Bull. Korean Math. Soc. **57** (2020), No. 6, pp. 1367–1382
https://doi.org/10.4134/BKMS.b191032
pISSN: 1015-8634 / eISSN: 2234-3016

SOME RIGIDITY CHARACTERIZATIONS OF EINSTEIN METRICS AS CRITICAL POINTS FOR QUADRATIC CURVATURE FUNCTIONALS

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Abstract. We study rigidity results for the Einstein metrics as the critical points of a family of known quadratic curvature functionals involving the scalar curvature, the Ricci curvature and the Riemannian curvature tensor, characterized by some pointwise inequalities involving the Weyl curvature and the traceless Ricci curvature. Moreover, we also provide a few rigidity results for locally conformally flat critical metrics.

1. Introduction

A well-known example of a Riemannian functional is the Einstein-Hilbert functional

$$\mathcal{H}=\int_M R$$

on $\mathscr{M}_1(M^n)$, where R denotes the scalar curvature and $\mathscr{M}_1(M^n)$ is the space of equivalence classes of smooth Riemannian metrics of volume one on closed Riemannian manifold M^n, $n\geq 3$. Furthermore, it is easy to see that Einstein metrics are critical for the functional $\mathcal{H}$ (see [3, 11]). In this paper, we are interested in studying the functional

$$\mathcal{F}_{t,s}(g)=\int_M |\mathrm{Ric}|^2+t\int_M R^2+s\int_M |\mathrm{Rm}|^2, \tag{1.1}$$

where t,s are real constants, Ric and Rm denote the Ricci curvature and the Riemannian curvature tensor, respectively. It is easy to observe from (2.4) that every Einstein metric is critical for $\mathcal{F}_{t,0}$. In [6], Catino considered the curvature functional $\mathcal{F}_{t,0}$ and obtained some conditions on the geometry of M^n such that critical metrics of $\mathcal{F}_{t,0}$ are Einstein. Certainly, there exist critical metrics which are not necessarily Einstein (for instance, see [3, Chapter 4] and [19]). For some development in this direction, see [1,5,9,10,12,13,15,18,22] and the references

Received November 21, 2019; Revised March 4, 2020; Accepted March 25, 2020.
2010 *Mathematics Subject Classification.* Primary 53C24; Secondary 53C21.
Key words and phrases. Critical metric, rigidity, Einstein.
This work is financially supported by NSFC (Nos. 11971153, 11671121).

therein. Therefore, it is natural to ask that under what conditions a critical metric for the functionals $\mathcal{F}_{t,s}(s \neq 0)$ must be an Einstein one.

Barros and Da Silva in [2] show that locally conformally flat critical metrics for $\mathcal{F}_{t,s}$ with $n+4(n-1)t+4s=0$ (when $n \geq 5$) and some additional conditions are space form metrics (see [2, Theorem 3]). In this paper, we give some new characterizations, by some pointwise inequalities involving the Weyl curvature and the traceless Ricci curvature, on critical metrics for $\mathcal{F}_{t,s}$ on $\mathscr{M}_1(M^n)$ with $n+4(n-1)t+4s \neq 0$. In order to state our results, throughout this paper, we denote by $\mathring{\mathrm{Ric}}$ and W the traceless Ricci tensor and the Weyl curvature, respectively. We denote by $\owedge$ the Kulkarni-Nomizu product.

Our main results are stated as follows:

Theorem 1.1. *Let M^n be a closed manifold of dimension $n \geq 5$ with positive scalar curvature and g be a critical metric for $\mathcal{F}_{t,s}$ on $\mathscr{M}_1(M^n)$. Suppose that*

$$\Big|W - \frac{(n-4)[4s+(n-2)]}{\sqrt{2n}(n-2)(8s+n-2)}\mathring{\mathrm{Ric}} \owedge g\Big||\mathring{\mathrm{R}}_{ij}| + \sqrt{\frac{2(n-1)^2}{n(n-2)}}\Big|\frac{2(n-2)s}{8s+n-2}\Big||W|^2$$
$$< -\sqrt{\frac{2(n-2)}{n-1}}\frac{[3n-4+2n(n-1)t+8s]}{n(8s+n-2)}R|\mathring{\mathrm{R}}_{ij}|, \tag{1.2}$$

where t, s satisfy

$$\begin{cases} s \geq -\frac{1}{4} \\ n+4(n-1)t+4s \leq 0 \\ 3n-4+2n(n-1)t+8s < 0 \end{cases} \tag{1.3}$$

or

$$\begin{cases} s < -\frac{n-2}{8} \\ n+4(n-1)t+4s \geq 0 \\ 3n-4+2n(n-1)t+8s > 0. \end{cases} \tag{1.4}$$

Then M^n is Einstein.

Theorem 1.2. *Let M^n be a closed manifold of dimension $n \geq 5$ with positive scalar curvature and g be a critical metric for $\mathcal{F}_{t,s}$ on $\mathscr{M}_1(M^n)$. Suppose that*

$$\Big|W - \frac{(n-4)[4s+(n-2)]}{\sqrt{2n}(n-2)(8s+n-2)}\mathring{\mathrm{Ric}} \owedge g\Big||\mathring{\mathrm{R}}_{ij}| + \sqrt{\frac{2(n-1)^2}{n(n-2)}}\Big|\frac{2(n-2)s}{8s+n-2}\Big||W|^2$$
$$< \sqrt{\frac{2(n-2)}{n-1}}\frac{[3n-4+2n(n-1)t+8s]}{n(8s+n-2)}R|\mathring{\mathrm{R}}_{ij}|, \tag{1.5}$$

where t, s satisfy

$$\begin{cases} -\frac{n-2}{8} < s \leq -\frac{1}{4} \\ n+4(n-1)t+4s \geq 0 \\ 3n-4+2n(n-1)t+8s > 0. \end{cases} \tag{1.6}$$

Then M^n is Einstein.

