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A NOTE ON w-GD DOMAINS

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Abstract. Let S and T be w-linked extension domains of a domain R with $S \subseteq T$. In this paper, we define what satisfying the w_R-GD property for $S \subseteq T$ means and what being w_R- or w-GD domains for T means. Then some sufficient conditions are given for the w_R-GD property and w_R-GD domains. For example, if T is w_R-integral over S and S is integrally closed, then the w_R-GD property holds. It is also given that S is a w_R-GD domain if and only if $S \subseteq T$ satisfies the w_R-GD property for each w_R-linked valuation overring T of S, if and only if $S \subseteq (S[u])_w$ satisfies the w_R-GD property for each element u in the quotient field of S, if and only if $S_\mathfrak{m}$ is a GD domain for each maximal w_R-ideal $\mathfrak{m}$ of S. Then we focus on discussing the relationship among GD domains, w-GD domains, w_R-GD domains, Prüfer domains, PvMDs and Pw_RMDs, and also provide some relevant counterexamples. As an application, we give a new characterization of Pw_RMDs. We show that S is a Pw_RMD if and only if S is a w_R-GD domain and every w_R-linked overring of S that satisfies the w_R-GD property is w_R-flat over S. Furthermore, examples are provided to show these two conditions are necessary for Pw_RMDs.

1. Introduction

In this paper, we assume that R is an integral domain with quotient field K. An overring of R means a subring of K containing R. In 1974, Dobbs ([6]) introduced the notion of GD domains, i.e., an integral domain R is called a *GD domain* if $R \subseteq T$ satisfies the going-down (GD for short) property for each overring T of R. In 1976, Dobbs proved that R is a GD domain if and only if $R \subseteq T$ satisfies the GD property for each integral domain T containing R ([8, Theorem 1]). Examples of GD domains are Prüfer domains and arbitrary domains of Krull dimension 1. GD has been figured prominently in the characterization of several kinds of domains. For example, R is a Bézout domain if and only if R is a GCD and $R \subseteq R[u]$ satisfies GD for all $u \in K$ ([4, Corollary 4.3]). And R is Prüfer if and only if R is an integrally closed

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FC domain (i.e., domains for which every intersection of two principal ideals is finitely generated) and $R \subseteq R[u]$ satisfies GD for all $u \in K$ ([5, Corollary 4]).

Since (semi)star operations $*$ on domains were introduced several decades ago, many researchers have been studying the $*$-version of classical theorems on domains. In 2009, Dobbs and Sahandi ([9]) introduced $*$-GD domains: R is called a *$*$-GD domain* if for every overring T of R and every semistar operation $*'$ on T, the extension $R \subseteq T$ satisfies $(*, \widetilde{*'})$-GD property ([9, Definition 3.1]). Here an extension $R \subseteq T$ of domains is said to satisfy *the $(*, *')$-GD property* if whenever $P_0 \subseteq P$ are quasi-$*$-prime ideals of R and Q is a quasi-$*'$-prime ideal of T such that $Q \cap R = P$, there exists a quasi-$*'$-prime ideal Q_0 of T such that $Q_0 \subseteq Q$ and $Q_0 \cap R = P_0$, where $*$ and $*'$ are semistar operations on R and T respectively ([9, Definition 2.1]). And $*$-GD domains are discussed mainly by the aid of $*$-Nagata domains in the three papers [9, 10, 15].

