}(\varepsilon), \quad p \mid a_{n_t}(\varepsilon),$$

it follows that

$$g_{j+1,d}(n_t) \equiv 0 \pmod{p}.$$

Noting that $y < p$ and there are at most $d-1$ solutions of the equation

$$g_{j+1,d}(x) \equiv 0 \pmod{p}.$$

It follows that, for every integer $0 \le j \le s-1$, there are at most $d-1$ integers $n_t \in A_j$ with $n_{t+1} - n_t = d$. Therefore, for any integer $d \ge 1$, we have

$$|\{i : n_i - n_{i-1} = d\}| \le s(d-1).$$

This completes the proof of Lemma 2.1. □

Lemma 2.2. *Let $\varepsilon = \{\varepsilon_i\}_{i=1}^{\infty}$ be a pure recurring sequence with $\varepsilon_i \in \{-1, 1\}$ and s be a period of the sequence. For any prime number p and any real numbers x, y with $1 \le y < p$, we have*

$$|S_{p,\varepsilon} \cap [x, x+y]| \le 3 s^{\frac{1}{3}} y^{\frac{2}{3}}.$$

Proof. Let

$$S_{p,\varepsilon} \cap [x, x+y] = \{n_1 < n_2 < \cdots < n_l\}$$

and

$$a_d = |\{i : n_i - n_{i-1} = d\}|.$$

Let $z = (y/s)^{1/3}$. By Lemma 2.1 we have

$$\sum_{d \le z} a_d \le s \sum_{d \le z} (d-1) \le s z^2 = s^{1/3} y^{2/3}.$$

Clearly,

$$\frac{y^{1/3}}{s^{1/3}} \sum_{d > z} a_d = z \sum_{d > z} a_d \le \sum_d d a_d = \sum_{i=1}^{l-1} (n_{i+1} - n_i) = n_l - n_1 \le y.$$

By the definition of a_d we have

$$|S_{p,\varepsilon} \cap [x, x+y]| = l = 1 + \sum_d a_d \le 1 + 2s^{1/3}y^{2/3} \le 3s^{1/3}y^{2/3}.$$

This completes the proof of Lemma 2.2. □

Lemma 2.3. *For any prime number* p*, let* δ_j $(j = 0, 1, \dots)$ *be sequences defined in* (2.1)*. For any real number* $x \ge 1$*, we have*

$$S_{p,\delta_j}(x) \le \min\left\{4s^{\frac{1}{3}}x^{\frac{2}{3}+\frac{\log(64s)}{3\log p}}, p^2x^{1-\frac{\log\frac{3}{2}}{\log p}}\right\}.$$

Proof. Firstly, we prove that $S_{p,\delta_j}(x) \le 4s^{\frac{1}{3}}x^{\frac{2}{3}+\frac{\log(64s)}{3\log p}}$. Let m be an integer with $m \in S_{p,\delta_j}$ and $m \ge p$, then $p \mid a_m(\delta_j)$. Write $m = pm_1 + r$, where $m_1 \ge 1$ and $0 \le r < p$. Since

$$H_{m,\delta_j} = \sum_{\substack{i=1\\ p\nmid i}}^{m} \frac{\delta_{j,i}}{i} + \sum_{\substack{i=1\\ p\mid i}}^{m} \frac{\delta_{j,i}}{i} = \frac{b}{a} + \frac{1}{p}H_{m_1,\delta_{j+1}} = \frac{pbb_{m_1}(\delta_{j+1}) + aa_{m_1}(\delta_{j+1})}{pab_{m_1}(\delta_{j+1})},$$

where a and b are coprime integers, it follows that $p \mid pbb_{m_1}(\delta_{j+1}) + aa_{m_1}(\delta_{j+1})$. Noting that $p \nmid a$, we have $p \mid a_{m_1}(\delta_{j+1})$ and so $m_1 \in S_{p,\delta_{j+1}}$.