Theorem 1.3. *Let M^n be a closed manifold of dimension $n \geq 5$ with positive scalar curvature and g be a critical metric for $\mathcal{F}_{t,s}$ on $\mathscr{M}_1(M^n)$, where $1+2t+2s=0$. Suppose that*

$$\Big|W+\frac{2s(n^2-3n+4)+2(n-2)}{\sqrt{2n}(n-2)[(n-2)+2ns]}\mathring{\mathrm{Ric}}\owedge g\Big||\mathring{\mathrm{R}}_{ij}|+\sqrt{\frac{2(n-1)^2}{n(n-2)}}\Big|\frac{(n-2)s}{(n-2)+2ns}\Big||W|^2$$
$$\leq\sqrt{\frac{2(n-2)}{n-1}}\frac{[n^2-3n+4+2(n^2+n-4)s]}{2n[(n-2)+2ns]}R|\mathring{\mathrm{R}}_{ij}|, \tag{1.7}$$

where s satisfy

$$s>-\frac{1}{4} \tag{1.8}$$

or

$$s<-\frac{n-2}{2n}. \tag{1.9}$$

Then M^n is Einstein as long as there exists a point such that the inequality in (1.7) *is strict.*

Theorem 1.4. *Let M^n be a closed manifold of dimension $n \geq 5$ with positive scalar curvature and g be a critical metric for $\mathcal{F}_{t,s}$ on $\mathscr{M}_1(M^n)$, where $1+2t+2s=0$. Suppose that*

$$\Big|W+\frac{2s(n^2-3n+4)+2(n-2)}{\sqrt{2n}(n-2)[(n-2)+2ns]}\mathring{\mathrm{Ric}}\owedge g\Big||\mathring{\mathrm{R}}_{ij}|+\sqrt{\frac{2(n-1)^2}{n(n-2)}}\Big|\frac{(n-2)s}{(n-2)+2ns}\Big||W|^2$$
$$\leq-\sqrt{\frac{2(n-2)}{n-1}}\frac{[n^2-3n+4+2(n^2+n-4)s]}{2n[(n-2)+2ns]}R|\mathring{\mathrm{R}}_{ij}|, \tag{1.10}$$

where s satisfy

$$-\frac{n-2}{2n}<s<-\frac{n^2-3n+4}{2(n^2+n-4)}. \tag{1.11}$$

Then M^n is Einstein as long as there exists a point such that the inequality in (1.10) *is strict.*

Theorem 1.5. *Let M^n be a locally conformally flat closed manifold of dimension $n \geq 4$ with positive scalar curvature and g be a critical metric for $\mathcal{F}_{t,s}$ on $\mathscr{M}_1(M^n)$.*

(1) *If $n=4$ and $3t+s+1\neq 0$, then M^4 is of positive constant sectional curvature;*

(2) *If $n\geq 5$ and t,s satisfy*

$$\begin{cases} s\geq -\frac{n-2}{4} \\ (n-1)(n-2)t+2s+(n-2)<0 \\ 2n(n-1)t+4(n-2)s+(n^2-3n+4)\leq 0 \end{cases} \tag{1.12}$$

or

$$\begin{cases} s \leq -\frac{n-2}{4} \\ (n-1)(n-2)t + 2s + (n-2) > 0 \\ 2n(n-1)t + 4(n-2)s + (n^2 - 3n + 4) \geq 0, \end{cases} \tag{1.13}$$

then M^n is of positive constant sectional curvature.

Taking $s = -\frac{n-2}{4}$ in (1.12) and (1.13), respectively, we obtain the following.

Corollary 1.6. *Let M^n be a locally conformally flat closed manifold of dimension $n \geq 5$ with positive scalar curvature and g be a critical metric for $\mathcal{F}_{t,s}$ on $\mathscr{M}_1(M^n)$. If $s = -\frac{n-2}{4}$ and $t \neq -\frac{1}{2(n-1)}$, then M^n is of positive constant sectional curvature.*

Next, we give some rigidity results for $n = 3$:

Theorem 1.7. *Let M^3 be a closed manifold with positive scalar curvature and g be a critical metric for $\mathcal{F}_{t,s}$ on $\mathscr{M}_1(M^3)$, where t, s satisfy*

$$\begin{cases} s \leq -\frac{1}{4} \\ 2t + 2s + 1 < 0 \\ 3t + s + 1 \leq 0 \end{cases} \tag{1.14}$$

or

$$\begin{cases} s \geq -\frac{1}{4} \\ 2t + 2s + 1 > 0 \\ 3t + s + 1 \geq 0. \end{cases} \tag{1.15}$$

Suppose that the divergence of Cotton tensor is zero (that is, $C_{ijk,i} = 0$). Then M^3 is of positive constant sectional curvature.

Taking $s = -\frac{1}{4}$ in (1.14) and (1.15), respectively, we obtain the following.

Corollary 1.8. *Let M^3 be a closed manifold with positive scalar curvature and g be a critical metric for $\mathcal{F}_{t,s}$ on $\mathscr{M}_1(M^3)$, where $s = -\frac{1}{4}$ and $t \neq -\frac{1}{4}$. Then M^3 is of positive constant sectional curvature provided that the divergence of Cotton tensor is zero.*

Remark 1.9. In particular, when $n = 3$, we have $W = 0$ automatically. Hence it is seen from (2.1) that an Einstein manifold M^3 with positive scalar curvature must be of positive constant sectional curvature.

Remark 1.10. When $s = 0$, it is easy to check that our Theorems 1.1 and 1.3 become Theorems 1.1 and 1.3 of [21], respectively.