In this paper, we pay close attention to the corresponding GD domains of a specific star operation, i.e., the w-operation. Analogously to the GD-property, a w-linked extension $S \subseteq T$ of domains over R is said to satisfy *the w_R-GD property* if given $P_1, P_2 \in w_R$-Spec(S) with $P_1 \subseteq P_2$ and $Q_2 \in w_R$-Spec(T) with $Q_2 \cap S = P_2$, there exists some $Q_1 \in w_R$-Spec(T) such that $Q_1 \subseteq Q_2$ and $Q_1 \cap S = P_1$. In particular, when $S = R$, then $R \subseteq T$ is said to satisfy the w-GD property. Finally S (resp., R) is called a *w_R-GD* (resp., *w-GD*) *domain* if $S \subseteq T$ (resp., $R \subseteq T$) satisfies the w_R-GD (resp., w-GD) property for each w_R-linked (resp., w-linked) extension T over S (resp., R). Then it is natural to ask whether the definition of w-GD domains here is the same as that of the specific w-case of $*$-GD domains introduced by Dobbs and Sahandi ([9]). Of course, the answer is positive. It depends on the following characterizations of w_R-GD domains. Let S be a w-linked extension domain over R and let F be the quotient field of S. Then S is a w_R-GD domain if and only if $S \subseteq T$ satisfies the w_R-GD property for each w_R-linked valuation overring T of S, if and only if $S \subseteq (S[u])_w$ satisfies the w_R-GD property for each $u \in F$, if and only if $S_{\mathfrak{m}}$ is a GD domain for each maximal w_R-ideal $\mathfrak{m}$ of S (Theorem 3.2). In Section 3, we also point out the relationship among GD domains, w-GD domains, w_R-GD domains, Prüfer domains, PvMDs and Pw_RMDs, and provide the relative counterexamples. In Section 4, we discuss the ring S whose w_R-linked overring that satisfies the w_R-GD property is w_R-flat over S. It is easy to show that a Pw_RMD is such a ring, but the converse does not hold. Indeed, S is a Pw_RMD if and only if S is not only such a ring, but also a w_R-GD domain.

Now we recall some notions. Let $\overline{F}(R)$ be the set of all nonzero R-submodules of K and let $F(R)$ be the set of nonzero fractional ideals of R. A mapping $\overline{F}(R) \to \overline{F}(R), A \mapsto A_*$ is called a *semistar operation* on R if for any nonzero $x \in K$ and $A, B \in \overline{F}(R)$, the following conditions hold: (1) $(xA)_* = xA_*$. (2) $A \subseteq A_*$ and $A \subseteq B$ implies that $A_* \subseteq B_*$. (3) $(A_*)_* = A_*$. A star operation on R is exactly the restriction on $F(R)$ of a semistar operation on R with $R_* = R$. Let $*$ be a semistar (resp., star) operation. Then an ideal I of R is called a *quasi-$*$-ideal* (resp., *$*$-ideal*) if $I_* \cap R = I$ (resp., $I = I_*$). A prime ideal P of R is a *quasi-$*$-prime ideal* (resp., *prime $*$-ideal*) if P is a quasi-$*$-ideal (resp., $*$-ideal). An ideal $\mathfrak{m}$ of R is a *quasi-$*$-maximal ideal* (resp., *a maximal $*$-ideal*) if $\mathfrak{m}$ is maximal in the set of all proper quasi-$*$-ideals (resp., $*$-ideals) of R. Note that each quasi-$*$-maximal ideal (resp., maximal $*$-ideal) is prime. For an $A \in F(R)$, define $A^{-1} = \{x \in K \mid xA \subseteq R\}$ and $A_v = (A^{-1})^{-1}$. A finitely generated ideal J of R is called a GV-*ideal* if $J^{-1} = R$, denoted by $J \in \mathrm{GV}(R)$. The w-envelope of a torsion-free R-module M is the set given by

$$M_w = \{x \in E(M) \mid Jx \subseteq M \text{ for some } J \in \mathrm{GV}(R)\},$$

where $E(M)$ is the injective hull of M. Obviously both v and w are star operations on R. A torsion-free module M is called a *w-module* if $M_w = M$. Let $R \subseteq T$ be an extension of domains. Then T is called a *w-linked extension* of R if T is a w-module as an R-module. In the case that $R \subseteq T \subseteq K$, we say that T is a *w-linked overring* of R. For any undefined terminology and notation we refer to [9, 18].

2. Preliminaries

For an extension $R \subseteq T$ of domains and a T-module M, we distinguish M_w, the w-envelope of M as an R-module, with $M_{w(T)}$, the w-envelope of M as a T-module. That is to say, $w(T)$ stands for the w-operation on T. Let T be w-linked over R. For any fractional ideal A of T, define $w_R : A \mapsto A_w$. Then w_R is a star-operation on T. Let w-Spec(R) (resp., w-Max(R)) denote the set of prime w-ideals (resp., maximal w-ideals) of R and let w_R-Spec(T) (resp., w_R-Max(T)) denote the set of prime w_R-ideals (resp., maximal w_R-ideals) of T.