Let k be the integer with $p^k \le x < p^{k+1}$. Let $A = 3s^{\frac{1}{3}}(p-1)^{\frac{2}{3}}$. In view of Lemma 2.2,

$$\begin{aligned}
S_{p,\delta_j}(x) &\le |S_{p,\delta_j} \cap [1, p-1]| + \sum_{n \in S_{p,\delta_{j+1}} \cap [1, \frac{x}{p}]} |S_{p,\delta_j} \cap [pn, pn+p-1]| \\
&\le A + A|S_{p,\delta_{j+1}} \cap [1, \frac{x}{p}]| \\
&\le A + A^2 + A^2|S_{p,\delta_{j+2}} \cap [1, \frac{x}{p^2}]| \\
&\quad \vdots \\
&\le A + \cdots + A^k + A^k|S_{p,\delta_{j+k}} \cap [1, \frac{x}{p^k}]| \\
&\le (A+1)^k + A^k \cdot \left(3s^{1/3}\left(\frac{x}{p^k}\right)^{2/3}\right) \\
&\le (A+1)^k \cdot \left(1 + 3s^{1/3}\left(\frac{x}{p^k}\right)^{2/3}\right) \\
&\le \left(4s^{1/3}(p-1)^{2/3}\right)^k \left(4s^{1/3}\left(\frac{x}{p^k}\right)^{2/3}\right) \\
&\le (4s^{\frac{1}{3}})^{k+1}x^{\frac{2}{3}}.
\end{aligned}$$

By $p^k \le x < p^{k+1}$ we have

$$k \le \frac{\log x}{\log p}.$$

Therefore

$$S_{p,\delta_j}(x) \le (4s^{\frac{1}{3}})^{k+1} x^{\frac{2}{3}} \le 4s^{\frac{1}{3}} x^{\frac{2}{3}+\frac{\log(64s)}{3\log p}}.$$

Now, we prove that $S_{p,\delta_j}(x) \le p^2 x^{1-\frac{\log\frac{3}{2}}{\log p}}$. It is easy to prove that $S_{2,\delta_j} = \emptyset$. We may assume that $p \ge 3$. In view of Lemma 2.1, there is no integer n such that both n and $n+1$ are in S_{p,δ_j}. It follows that

$$\begin{aligned}
S_{p,\delta_j}(x) &\le |S_{p,\delta_j} \cap [1, p-1]| + \sum_{n \in S_{p,\delta_{j+1}} \cap [1,\frac{x}{p}]} |S_{p,\delta_j} \cap [pn, pn+p-1]| \\
&\le \frac{p+1}{2} + \frac{p+1}{2} |S_{p,\delta_{j+1}} \cap [1, \frac{x}{p}]| \\
&\le \frac{p+1}{2} + \left(\frac{p+1}{2}\right)^2 + \left(\frac{p+1}{2}\right)^2 |S_{p,\delta_{j+2}} \cap [1, \frac{x}{p^2}]| \\
&\ \ \vdots \\
&\le \frac{p+1}{2} + \cdots + \left(\frac{p+1}{2}\right)^k + \left(\frac{p+1}{2}\right)^k |S_{p,\delta_{j+k}} \cap [1, \frac{x}{p^k}]| \\
&\le \frac{p+1}{2} + \cdots + \left(\frac{p+1}{2}\right)^{k+1} \\
&\le \left(\frac{p+1}{2}\right)^{k+2} \\
&\le \left(\frac{2p}{3}\right)^{k+2} \\
&\le \left(\frac{2}{3}\right)^{k+1} p^2 x.
\end{aligned}$$

Since

$$\left(\frac{3}{2}\right)^{k+1} > x^{\frac{\log\frac{3}{2}}{\log p}},$$

it follows that

$$S_{p,\delta_j}(x) \le p^2 x^{1-\frac{\log\frac{3}{2}}{\log p}}.$$

This completes the proof of Lemma 2.3. □

Lemma 2.4. *For any prime number p, let δ_j $(j = 0, 1, \ldots)$ be sequences defined in* (2.1). *For any positive integer k, we have*

$$I_{p^k,\varepsilon} = \{p^k n_1 + r : n_1 \in S_{p,\delta_k} \cup \{0\},\ 0 \le r \le p^k - 1\} \setminus \{0\}.$$

Proof. It is clear that $n \in I_{p^k,\varepsilon}$ if and only if $\nu_p(H_{n,\varepsilon}) = \nu_p(a_n(\varepsilon)) - \nu_p(b_n(\varepsilon)) > -k$.