Remark 1.11. For $n \geq 4$, the Bach tensor is defined (see [4, 16]) by

$$B_{ij} = \frac{1}{n-3} W_{ikjl,lk} + \frac{1}{n-2} W_{ikjl} R^{kl}. \tag{1.16}$$

By virtue of (2.3), we have that (1.16) can be written as

$$B_{ij} = \frac{1}{n-2}(C_{kij,k} + W_{ikjl}R^{kl}). \tag{1.17}$$

Therefore, we can define the Bach tensor on M^3 by

$$B_{ij} = C_{kij,k}. \tag{1.18}$$

Thus, when $n = 3$, $C_{ijk,i} = 0$ is equivalent to $B_{ij} = 0$. In [22], Sheng and Wang studied the case that the critical metrics are Bach-flat (that is, $B_{ij} = 0$). Our Theorem 1.7 generalizes partially the results of Sheng and Wang in [22].

Remark 1.12. Corollary 1.6 is exactly Theorem 4 with positive scalar curvature of Barros and Da Silva [2]. Therefore, our Theorem 1.5 generalizes Theorem 4 of Barros and Da Silva [2]. Moreover, for $n = 3, 4$, our Theorem 1.5 and Corollary 1.8 can be seen as a supplement to Theorem 4 of Barros and Da Silva in [2].

Remark 1.13. By the definition of the Cotton tensor given by (2.2), we have

$$\int_M C_{ijk,i}R_{jk} = -\int_M C_{ijk}R_{jk,i} = -\frac{1}{2}\int_M |C_{ijk}|^2, \tag{1.19}$$

which shows that if $C_{ijk,i} = 0$, then we have $C_{ijk} = 0$. Therefore, if we add the assumption on $C_{ijk,i} = 0$ for Theorem 1.1, then M^n is Einstein as long as we replace (1.3) with

$$\begin{cases} s > -\frac{n-2}{8} \\ n + 4(n-1)t + 4s \leq 0 \\ 3n - 4 + 2n(n-1)t + 8s < 0. \end{cases} \tag{1.20}$$

Similarly, if we add the assumption on $C_{ijk,i} = 0$ for Theorem 1.2, then M^n is Einstein as long as we replace (1.6) with

$$\begin{cases} s < -\frac{n-2}{8} \\ n + 4(n-1)t + 4s \leq 0 \\ 3n - 4 + 2n(n-1)t + 8s < 0; \end{cases} \tag{1.21}$$

or

$$\begin{cases} s > -\frac{n-2}{8} \\ n + 4(n-1)t + 4s \geq 0 \\ 3n - 4 + 2n(n-1)t + 8s > 0. \end{cases} \tag{1.22}$$

Acknowledgment. We would like to thank the referee for suggestions which make the paper more readable.

2. Preliminaries

For $n \geq 3$, it is well-known that the Weyl curvature tensor and the Cotton tensor are defined by

$$
\begin{aligned}
W_{ijkl} &= R_{ijkl} - \frac{1}{n-2}(R_{ik}g_{jl} - R_{il}g_{jk} + R_{jl}g_{ik} - R_{jk}g_{il}) \\
&\quad + \frac{R}{(n-1)(n-2)}(g_{ik}g_{jl} - g_{il}g_{jk}) \\
&= R_{ijkl} - \frac{1}{n-2}(\mathring{R}_{ik}g_{jl} - \mathring{R}_{il}g_{jk} + \mathring{R}_{jl}g_{ik} - \mathring{R}_{jk}g_{il}) \\
&\quad - \frac{R}{n(n-1)}(g_{ik}g_{jl} - g_{il}g_{jk}),
\end{aligned} \tag{2.1}
$$

and

$$
\begin{aligned}
C_{ijk} &= R_{kj,i} - R_{ki,j} - \frac{1}{2(n-1)}(R_{,i}g_{jk} - R_{,j}g_{ik}) \\
&= \mathring{R}_{kj,i} - \mathring{R}_{ki,j} + \frac{n-2}{2n(n-1)}(R_{,i}g_{jk} - R_{,j}g_{ik}),
\end{aligned} \tag{2.2}
$$

respectively. Here $\mathring{R}_{ij} = R_{ij} - \frac{1}{n}Rg_{ij}$ denotes the traceless Ricci tensor and the indices after a comma denote the covariant derivatives. From the definition of the Cotton tensor, it is easy to see

$$C_{ijk} = -C_{jik}, \qquad g^{ij}C_{ijk} = g^{ik}C_{ijk} = g^{jk}C_{ijk} = 0$$

and

$$C_{ijk,k} = 0, \qquad C_{ijk} + C_{jki} + C_{kij} = 0.$$

For $n \geq 4$, the divergence of the Weyl curvature tensor is related to the Cotton tensor by

$$-\frac{n-3}{n-2}C_{ijk} = W_{ijkl,l}. \tag{2.3}$$

Moreover, $W_{ijkl} = 0$ holds naturally on (M^3, g), and (M^3, g) is locally conformally flat if and only if $C_{ijk} = 0$. For $n \geq 4$, (M^n, g) is locally conformally flat if and only if $W_{ijkl} = 0$.

It has been proved by Catino in [6] (see [6, Proposition 6.1]) that a metric g is critical for $\mathcal{F}_{t,s}$ on $\mathscr{M}_1(M^n)$ if and only if it satisfies the equations

$$
\begin{aligned}
(1+4s)\Delta\mathring{R}_{ij} &= (1+2t+2s)R_{,ij} - \frac{1+2t+2s}{n}(\Delta R)g_{ij} - 2(1+2s)R_{ikjl}\mathring{R}_{kl} \\
&\quad - \frac{2+2nt-4s}{n}R\mathring{R}_{ij} + \frac{2}{n}(|\mathring{R}_{ij}|^2 + s|\mathrm{Rm}|^2)g_{ij} \\
&\quad - 2sR_{ikpq}R_{jkpq} + 4s\mathring{R}_{ik}\mathring{R}_{jk}
\end{aligned} \tag{2.4}
$$

and

$$[n+4(n-1)t+4s]\Delta R = (n-4)(|R_{ij}|^2 + tR^2 + s|\mathrm{Rm}|^2 - \lambda)$$

$$
\begin{aligned}
&= (n-4)\Big[s|W|^2 + \frac{n-2+4s}{n-2}|\mathring{\mathrm{R}}_{ij}|^2 \\
&\quad + \frac{n-1+n(n-1)t+2s}{n(n-1)}R^2 - \lambda\Big],
\end{aligned}
\tag{2.5}
$$

where $\lambda = \mathcal{F}_{t,s}(g)$ and we used the fact

$$
|\mathrm{Rm}|^2 = |W|^2 + \frac{4}{n-2}|\mathring{\mathrm{R}}_{ij}|^2 + \frac{2}{n(n-1)}R^2 \tag{2.6}
$$

from (2.1).