Lemma 2.1 ([18, Theorem 7.7.4 and Theorem 7.7.7])**.** *The following statements are equivalent for an extension $R \subseteq T$ of domains.*

(1) *T is w-linked over R.*

(2) *$A \cap R$ is a w-ideal of R for any $w(T)$-ideal A of T.*

(3) *If $J \in \mathrm{GV}(R)$, then $JT \in \mathrm{GV}(T)$.*

If one of the above statements holds, then so do the following statements.

(1) *If $Q \in w_R$-Spec(T), then $Q \cap R \in w$-Spec(R).*

(2) *If $Q \in$ Spec(T) and $Q \cap R \in w$-Spec(R), then $Q \in w_R$-Spec(T).*

Clearly if A is a nonzero ideal of T, then $A \subseteq A_{w_R} = A_w \subseteq A_{w(T)}$.

Definition 2.2. Let S and T be w-linked extension domains of R with $S \subseteq T$. Then $S \subseteq T$ is said to satisfy the w_R-GD property if given P, $P_1 \in w_R$-Spec(S) with $P \subseteq P_1$ and $Q_1 \in w_R$-Spec(T) with $Q_1 \cap R = P_1$, there exists some $Q \in w_R$-Spec(T) such that $Q \subseteq Q_1$ and $Q \cap R = P$. Specially, we say that $R \subseteq T$ satisfies the w-GD property when $S = R$.

By Lemma 2.1, the w-GD property of Definition 2.2 is equivalent to the statement: Let T be w-linked over R. If given P, $P_1 \in w$-Spec(R) with $P \subseteq P_1$

and $Q_1 \in \mathrm{Spec}(T)$ with $Q_1 \cap R = P_1$, there exists some $Q \in \mathrm{Spec}(T)$ such that $Q \subseteq Q_1$ and $Q \cap R = P$.

Proposition 2.3. *Let S and T be w-linked extension domains of R with $S \subseteq T$. Then the following statements are equivalent.*

(1) *The w_R-GD property holds.*

(2) *For $P \in w_R\text{-}\mathrm{Spec}(S)$, any prime w_R-ideal Q of T minimal over PT contracts to P.*

Proof. (1)⇒(2) It is clear that $P \subseteq Q \cap S$. Since $Q \in w_R\text{-}\mathrm{Spec}(T)$, $Q \cap S \in w_R\text{-}\mathrm{Spec}(S)$. If $Q \cap S \neq P$, then $Q_1 \cap S = P$ with $Q_1 \subseteq Q$ for some $Q_1 \in w_R\text{-}\mathrm{Spec}(T)$. Hence $PT \subseteq Q_1$, which is a contradiction to the minimality of Q. So $Q \cap S = P$.

(2)⇒(1) For prime w_R-ideals P, P_1 of S with $P \subseteq P_1$ and for a prime w_R-ideal Q_1 of T with $Q_1 \cap S = P_1$, there exists a prime w_R-ideal Q of T contained in Q_1 such that Q is minimal over PT. Hence $Q \cap S = P$ by (2). So w_R-GD holds. □

The following result shows that the w_R-GD property is local in some sense.

Theorem 2.4. *Let S and T be w-linked extension domains of R with $S \subseteq T$. Then the following statements are equivalent.*

(1) *$S \subseteq T$ satisfies the w_R-GD property.*

(2) *$S_{\mathfrak{p}} \subseteq T_{\mathfrak{p}}$ satisfies the GD property for any $\mathfrak{p} \in w_R\text{-}\mathrm{Spec}(S)$, where $T_{\mathfrak{p}} = T_{S\setminus\mathfrak{p}}$.*