If $n < p^k$, then $\nu_p(H_{n,\varepsilon}) \geq -\nu_p([1,2,\ldots,n]) > -k$. So $n \in I_{p^k,\varepsilon}$. In the following, we assume that $n \geq p^k$. Let

$$n = p^k n_1 + r,\ 0 \leq r \leq p^k - 1,\ n_1 \geq 1,\ n_1, r \in \mathbb{Z}.$$

Write

$$H_{n,\varepsilon} = \sum_{\substack{m=1 \\ p^k \nmid m}}^{n} \frac{\varepsilon_m}{m} + \frac{1}{p^k} H_{n_1,\delta_k} = \frac{c}{dp^{k-1}} + \frac{a_{n_1}(\delta_k)}{p^k b_{n_1}(\delta_k)} = \frac{pcb_{n_1}(\delta_k) + da_{n_1}(\delta_k)}{p^k db_{n_1}(\delta_k)},$$

where $p \nmid d$ and $(a_{n_1}(\delta_k), b_{n_1}(\delta_k)) = 1$.

If $n_1 \in S_{p,\delta_k}$, then $p \mid a_{n_1}(\delta_k)$ and $p \nmid b_{n_1}(\delta_k)$. Thus, $p \mid pcb_{n_1}(\delta_k) + da_{n_1}(\delta_k)$ and $\nu_p(p^k db_{n_1}(\delta_k)) = k$. So $\nu_p(H_{n,\varepsilon}) > -k$, and then $n \in I_{p^k,\varepsilon}$.

If $n_1 \notin S_{p,\delta_k}$, then $p \nmid a_{n_1}(\delta_k)$ and $p \nmid pcb_{n_1}(\delta_k) + da_{n_1}(\delta_k)$. Thus, $\nu_p(H_{n,\varepsilon}) \leq -k$, and then $n \notin I_{p^k,\varepsilon}$.

Therefore $n \in I_{p^k,\varepsilon}$ if and only if $n_1 \in S_{p,\delta_k} \cup \{0\}$. This completes the proof of Lemma 2.4. □

Lemma 2.5. *Let m_n be the least common multiple of $1, 2, \ldots, \lfloor n^{1/4} \rfloor$ and let $T_\varepsilon = \{n : m_n \nmid b_n(\varepsilon)\}$. Then*

$$T_\varepsilon(x) \ll \frac{x}{\log x}.$$

Proof. By Lemmas 2.3 and 2.4, for any prime p and any positive integer k, we can prove that for $x \geq p^k$,

$$I_{p^k,\varepsilon}(x) \leq 5s^{\frac{1}{3}} p^k \left(\frac{x}{p^k}\right)^{\frac{2}{3} + \frac{\log(64s)}{3\log p}} \leq 5s^{\frac{1}{3}} p^{\frac{k}{3}} x^{\frac{2}{3} + \frac{\log(64s)}{3\log p}},$$

and for $x < p^k$,

$$I_{p^k,\varepsilon}(x) \leq x \leq p^{\frac{k}{3}} x^{\frac{2}{3}} \leq 5s^{\frac{1}{3}} p^{\frac{k}{3}} x^{\frac{2}{3} + \frac{\log(64s)}{3\log p}}.$$

Therefore,

$$I_{p^k,\varepsilon}(x) \leq 5s^{\frac{1}{3}} p^{\frac{k}{3}} x^{\frac{2}{3} + \frac{\log(64s)}{3\log p}}. \tag{2.2}$$

Similarly, by Lemmas 2.3 and 2.4 we can prove that,

$$I_{p^k,\varepsilon}(x) \leq 2p^{2 + k\frac{\log \frac{3}{2}}{\log p}} x^{1 - \frac{\log \frac{3}{2}}{\log p}}. \tag{2.3}$$

For any prime p and any positive real number x with $p \leq x^{1/4}$, let α_p be the integer such that $p^{\alpha_p} \leq x^{1/4} < p^{\alpha_p + 1}$. By the definition of m_n and T_ε, we know that

$$T_\varepsilon(x) \leq \sum_{p \leq x^{1/4}} I_{p^{\alpha_p},\varepsilon}(x) := I_1 + I_2 + I_3,$$

where

$$I_1=\sum_{p\le(64s)^2} I_{p^{\alpha_p},\varepsilon}(x),\ I_2=\sum_{(64s)^2<p\le x^{1/12}} I_{p^{\alpha_p},\varepsilon}(x),\ I_3=\sum_{x^{1/12}<p\le x^{1/4}} I_{p^{\alpha_p},\varepsilon}(x).$$