Using the formula (2.1), we can also derive

$$
\mathring{\mathrm{R}}_{kl}R_{ikjl} = \mathring{\mathrm{R}}_{kl}W_{ikjl} + \frac{1}{n-2}(|\mathring{\mathrm{R}}_{ij}|^2 g_{ij} - 2\mathring{\mathrm{R}}_{ik}\mathring{\mathrm{R}}_{jk}) - \frac{1}{n(n-1)}R\mathring{\mathrm{R}}_{ij} \tag{2.7}
$$

and

$$
\begin{aligned}
R_{ikpq}R_{jkpq} &= W_{ikpq}W_{jkpq} + \frac{4}{n-2}W_{ikjl}\mathring{\mathrm{R}}_{kl} + \frac{2(n-4)}{(n-2)^2}\mathring{\mathrm{R}}_{ik}\mathring{\mathrm{R}}_{jk} \\
&\quad + \frac{2}{(n-2)^2}|\mathring{\mathrm{R}}_{ij}|^2 g_{ij} + \frac{2}{n^2(n-1)}R^2 g_{ij} + \frac{4}{n(n-1)}R\mathring{\mathrm{R}}_{ij}.
\end{aligned}
\tag{2.8}
$$

Therefore, (2.4) can be written as

$$
\begin{aligned}
(1+4s)\Delta\mathring{\mathrm{R}}_{ij} &= (1+2t+2s)\mathring{R}_{,ij} - 2(1+2s)R_{ikjl}\mathring{\mathrm{R}}_{kl} \\
&\quad - \frac{2+2nt-4s}{n}R\mathring{\mathrm{R}}_{ij} + \frac{2}{n}(|\mathring{\mathrm{R}}_{ij}|^2 + s|\mathrm{Rm}|^2)g_{ij} \\
&\quad - 2sR_{ikpq}R_{jkpq} + 4s\mathring{\mathrm{R}}_{ik}\mathring{\mathrm{R}}_{jk} \\
&= (1+2t+2s)\mathring{R}_{,ij} - \frac{2(n-2)+4ns}{n-2}W_{ikjl}\mathring{\mathrm{R}}_{kl} - 2sW_{ikpq}W_{jkpq} \\
&\quad + \Big[-\frac{4s(n^2-3n+4)+4(n-2)}{n(n-2)^2}|\mathring{\mathrm{R}}_{ij}|^2 + \frac{2s}{n}|W|^2\Big]g_{ij} \\
&\quad + \frac{4s(n^2-3n+4)+4(n-2)}{(n-2)^2}\mathring{\mathrm{R}}_{ik}\mathring{\mathrm{R}}_{jk} \\
&\quad + \frac{4-2n-2n(n-1)t+4(n-2)s}{n(n-1)}R\mathring{\mathrm{R}}_{ij},
\end{aligned}
\tag{2.9}
$$

where $\mathring{R}_{,ij} = R_{,ij} - \frac{1}{n}(\Delta R)g_{ij}$. It follows from (2.9) that

$$
\begin{aligned}
\frac{1+4s}{2}\Delta|\mathring{\mathrm{R}}_{ij}|^2 &= (1+4s)|\nabla\mathring{\mathrm{R}}_{ij}|^2 + (1+4s)\mathring{\mathrm{R}}_{ij}\Delta\mathring{\mathrm{R}}_{ij} \\
&= (1+4s)|\nabla\mathring{\mathrm{R}}_{ij}|^2 + (1+2t+2s)R_{,ij}\mathring{\mathrm{R}}_{ij} \\
&\quad - \frac{2(n-2)+4ns}{n-2}W_{ikjl}\mathring{\mathrm{R}}_{kl}\mathring{\mathrm{R}}_{ij} - 2sW_{ikpq}W_{jkpq}\mathring{\mathrm{R}}_{ij} \\
&\quad + \frac{4s(n^2-3n+4)+4(n-2)}{(n-2)^2}\mathring{\mathrm{R}}_{ik}\mathring{\mathrm{R}}_{kj}\mathring{\mathrm{R}}_{ji}
\end{aligned}
$$

$$+\frac{4-2n-2n(n-1)t+4(n-2)s}{n(n-1)}R|\mathring{R}_{ij}|^2. \tag{2.10}$$

Integrating both sides of (2.10) yields

$$\begin{aligned}
0=(1+4s)\int_M|\nabla\mathring{R}_{ij}|^2+\int_M\Big(&-\frac{(n-2)(1+2t+2s)}{2n}|\nabla R|^2\\
&-\frac{2(n-2)+4ns}{n-2}W_{ikjl}\mathring{R}_{kl}\mathring{R}_{ij}-2sW_{ikpq}W_{jkpq}\mathring{R}_{ij}\\
&+\frac{4s(n^2-3n+4)+4(n-2)}{(n-2)^2}\mathring{R}_{ik}\mathring{R}_{kj}\mathring{R}_{ji}\\
&+\frac{4-2n-2n(n-1)t+4(n-2)s}{n(n-1)}R|\mathring{R}_{ij}|^2\Big),
\end{aligned}\tag{2.11}$$

where we used the second Bianchi identity $\mathring{R}_{kj,k}=\frac{n-2}{2n}R_{,j}$. Hence, we obtain the following result:

Lemma 2.1. *Let M^n be a closed manifold and g be a critical metric for $\mathcal{F}_{t,s}$ on $\mathscr{M}_1(M^n)$. Then*

$$\begin{aligned}
(1+4s)\int_M|\nabla\mathring{R}_{ij}|^2=\int_M\Big(&\frac{(n-2)(1+2t+2s)}{2n}|\nabla R|^2\\
&+\frac{2(n-2)+4ns}{n-2}W_{ikjl}\mathring{R}_{kl}\mathring{R}_{ij}+2sW_{ikpq}W_{jkpq}\mathring{R}_{ij}\\
&-\frac{4s(n^2-3n+4)+4(n-2)}{(n-2)^2}\mathring{R}_{ik}\mathring{R}_{kj}\mathring{R}_{ji}\\
&-\frac{4-2n-2n(n-1)t+4(n-2)s}{n(n-1)}R|\mathring{R}_{ij}|^2\Big).
\end{aligned}\tag{2.12}$$

For any closed manifold, we also have the following result (see [21, Lemma 2.2])

Lemma 2.2. *Let M^n be a closed manifold. Then*

$$\begin{aligned}
\int_M|\nabla\mathring{R}_{ij}|^2=\int_M\Big(&W_{ijkl}\mathring{R}_{jl}\mathring{R}_{ik}-\frac{n}{n-2}\mathring{R}_{ij}\mathring{R}_{jk}\mathring{R}_{ki}\\
&-\frac{1}{n-1}R|\mathring{R}_{ij}|^2+\frac{(n-2)^2}{4n(n-1)}|\nabla R|^2+\frac12|C_{ijk}|^2\Big).
\end{aligned}\tag{2.13}$$

The next lemma comes from [8,14,20] (for the case of $\lambda=\frac{2}{n-2}$, see [7]):

Lemma 2.3. *For every Riemannian manifold (M^n,g) and any $\lambda\in\mathbb{R}$, the following estimate holds*

$$\begin{aligned}
&\Big|-W_{ijkl}\mathring{R}_{jl}\mathring{R}_{ik}+\lambda\mathring{R}_{ij}\mathring{R}_{jk}\mathring{R}_{ki}\Big|\\
\le&\sqrt{\frac{n-2}{2(n-1)}}\Big(|W|^2+\frac{2(n-2)\lambda^2}{n}|\mathring{R}_{ij}|^2\Big)^{\frac12}|\mathring{R}_{ij}|^2
\end{aligned}$$

$$= \sqrt{\frac{n-2}{2(n-1)}}\Big|W + \frac{\lambda}{\sqrt{2n}}\mathring{\mathrm{Ric}} \owedge g\Big||\mathring{R}_{ij}|^2. \tag{2.14}$$

3. Proof of main results

3.1. Proof of Theorem 1.1

Notice that (2.13) can be written as

$$\begin{aligned}(1+4s)\int_M |\nabla \mathring{R}_{ij}|^2 = (1+4s)\int_M \Big(&W_{ijkl}\mathring{R}_{jl}\mathring{R}_{ik} - \frac{n}{n-2}\mathring{R}_{ij}\mathring{R}_{jk}\mathring{R}_{ki} \\ &- \frac{1}{n-1}R|\mathring{R}_{ij}|^2 + \frac{(n-2)^2}{4n(n-1)}|\nabla R|^2 + \frac{1}{2}|C_{ijk}|^2\Big).\end{aligned} \tag{3.1}$$

Combining (3.1) with (2.12), we have

$$\begin{aligned}0 = \int_M \Big[&\frac{n-2+8s}{n-2}W_{ijkl}\mathring{R}_{jl}\mathring{R}_{ik} + \frac{(n-4)[4s+(n-2)]}{(n-2)^2}\mathring{R}_{ij}\mathring{R}_{jk}\mathring{R}_{ki} \\ &+ 2sW_{ikpq}W_{jkpq}\mathring{R}_{ij} + \frac{3n-4+2n(n-1)t+8s}{n(n-1)}R|\mathring{R}_{ij}|^2 \\ &+ \frac{(n-2)[n+4(n-1)t+4s]}{4n(n-1)}|\nabla R|^2 - \frac{1+4s}{2}|C_{ijk}|^2\Big],\end{aligned} \tag{3.2}$$

which is equivalent to

$$\begin{aligned}0 = \int_M \Big[&-W_{ijkl}\mathring{R}_{jl}\mathring{R}_{ik} - \frac{(n-4)[4s+(n-2)]}{(n-2)(8s+n-2)}\mathring{R}_{ij}\mathring{R}_{jk}\mathring{R}_{ki} - \frac{2(n-2)s}{8s+n-2} \\ &\times W_{ikpq}W_{jkpq}\mathring{R}_{ij} - \frac{(n-2)[3n-4+2n(n-1)t+8s]}{n(n-1)(8s+n-2)}R|\mathring{R}_{ij}|^2 \\ &- \frac{(n-2)^2[n+4(n-1)t+4s]}{4n(n-1)(8s+n-2)}|\nabla R|^2 + \frac{(n-2)(1+4s)}{2(8s+n-2)}|C_{ijk}|^2\Big]\end{aligned} \tag{3.3}$$

as long as $8s+n-2 \neq 0$. Substituting the estimate (2.14) with

$$\lambda = -\frac{(n-4)[4s+(n-2)]}{(n-2)(8s+n-2)}$$

and

$$|W_{ikpq}W_{jkpq}\mathring{R}_{ij}| \leq \sqrt{\frac{n-1}{n}}|W|^2|\mathring{R}_{ij}| \tag{3.4}$$

into (3.3) gives

$$\begin{aligned}0 \geq \int_M \Bigg[&-\sqrt{\frac{n-2}{2(n-1)}}\Big|W - \frac{(n-4)[4s+(n-2)]}{\sqrt{2n}(n-2)(8s+n-2)}\mathring{\mathrm{Ric}} \owedge g\Big||\mathring{R}_{ij}|^2 \\ &- \sqrt{\frac{n-1}{n}}\Big|\frac{2(n-2)s}{8s+n-2}\Big||W|^2|\mathring{R}_{ij}| - \frac{(n-2)[3n-4+2n(n-1)t+8s]}{n(n-1)(8s+n-2)}\end{aligned}$$

$$\times R|\mathring{\mathrm{R}}_{ij}|^2 - \frac{(n-2)^2[n+4(n-1)t+4s]}{4n(n-1)(8s+n-2)}|\nabla R|^2$$

$$+ \frac{(n-2)(1+4s)}{2(8s+n-2)}|C_{ijk}|^2\Bigg]. \tag{3.5}$$

For the proof of (3.4), we refer to [17, Lemma 2.4]. Noticing that if t, s satisfy (1.3), then we have

$$\begin{cases} 1+4s \geq 0 \\ 8s+n-2 > 0 \\ n+4(n-1)t+4s \leq 0 \\ 3n-4+2n(n-1)t+8s < 0. \end{cases} \tag{3.6}$$