(3) *$S_{\mathfrak{m}} \subseteq T_{\mathfrak{m}}$ satisfies the GD property for any $\mathfrak{m} \in w_R\text{-}\mathrm{Max}(S)$, where $T_{\mathfrak{m}} = T_{S\setminus\mathfrak{m}}$.*

Proof. (1) ⇒ (2) Let $\mathfrak{p} \in w_R\text{-}\mathrm{Spec}(S)$. For prime ideals $P_{\mathfrak{p}}, (P_1)_{\mathfrak{p}}$ of $S_{\mathfrak{p}}$ with $P_{\mathfrak{p}} \subseteq (P_1)_{\mathfrak{p}}$ and a prime ideal $(Q_1)_{\mathfrak{p}}$ of $T_{\mathfrak{p}}$ with $(Q_1)_{\mathfrak{p}} \cap S_{\mathfrak{p}} = (P_1)_{\mathfrak{p}}$, it is easy to verify that $P = P_{\mathfrak{p}} \cap S$ and $P_1 = (P_1)_{\mathfrak{p}} \cap S$ are both prime w_R-ideals of S and $P \subseteq P_1$. Because $Q_1 = (Q_1)_{\mathfrak{p}} \cap T$, $Q_1 \cap S = (Q_1)_{\mathfrak{p}} \cap T \cap S \cap S_{\mathfrak{p}} = (P_1)_{\mathfrak{p}} \cap S = P_1$. By (1), there exists $Q \in w_R\text{-}\mathrm{Spec}(T)$ with $Q \subseteq Q_1$ such that $Q \cap S = P$. Hence $Q_{\mathfrak{p}} \subseteq (Q_1)_{\mathfrak{p}}$ and $Q_{\mathfrak{p}} \cap S_{\mathfrak{p}} = P_{\mathfrak{p}}$.

(2) ⇒ (3) This is clear.

(3) ⇒ (1) Let P, P_1 be prime w_R-ideals of S with $P \subseteq P_1$ and Q_1 be a prime ideal of T with $Q_1 \cap S = P_1$. Then $P \subseteq P_1 \subseteq \mathfrak{m}$ for some $\mathfrak{m} \in w_R\text{-}\mathrm{Max}(S)$. So $P_{\mathfrak{m}}, (P_1)_{\mathfrak{m}}$ are prime ideals of $S_{\mathfrak{m}}$ with $P_{\mathfrak{m}} \subseteq (P_1)_{\mathfrak{m}}$ and $(Q_1)_{\mathfrak{m}}$ is a prime ideal of $T_{\mathfrak{m}}$ with $(Q_1)_{\mathfrak{m}} \cap S_{\mathfrak{m}} = (P_1)_{\mathfrak{m}}$. By (3), there exists a prime ideal $Q_{\mathfrak{m}}$ of $T_{\mathfrak{m}}$ such that $Q_{\mathfrak{m}} \subseteq (Q_1)_{\mathfrak{m}}$ and $Q_{\mathfrak{m}} \cap S_{\mathfrak{m}} = P_{\mathfrak{m}}$. Hence $Q \subseteq Q_1$ and $Q \cap S = P$. □

Let $R[X]$ be the polynomial ring over R and $c(f)$ be the ideal of R generated by the coefficients of $f \in R[X]$. Let $*$ be a star-operation on R and $N_* = \{f \in R[X] \mid c(f)_* = R\}$. In [1], an overring T is called $*$-*linked* over R if $T = T[X]_{N_*} \cap K$; equivalently, $I_* = R$ for a finitely generated fractional ideal I implies $(IT)_v = T$. Following this, if S is w-linked over R, then T is called *a w_R-linked overring* of S if T is an overring of S and $I_{w_R} = S$ for a finitely generated fractional ideal I of S implies $(IT)_v = T$.

Proposition 2.5. *Let S be w-linked over R. Then an overring T of S is a w_R-linked overring if and only if T is a w-module as an R-module.*

Proof. Assume that T is a w_R-linked overring of S. For any $x \in T_w$, there exists some $J \in \mathrm{GV}(R)$ such that $xJ \subseteq T$. Set $W = R\backslash\{0\}$. Then $T_W = E(T)$. Thus $T_w \subseteq T_W \subseteq F$, where F denotes the quotient field of S. So $x \in F$. Since $xJT \subseteq T$, $x \in (JT)^{-1}$. Obviously $(JS)_{w_R} = S$. By assumption, $(JT)^{-1} = T$. Thus $x \in T$, which implies $T_w \subseteq T$. Hence T is a w-module as an R-module.