For I_1, by (2.3) and $p^{\alpha_p}\le x^{1/4}$, we have

$$\begin{aligned} I_1 &= \sum_{p\le(64s)^2} I_{p^{\alpha_p},\varepsilon}(x) \\ &\le \sum_{p\le(64s)^2} 2p^{2+\alpha_p\frac{\log\frac{3}{2}}{\log p}}x^{1-\frac{\log\frac{3}{2}}{\log p}} \\ &\ll x^{1-\frac{3}{8}\cdot\frac{\log\frac{3}{2}}{\log(64s)}} \\ &\ll \frac{x}{\log x}. \end{aligned}$$

It follows from (2.2) and $p^{\alpha_p}\le x^{1/4}$ that

$$I_2\le \sum_{(64s)^2<p\le x^{1/12}} 5s^{\frac{1}{3}}p^{\frac{\alpha_p}{3}}x^{\frac{2}{3}+\frac{\log(64s)}{3\log p}} \ll \sum_{p\le x^{1/12}} x^{\frac{1}{12}+\frac{2}{3}+\frac{1}{6}} \ll \frac{x}{\log x}.$$

If $p>x^{1/12}$, then

$$x^{\frac{\log(64s)}{3\log p}}=e^{\frac{\log(64s)\log x}{3\log p}}\le e^{\frac{4\log(64s)\log x}{\log x}}=e^{4\log(64s)}.$$

For I_3, by (2.2) and $p^{\alpha_p}\le x^{1/4}$, we have

$$I_3\le \sum_{x^{1/12}<p\le x^{1/4}} 5s^{\frac{1}{3}}p^{\frac{\alpha_p}{3}}x^{\frac{2}{3}+\frac{\log(64s)}{3\log p}} \ll \sum_{p\le x^{1/4}} x^{\frac{1}{12}+\frac{2}{3}} \ll \frac{x}{\log x}.$$

Thus

$$I_1+I_2+I_3\ll\frac{x}{\log x}.$$

Therefore,

$$T_\varepsilon(x)\ll\frac{x}{\log x}.$$

This completes the proof of Lemma 2.5. □

Proof of Theorem 1.1. For any prime number p, let $a_0(\delta_1)=0$. Let

$$\begin{gathered} A_\varepsilon=\{n: b_{n+1}(\varepsilon)=b_n(\varepsilon)\}, \\ B=\{n: p^2\mid n+1 \text{ for some prime } p>n^{1/9}\}, \\ C=\{n: n+1=pk, p\mid a_k(\delta_1)a_{k-1}(\delta_1) \text{ for some prime } p>n^{1/9}\}, \\ D=\{n: \nu_p(n+1)\ge\nu_p(m_n) \text{ for some prime } p\le n^{1/9}\}. \end{gathered}$$

The proof is similar to Theorem 1.1 in [6]. For the readability of the paper, we show the whole process. Let n be a positive integer with $n\notin B\cup C\cup D\cup T_\varepsilon$. Now we prove that $b_{n+1}(\varepsilon)=b_n(\varepsilon)$. It suffices to prove that $\nu_p(b_{n+1}(\varepsilon))=\nu_p(b_n(\varepsilon))$ for any prime p.

Let p be a prime. We divide into the following cases:

Case 1: $p \nmid n+1$. Noting that

$$\frac{a_n(\varepsilon)}{b_n(\varepsilon)} + \frac{\varepsilon_{n+1}}{n+1} = \frac{a_{n+1}(\varepsilon)}{b_{n+1}(\varepsilon)},$$

we have $\nu_p(b_{n+1}(\varepsilon)) = \nu_p(b_n(\varepsilon))$.