Therefore, applying (1.2) and (3.6) into (3.5) gives

$$0 \geq \int_M \Bigg[-\sqrt{\frac{n-2}{2(n-1)}}\Big|W - \frac{(n-4)[4s+(n-2)]}{\sqrt{2n}(n-2)(8s+n-2)}\mathring{\mathrm{Ric}} \owedge g\Big||\mathring{\mathrm{R}}_{ij}|^2$$

$$- \sqrt{\frac{n-1}{n}}\Big|\frac{2(n-2)s}{8s+n-2}\Big||W|^2|\mathring{\mathrm{R}}_{ij}| - \frac{(n-2)[3n-4+2n(n-1)t+8s]}{n(n-1)(8s+n-2)}$$

$$\times R|\mathring{\mathrm{R}}_{ij}|^2 - \frac{(n-2)^2[n+4(n-1)t+4s]}{4n(n-1)(8s+n-2)}|\nabla R|^2$$

$$+ \frac{(n-2)(1+4s)}{2(8s+n-2)}|C_{ijk}|^2\Bigg] \geq 0, \tag{3.7}$$

which shows $\mathring{\mathrm{R}}_{ij} = 0$ and hence M^n is Einstein.

Similarly, if t, s satisfy (1.4), then we have

$$\begin{cases} 1+4s \leq 0 \\ 8s+n-2 < 0 \\ n+4(n-1)t+4s \geq 0 \\ 3n-4+2n(n-1)t+8s > 0. \end{cases} \tag{3.8}$$

Therefore, applying (1.2) and (3.8) into (3.5) also yields the estimate (3.7) and the desired Theorem 1.1 follows.

3.2. Proof of Theorem 1.2

When $8s+n-2 \neq 0$, inserting the estimate (2.14) with $\lambda = -\frac{(n-4)[4s+(n-2)]}{(n-2)(8s+n-2)}$ and (3.4) into (3.3), we deduce

$$0 \leq \int_M \Bigg[\sqrt{\frac{n-2}{2(n-1)}}\Big|W - \frac{(n-4)[4s+(n-2)]}{\sqrt{2n}(n-2)(8s+n-2)}\mathring{\mathrm{Ric}} \owedge g\Big||\mathring{\mathrm{R}}_{ij}|^2$$

$$+ \sqrt{\frac{n-1}{n}}\Big|\frac{2(n-2)s}{8s+n-2}\Big||W|^2|\mathring{\mathrm{R}}_{ij}| - \frac{(n-2)[3n-4+2n(n-1)t+8s]}{n(n-1)(8s+n-2)}$$

$$\times R|\mathring{\mathrm{R}}_{ij}|^2 - \frac{(n-2)^2[n+4(n-1)t+4s]}{4n(n-1)(8s+n-2)}|\nabla R|^2$$

$$+ \frac{(n-2)(1+4s)}{2(8s+n-2)}|C_{ijk}|^2\Bigg]. \tag{3.9}$$

If t, s satisfy (1.6), then we have

$$\begin{cases} 1+4s \le 0 \\ 8s+n-2 > 0 \\ n+4(n-1)t+4s \ge 0 \\ 3n-4+2n(n-1)t+8s > 0. \end{cases} \tag{3.10}$$

Applying (3.10) and (1.5) into (3.9) also yields the following estimate

$$0 \le \int_M \Bigg[\sqrt{\frac{n-2}{2(n-1)}}\Big|W - \frac{(n-4)[4s+(n-2)]}{\sqrt{2n}(n-2)(8s+n-2)}\mathring{\mathrm{Ric}} \owedge g\Big||\mathring{\mathrm{R}}_{ij}|^2$$

$$+ \sqrt{\frac{n-1}{n}}\Big|\frac{2(n-2)s}{8s+n-2}\Big||W|^2|\mathring{\mathrm{R}}_{ij}| - \frac{(n-2)[3n-4+2n(n-1)t+8s]}{n(n-1)(8s+n-2)}$$

$$\times R|\mathring{\mathrm{R}}_{ij}|^2 - \frac{(n-2)^2[n+4(n-1)t+4s]}{4n(n-1)(8s+n-2)}|\nabla R|^2$$

$$+ \frac{(n-2)(1+4s)}{2(8s+n-2)}|C_{ijk}|^2\Bigg] \le 0, \tag{3.11}$$

and the desired Theorem 1.2 follows.