Conversely, assume that T is a w-module over R. Let I be a finitely generated ideal of S with $I_{w_R} = S$. Then there exists some $J \in \mathrm{GV}(R)$ such that $J \subseteq I$. So $R = J_w \subseteq I_w$. Thus $(IT)_w = (I_wT_w)_w = T_w$. Since $T = T_w = (IT)_w = (IT)_{w_R} \subseteq (IT)_v \subseteq T_v = T$, $T = (IT)_v$. By definition, T is a w_R-linked overring of S. □

By Proposition 2.5, the definition of w-linked overrings in [1] is exactly that of w-linked overrings in the introduction. Now we can define w_R-linked extensions. Let S be w-linked over R and let $S \subseteq T$ be an extension of domains. Then T is called a *w_R-linked extension* of S if T is a w-module as an R-module. In the case that $S \subseteq T \subseteq F$ where F is the quotient field of S, T is exactly a w_R-linked overring of S by Proposition 2.5.

Let $*$ be a star operation on R. An overring V of R is called a *$*$-linked valuation overring* of R if V is a $*$-linked overring of R and V is a valuation domain.

Lemma 2.6 ([2, Lemma 3.3]). *The set of $*$-linked valuation overrings of R is the set $\{W \cap K \mid W$ is a valuation overring of $R[X]_{N_*}\}$.*

Lemma 2.7. *Let T be w-linked over R and Q a prime w_R-ideal of T. Then there exists some w_R-linked valuation overring V of T such that the maximal ideal of V contracts to Q.*

Proof. Set $N_{w_R} = \{f \in T[X] \mid c(f)_{w_R} = T\}$. For any $f \in QT[X]$, we have $c(f)_{w_R} \subseteq Q_{w_R} = Q \neq T$. Hence $QT[X] \cap N_{w_R} = \emptyset$, which implies that $QT[X]_{N_{w_R}}$ is a prime ideal of $T[X]_{N_{w_R}}$. By [11, Theorem 19.6], there exists a valuation overring V' of $T[X]_{N_{w_R}}$ whose maximal ideal M' lies over $QT[X]_{N_{w_R}}$. Let $V = V' \cap F$, where F denotes the quotient field of T. By Lemma 2.6, V is a w_R-linked valuation overring of T whose maximal ideal is $M' \cap F$. Obviously the maximal ideal of V contracts to Q. □

Proposition 2.8. *Let S be w-linked over R. Then the following statements are equivalent.*

(1) *$S \subseteq T$ satisfies the w_R-GD property for every w_R-linked overring T of S.*

(2) *$S \subseteq V$ satisfies the w_R-GD property for every w_R-linked valuation overring V of S.*

Proof. (1) $\Rightarrow$ (2) This is clear.

(2) $\Rightarrow$ (1) Let T be a w_R-linked overring of S and let P and P_1 be prime w_R-ideals of S with $P \subseteq P_1$ and Q_1 a prime ideal of T with $Q_1 \cap S = P_1$. By Lemma 2.7, there exists some w_R-linked valuation overring V of T such that the maximal ideal M_1 of V contracts to Q_1. Obviously V is also a w_R-linked valuation overring of S. By (2), there exists some $M \in \mathrm{Spec}(V)$ with $M \subseteq M_1$ such that $V \cap S = P$. Set $Q = V \cap T$. Then $Q \in \mathrm{Spec}(T)$ with $Q \subseteq Q_1$ and $Q \cap S = P$. Thus $S \subseteq T$ satisfies the w_R-GD property. $\square$