Case 2: $p \mid n+1$ and $p > n^{1/9}$. Let $n+1 = pk$. Since $n \notin B \cup C$, it follows that $k \geq 2$ and $p \nmid k a_k(\delta_1) a_{k-1}(\delta_1)$. Noting that $p \nmid k$ and

$$\frac{a_{k-1}(\delta_1)}{b_{k-1}(\delta_1)} + \frac{\varepsilon_k}{k} = \frac{a_k(\delta_1)}{b_k(\delta_1)},$$

we have $\nu_p(b_{k-1}(\delta_1)) = \nu_p(b_k(\delta_1))$. Since

$$H_{n,\varepsilon} = \sum_{\substack{m=1 \\ p\nmid m}}^{n} \frac{\varepsilon_m}{m} + \frac{1}{p} H_{k-1,\delta_1} = \frac{b}{a} + \frac{a_{k-1}(\delta_1)}{p b_{k-1}(\delta_1)} = \frac{p b b_{k-1}(\delta_1) + a a_{k-1}(\delta_1)}{p a b_{k-1}(\delta_1)} = \frac{a_n(\varepsilon)}{b_n(\varepsilon)}$$

and

$$H_{n+1,\varepsilon} = \sum_{\substack{m=1 \\ p\nmid m}}^{n} \frac{\varepsilon_m}{m} + \frac{1}{p} H_{k,\delta_1} = \frac{b}{a} + \frac{a_k(\delta_1)}{p b_k(\delta_1)} = \frac{p b b_k(\delta_1) + a a_k(\delta_1)}{p a b_k(\delta_1)} = \frac{a_{n+1}(\varepsilon)}{b_{n+1}(\varepsilon)},$$

where a, b are positive integers with $p \nmid a$, it follows from $p \nmid k a_k(\delta_1) a_{k-1}(\delta_1)$ and $\nu_p(b_{k-1}(\delta_1)) = \nu_p(b_k(\delta_1))$ that

$$\nu_p(b_{n+1}(\varepsilon)) = \nu_p(b_n(\varepsilon)).$$

Case 3: $p \mid n+1$ and $p \leq n^{1/9}$. By $n \notin D \cup T_\varepsilon$, we have

$$\nu_p(n+1) < \nu_p(m_n) \leq \nu_p(b_n(\varepsilon)).$$

It follows from

$$\frac{a_n(\varepsilon)}{b_n(\varepsilon)} + \frac{\varepsilon_{n+1}}{n+1} = \frac{a_{n+1}(\varepsilon)}{b_{n+1}(\varepsilon)},$$

that $\nu_p(b_{n+1}(\varepsilon)) = \nu_p(b_n(\varepsilon))$.

Up to now, we have proved that $b_{n+1}(\varepsilon) = b_n(\varepsilon)$ for any positive integer $n \notin B \cup C \cup D \cup T_\varepsilon$.

Now we prove that

$$B(x) + C(x) + D(x) + T_\varepsilon(x) \ll \frac{x}{\log x}.$$

As in [6, Theorem 1.1], we have

$$B(x) \ll x^{\frac{17}{18}}, \; D(x) \ll x^{\frac{26}{27}}.$$

By the definition of C and Lemma 2.3 we have

$$C(x) \leq C(\sqrt{x}) + 2 \sum_{x^{1/18} < p \leq x} S_{p,\delta_1}\left(\frac{x+1}{p}\right) + \pi(x+1)$$

$$\ll \sqrt{x} + \sum_{x^{1/18}<p\le x} \left(\frac{x+1}{p}\right)^{\frac{2}{3}+\frac{\log(64s)}{3\log p}} + \frac{x}{\log x}$$
$$\ll \sqrt{x} + \sum_{x^{1/18}<p\le x} \left(\frac{x}{p}\right)^{\frac{2}{3}} + \frac{x}{\log x}$$
$$\ll \sqrt{x} + \sum_{p\le x} \left(\frac{x}{p}\right)^{\frac{2}{3}} + \frac{x}{\log x}.$$

It follows from the proof of [6, Theorem 1.1] that

$$\sum_{p\le x} \left(\frac{1}{p}\right)^{\frac{2}{3}} \ll \frac{x^{1/3}}{\log x},$$

Hence,

$$C(x) \ll \frac{x}{\log x}.$$

Therefore, it follows from Lemma 2.5 that

$$A_\varepsilon(x) \ge x - B(x) - C(x) - D(x) - T_\varepsilon(x) \ge x - c\frac{x}{\log x}$$

for a positive constant c.

For any prime p, we have $b_p(\varepsilon) = pb_{p-1}(\varepsilon)$. Thus $p-1 \notin A_\varepsilon$ for any prime p. Hence

$$A_\varepsilon(x) \le x - \pi(x) = x - (1+o(1))\frac{x}{\log x}.$$

This completes the proof of Theorem 1.1. □

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