3.3. Proof of Theorem 1.3

When t, s satisfy $1+2t+2s=0$, then the formula (2.10) becomes

$$\frac{1+4s}{2}\Delta|\mathring{\mathrm{R}}_{ij}|^2 = (1+4s)|\nabla\mathring{\mathrm{R}}_{ij}|^2 - \frac{2(n-2)+4ns}{n-2}W_{ikjl}\mathring{\mathrm{R}}_{kl}\mathring{\mathrm{R}}_{ij}$$

$$- 2sW_{ikpq}W_{jkpq}\mathring{\mathrm{R}}_{ij} + \frac{4s(n^2-3n+4)+4(n-2)}{(n-2)^2}\mathring{\mathrm{R}}_{ik}\mathring{\mathrm{R}}_{kj}\mathring{\mathrm{R}}_{ji}$$

$$+ \frac{4-2n-2n(n-1)t+4(n-2)s}{n(n-1)}R|\mathring{\mathrm{R}}_{ij}|^2, \tag{3.12}$$

which gives

$$\frac{(n-2)(1+4s)}{4[(n-2)+2ns]}\Delta|\mathring{\mathrm{R}}_{ij}|^2$$

$$= \frac{(n-2)(1+4s)}{2[(n-2)+2ns]}|\nabla\mathring{\mathrm{R}}_{ij}|^2 - W_{ikjl}\mathring{\mathrm{R}}_{kl}\mathring{\mathrm{R}}_{ij} - \frac{(n-2)s}{(n-2)+2ns}W_{ikpq}W_{jkpq}\mathring{\mathrm{R}}_{ij}$$

$$+ \frac{2s(n^2-3n+4)+2(n-2)}{(n-2)[(n-2)+2ns]}\mathring{\mathrm{R}}_{ik}\mathring{\mathrm{R}}_{kj}\mathring{\mathrm{R}}_{ji}$$

$$+\frac{(n-2)[2-n-n(n-1)t+2(n-2)s]}{n(n-1)[(n-2)+2ns]}R|\mathring{\mathrm{R}}_{ij}|^2$$

$$\begin{aligned}\geq{}& \frac{(n-2)(1+4s)}{2[(n-2)+2ns]}|\nabla\mathring{\mathrm{R}}_{ij}|^2-\sqrt{\frac{n-2}{2(n-1)}}\Big|W+\frac{2s(n^2-3n+4)+2(n-2)}{\sqrt{2n}(n-2)[(n-2)+2ns]}\\
&\times\mathring{\mathrm{Ric}}\owedge g\Big||\mathring{\mathrm{R}}_{ij}|^2-\sqrt{\frac{n-1}{n}}\Big|\frac{(n-2)s}{(n-2)+2ns}\Big||W|^2|\mathring{\mathrm{R}}_{ij}|\\
&+\frac{(n-2)[2-n-n(n-1)t+2(n-2)s]}{n(n-1)[(n-2)+2ns]}R|\mathring{\mathrm{R}}_{ij}|^2\end{aligned}\tag{3.13}$$

provided $n-2+2ns\neq 0$. Noticing that

$$-\frac{n-2}{2n}<-\frac{n^2-3n+4}{2(n^2+n-4)}<-\frac{1}{4}$$

and if s satisfy (1.8), then we have

$$\begin{cases}1+4s>0\\ n-2+2ns>0\\ 2-n-n(n-1)t+2n(n-2)s>0.\end{cases}\tag{3.14}$$

Similarly, if t,s satisfy (1.9), then also we have

$$\begin{cases}1+4s<0\\ n-2+2ns<0\\ 2-n-n(n-1)t+2n(n-2)s<0.\end{cases}\tag{3.15}$$

Clearly, if (3.14) or (3.15) holds, then from (3.13) and (1.7) we both have

$$\frac{(n-2)(1+4s)}{4[(n-2)+2ns]}\Delta|\mathring{\mathrm{R}}_{ij}|^2\geq 0,\tag{3.16}$$

which shows that $|\mathring{\mathrm{R}}_{ij}|^2$ is subharmonic on M^n. Using the maximum principle, we obtain that $|\mathring{\mathrm{R}}_{ij}|^2$ is constant and $\nabla\mathring{\mathrm{R}}_{ij}=0$, implying that the Ricci curvature is parallel and the scalar curvature R is constant. In particular, (3.13) becomes

$$\begin{aligned}0=\Bigg[&-\sqrt{\frac{n-2}{2(n-1)}}\Big|W+\frac{2s(n^2-3n+4)+2(n-2)}{\sqrt{2n}(n-2)[(n-2)+2ns]}\\
&\times\mathring{\mathrm{Ric}}\owedge g\Big||\mathring{\mathrm{R}}_{ij}|-\sqrt{\frac{n-1}{n}}\Big|\frac{(n-2)s}{(n-2)+2ns}\Big||W|^2\\
&+\frac{(n-2)[2-n-n(n-1)t+2(n-2)s]}{n(n-1)[(n-2)+2ns]}R|\mathring{\mathrm{R}}_{ij}|\Bigg]|\mathring{\mathrm{R}}_{ij}|.\end{aligned}\tag{3.17}$$

If there exists a point p such that the inequality (1.7) is strict, then from (3.17) we have $|\mathring{\mathrm{R}}_{ij}|(p)=0$ which with the fact that $|\mathring{\mathrm{R}}_{ij}|$ constant shows that $\mathring{\mathrm{R}}_{ij}=0$, that is, M^n is Einstein, completing the proof of Theorem 1.3.

3.4. Proof of Theorem 1.4

When $1 + 2t + 2s = 0$, (3.12) can also be written as

$$\begin{aligned} -\frac{(n-2)(1+4s)}{4[(n-2)+2ns]}\Delta|\mathring{R}_{ij}|^2 = & -\frac{(n-2)(1+4s)}{2[(n-2)+2ns]}|\nabla\mathring{R}_{ij}|^2 + W_{ikjl}\mathring{R}_{kl}\mathring{R}_{ij} \\ & + \frac{(n-2)s}{(n-2)+2ns}W_{ikpq}W_{jkpq}\mathring{R}_{ij} \\ & - \frac{2s(n^2-3n+4)+2(n-2)}{(n-2)[(n-2)+2ns]}\mathring{R}_{ik}\mathring{R}_{kj}\mathring{R}_{ji} \\ & - \frac{(n-2)[2-n-n(n-1)t+2(n-2)s]}{n(n-1)[(n-2)+2ns]}R|\mathring{R}_{ij}|^2. \end{aligned} \tag{3.18}$$