It is well known that if S is an integral extension of an integrally closed domain R, then $R \subseteq S$ satisfies the GD property [18, Theorem 5.3.29]. Next we give a w_R-corresponding statement of this result. Let S and T be w-linked over R with $S \subseteq T$. An element $u \in T$ is said to be *w_R-integral* (resp., *w-integral*) over S (resp. R) if there is a nonzero finitely generated S (resp., R)-module $B \subseteq T$ such that $uB_w \subseteq B_w$. The set of elements of T which are w_R-integral (resp., w-integral) over S (resp., R) is called the *w_R-integral closure* of S (resp., *w-integral closure* of R) in T, denoted by $S_T^{w_R}$ (resp., R_T^w). It is easy to see that $S_T^{w_R}$ and R_T^w are subrings of T. In the case $T = F$, we write $S^{w_R} = S_T^{w_R}$ (resp., $R^w = R_T^w$), where F denotes the quotient field of S (resp., R). If $S_T^{w_R} = T$ (resp., $R_T^w = T$), we say that T is *w_R-integral* over S (resp., *w-integral* over R). R is integrally closed if and only if $R^w = R$ ([18]). For more details about w-integral elements, see [18].

Proposition 2.9. *Let S and T be w-linked extension domains of R with $S \subseteq T$ and let $u \in T$. Then the following statements are equivalent.*

(1) *u is w_R-integral over S.*

(2) *There exists some $J = (a_1, a_2, \ldots, a_t) \in \mathrm{GV}(R)$ such that each ua_i is integral over S.*

(3) *There exists some $J \in \mathrm{GV}(R)$ such that uJ is integral over S.*

Proof. (2)$\Leftrightarrow$(3) This is clear.

(1)$\Rightarrow$(2) Let u be w_R-integral over S. Then there is a nonzero finitely generated S-module $B \subseteq T$ such that $uB_w \subseteq B_w$, which implies that $uB \subseteq B_w$. So $uBJ \subseteq B$ for some $J \in \mathrm{GV}(R)$. Write $B = b_1S + b_2S + \cdots + b_nS$ and $J = (a_1, a_2, \ldots, a_t)$. Let $ub_ia_j = \sum_{s=1}^{n} r_{ijs}b_s$, where $1 \le i \le n, 1 \le j \le t, r_{ijs} \in S$. For any $1 \le j \le t$, we have

$$ua_j \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} r_{1j1} & r_{1j2} & \cdots & r_{1jn} \\ r_{2j1} & r_{2j2} & \cdots & r_{2jn} \\ \vdots & \vdots & & \vdots \\ r_{nj1} & r_{nj2} & \cdots & r_{njn} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

Let $A_j = \begin{pmatrix} r_{1j1} & r_{1j2} & \cdots & r_{1jn} \\ r_{2j1} & r_{2j2} & \cdots & r_{2jn} \\ \vdots & \vdots & & \vdots \\ r_{nj1} & r_{nj2} & \cdots & r_{njn} \end{pmatrix}$. Then $(ua_jE_n - A_j)\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, where E_n is the $n \times n$ identity matrix. Hence $(ua_jE_n - A_j)B = 0$. Because $B \neq 0$ and T is a domain, $\det(ua_jE_n - A_j) = 0$, which implies ua_j is integral over S.

(2)⇒(1) If there exists some $J = (a_1, a_2, \ldots, a_t) \in \mathrm{GV}(R)$ such that each ua_i is integral over S. Assume that n_i is the degree of the integrally dependent equation of ua_i over S. Let $B = \sum_{s_1,\ldots,s_t}(ua_1)^{s_1}(ua_2)^{s_2}\cdots(ua_t)^{s_t}S$ where $0 \le s_i \le n_i$ for each $1 \le i \le t$. Obviously B is a finitely generated S-module and $uJB \subseteq B$. Then $uB \subseteq B_w$. Hence $uB_w \subseteq B_w$. Then u is w_R-integral over S. □

Corollary 2.10. *Let S and T be w-linked extension domains of R with $S \subseteq T$ and S_T^c be the integral closure of S in T.*

(1) $S_T^c \subseteq S_T^{w_R} \subseteq S_T^{w(S)}$.
(2) $S_T^{w_R} = (S_T^c)_w$.

Proof. (1) It follows by the equivalence of (1) and (3) of Proposition 2.9.