Thus, we obtain

$$\begin{aligned} & -\frac{(n-2)(1+4s)}{4[(n-2)+2ns]}\Delta|\mathring{R}_{ij}|^2 \\ \geq & -\frac{(n-2)(1+4s)}{2[(n-2)+2ns]}|\nabla\mathring{R}_{ij}|^2 - \sqrt{\frac{n-2}{2(n-1)}}\Big|W + \frac{2s(n^2-3n+4)+2(n-2)}{\sqrt{2n}(n-2)[(n-2)+2ns]} \\ & \times \mathring{\mathrm{Ric}} \owedge g\Big||\mathring{R}_{ij}|^2 - \sqrt{\frac{n-1}{n}}\Big|\frac{(n-2)s}{(n-2)+2ns}\Big||W|^2|\mathring{R}_{ij}| \\ & - \frac{(n-2)[2-n-n(n-1)t+2(n-2)s]}{n(n-1)[(n-2)+2ns]}R|\mathring{R}_{ij}|^2. \end{aligned} \tag{3.19}$$

If t, s satisfy (1.11), then we have

$$\begin{cases} 1+4s<0 \\ n-2+2ns>0 \\ 2-n-n(n-1)t+2(n-2)s<0. \end{cases} \tag{3.20}$$

Therefore, applying (3.20) and (1.10) into (3.19) yields the estimate

$$-\frac{(n-2)(1+4s)}{4[(n-2)+2ns]}\Delta|\mathring{R}_{ij}|^2 \geq 0. \tag{3.21}$$

Then following the proof of Theorem 1.3 line by line we finish the proof of Theorem 1.4.

3.5. Proof of Theorem 1.5

When $W_{ijkl} = 0$, (2.12) becomes

$$\begin{aligned} (1+4s)\int_M |\nabla\mathring{R}_{ij}|^2 = & \int_M \Big(\frac{(n-2)(1+2t+2s)}{2n}|\nabla R|^2 \\ & - \frac{4s(n^2-3n+4)+4(n-2)}{(n-2)^2}\mathring{R}_{ik}\mathring{R}_{kj}\mathring{R}_{ji} \end{aligned}$$

$$- \frac{4-2n-2n(n-1)t+4(n-2)s}{n(n-1)} R|\mathring{R}_{ij}|^2 \Big) \tag{3.22}$$

and (2.13) becomes

$$\begin{aligned} \int_M |\nabla \mathring{R}_{ij}|^2 = \int_M \Big(&- \frac{n}{n-2} \mathring{R}_{ij}\mathring{R}_{jk}\mathring{R}_{ki} - \frac{1}{n-1} R|\mathring{R}_{ij}|^2 \\ &+ \frac{(n-2)^2}{4n(n-1)} |\nabla R|^2 + \frac{1}{2}|C_{ijk}|^2 \Big), \end{aligned} \tag{3.23}$$

respectively. Thus, combining (3.22) with (3.23), we obtain

$$\begin{aligned} 0 = &\frac{(n-4)[(n-2)+4s]}{n-2} \int_M |\nabla \mathring{R}_{ij}|^2 \\ &- \frac{2n[(n-1)(n-2)t+2s+(n-2)]}{(n-1)(n-2)} \int_M R|\mathring{R}_{ij}|^2 \\ &- \frac{(n-2)[2n(n-1)t+4(n-2)s+(n^2-3n+4)]}{2n(n-1)} \int_M |\nabla R|^2 \\ &+ \frac{2[(n-2)+(n^2-3n+4)s]}{n-2} \int_M |C_{ijk}|^2. \end{aligned} \tag{3.24}$$

For $n \geq 4$, from (2.3) we have $C_{ijk} = 0$ coming from $W_{ijkl} = 0$. In particular, when $n = 4$, (3.24) becomes

$$0 = (3t+s+1) \int_M (4R|\mathring{R}_{ij}|^2 + |\nabla R|^2), \tag{3.25}$$

which shows that if $3t + s + 1 \neq 0$, then we have $\mathring{R}_{ij} = 0$ and hence M^4 is Einstein. This combining with (2.1) gives that M^4 is of positive constant sectional curvature.

When $n \geq 5$, if t, s satisfy (1.12), then (3.24) yields

$$\begin{aligned} 0 = &\frac{(n-4)[(n-2)+4s]}{n-2} \int_M |\nabla \mathring{R}_{ij}|^2 \\ &- \frac{2n[(n-1)(n-2)t+2s+(n-2)]}{(n-1)(n-2)} \int_M R|\mathring{R}_{ij}|^2 \\ &- \frac{(n-2)[2n(n-1)t+4(n-2)s+(n^2-3n+4)]}{2n(n-1)} \int_M |\nabla R|^2 \geq 0, \end{aligned} \tag{3.26}$$

which concludes that M^n is Einstein. Similarly, if (1.13) is satisfied, we also have that M^n is Einstein and hence M^n is of positive constant sectional curvature.

3.6. Proof of Theorem 1.7

When $n = 3$, (3.24) becomes

$$0 = (1+4s) \int_M |\nabla \mathring{R}_{ij}|^2 + 3(2t+2s+1) \int_M R|\mathring{R}_{ij}|^2$$

$$+\frac{3t+s+1}{3}\int_M |\nabla R|^2 - 2(1+4s)\int_M |C_{ijk}|^2. \tag{3.27}$$

If $C_{ijk,i}=0$, then (1.19) shows that $C_{ijk}=0$ and (3.27) becomes

$$\begin{aligned} 0 = (1+4s)\int_M |\nabla \mathring{\mathrm{R}}_{ij}|^2 + 3(2t+2s+1)\int_M R|\mathring{\mathrm{R}}_{ij}|^2 \\ + \frac{3t+s+1}{3}\int_M |\nabla R|^2. \end{aligned} \tag{3.28}$$

Therefore, if t, s satisfy (1.14) or (1.15), we have $\mathring{\mathrm{R}}_{ij}=0$ and hence M^3 is of positive constant sectional curvature. The proof of Theorem 1.7 is finished.

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