(2) Let A be a nonzero finitely generated S-module. Then $A \subseteq A_{w_R} \subseteq A_{w(S)}$ by Lemma 2.1. Thus the result follows. □

Proposition 2.11. *Let S be w-linked over R. Then the following statements are equivalent.*

(1) *S is integrally closed.*
(2) *S is w_R-integrally closed.*
(3) *S is $w(S)$-integrally closed.*

Proof. (1) ⇔ (3) See [18, Example 7.7.14].

(1) ⇒ (2) If S is integrally closed, then S is $w(S)$-integrally closed. By (1) and Corollary 2.10, $S \subseteq (S^c)_w = S^{w_R} \subseteq S^{w(S)} = S$. Then $S^{w_R} = S$. So S is w_R-integrally closed.

(2) ⇒ (1) If S is w_R-integrally closed, then $S \subseteq S^c \subseteq S^{w_R} = S$. Thus $S^c = S$. □

Lemma 2.12. *Let T be w-linked over R and M a torsion-free T-module. Then the following statements hold.*

(1) *$M_Q = (M_w)_Q$ for any $Q \in w_R$-$\mathrm{Spec}(T)$.*
(2) *$M_w = \bigcap\{M_\mathfrak{m} \mid \mathfrak{m} \in w_R$-$\mathrm{Max}(T)\}$.*
(3) *If S is w-linked over R and $S \subseteq T$, then $(S_T^{w_R})_\mathfrak{m} = (S_\mathfrak{m})^c_{T_\mathfrak{m}}$, where $T_\mathfrak{m} = T_{S\setminus\mathfrak{m}}$ and $\mathfrak{m} \in w_R$-$\mathrm{Max}(S)$.*

Proof. (1) follows by the same way as the proof of [18, Theorem 6.2.16]. (2) follows by [18, Theorem 7.2.11(4)]. (3) follows by the same way as the proof of [18, Corollary 7.7.11]. □

Proposition 2.13. *Let S and T be w-linked extension domains of R with $S \subseteq T$. Then T is w_R-integral over S if and only if $T_{\mathfrak{m}}$ is integral over $S_{\mathfrak{m}}$ for any $\mathfrak{m} \in w_R\text{-Max}(S)$, where $T_{\mathfrak{m}} = T_{S\setminus\mathfrak{m}}$.*

Proof. If T is w_R-integral over S, then $S_T^{w_R} = T$. By Lemma 2.12, $(S_{\mathfrak{m}})^c_{T_{\mathfrak{m}}} = (S_T^{w_R})_{\mathfrak{m}} = T_{\mathfrak{m}}$. Then $T_{\mathfrak{m}}$ is integral over $S_{\mathfrak{m}}$.

Conversely, if for any $\mathfrak{m} \in w_R\text{-Max}(S)$, $T_{\mathfrak{m}}$ is integral over $S_{\mathfrak{m}}$, then $(S_{\mathfrak{m}})^c_{T_{\mathfrak{m}}} = T_{\mathfrak{m}}$. By Lemma 2.12(2), $(S^c_T)_w = \bigcap\{(S^c_T)_{\mathfrak{m}} \mid \mathfrak{m} \in w_R\text{-Max}(S)\}$ and $T = \bigcap\{T_{\mathfrak{m}} \mid \mathfrak{m} \in w_R\text{-Max}(S)\}$. Note that $(S^c_T)_{\mathfrak{m}} = (S_{\mathfrak{m}})^c_{T_{\mathfrak{m}}}$. So $S_T^{w_R} = (S^c_T)_w = \bigcap(S_{\mathfrak{m}})^c_{T_{\mathfrak{m}}} = \bigcap T_{\mathfrak{m}} = T$. □

Theorem 2.14. *Let S and T be w-linked extension domains of R with $S \subseteq T$. If T is w_R-integral over S and S is integrally closed, then $S \subseteq T$ satisfies the w_R-GD property.*

Proof. By Proposition 2.13, $T_{\mathfrak{m}}$ is integral over $S_{\mathfrak{m}}$ for any $\mathfrak{m} \in w_R\text{-Max}(S)$. Note that $S_{\mathfrak{m}}$ is integrally closed. Then $S_{\mathfrak{m}} \subseteq T_{\mathfrak{m}}$ satisfies the GD property. Thus $S \subseteq T$ satisfies the w_R-GD property by Theorem 2.4. □

3. w_R-GD domains

In [6], the definitions of GD domains and SGD domains were given by Dobbs: R is called a *GD domain* if $R \subseteq T$ satisfies GD for every overring T of R. R is called an *SGD domain* if $R \subseteq R[u]$ satisfies GD for each u in K. In [8], he proved that SGD domains are exactly GD domains. Examples of GD domains are Prüfer domains and arbitrary domains of Krull dimension 1. Now we use the w_R-operation to generalize GD domains.

Definition 3.1. Let S be w-linked over R. Then S is called a w_R-GD domain if $S \subseteq T$ satisfies the w_R-GD property for every w_R-linked extension T of S. In particular, in the case $S = R$, we call R a w-GD domain.

Theorem 3.2. *Let S be w-linked over R. Then the following statements are equivalent.*

(1) *S is a w_R-GD domain.*

(2) *$S \subseteq T$ satisfies w_R-GD for each w_R-linked valuation overring T.*

(3) *$S \subseteq (S[u])_w$ satisfies w_R-GD for each $u \in F$, where F is the quotient field of S.*

(4) *$S_{\mathfrak{m}}$ is a GD domain for each $\mathfrak{m} \in w_R\text{-Max}(S)$.*

(5) *$S_{\mathfrak{p}}$ is a GD domain for each $\mathfrak{p} \in w_R\text{-Spec}(S)$.*

Proof. (3) ⇔ (4) ⇔ (5) $S \subseteq (S[u])_w$ satisfies w_R-GD for each $u \in F$ if and only if $S_{\mathfrak{m}} \subseteq ((S[u])_w)_{\mathfrak{m}}$ satisfies the GD property for any $\mathfrak{m} \in w_R\text{-Max}(S)$ and any $u \in F$ by Theorem 2.4, if and only if $S_{\mathfrak{m}} \subseteq (S[u])_{\mathfrak{m}}$ satisfies the GD property for any $\mathfrak{m} \in w_R\text{-Max}(S)$ and any $u \in F$, if and only if $S_{\mathfrak{m}}$ is a GD domain for any $\mathfrak{m} \in w_R\text{-Max}(S)$ ([8, Theorem 1]), if and only if $S_{\mathfrak{p}}$ is a GD domain for any $\mathfrak{p} \in w_R\text{-Spec}(S)$.

(4) $\Rightarrow$ (1) $\Rightarrow$ (2) $\Rightarrow$ (3) These are clear by Theorem 2.4 and Proposition 2.8. $\square$

Let $*$ be a semistar operation on R and let $\mathrm{Na}(R,*)$ be the $*$-*Nagata ring of R with respect to* $*$, defined by $\mathrm{Na}(R,*) := R[X]_{N_*}$. Then $\widetilde{*}$ is also a semistar operation on R, which can be most concisely defined by $E_{\widetilde{*}} := E\mathrm{Na}(R,*) \cap K$ for all $E \in \overline{F}(R)$.

Dobbs and Sahandi ([10]) proved that R is a $\widetilde{*}$-GD domain if and only if $R_{\mathfrak{m}}$ is a GD domain for any quasi-$*$-maximal ideal $\mathfrak{m}$ ([10, Proposition 2.5]). Here $\widetilde{*}$-GD domains are the ones defined by Dobbs and Sahandi in [9]. Since $\widetilde{w} = w$ and $\widetilde{w_R} = w_R$, it follows that the two definitions of $\widetilde{w_R}$-GD domains in Definition 3.1 and [9, Definition 3.1] are the same. The discussion of $*$-GD domains is done mainly by the aid of $*$-Nagata rings in [9,10,15]. Then we can get the following three results.

Corollary 3.3 ([9, Corollary 3.14])**.** *If* $\mathrm{Na}(R,w)$ *is a GD domain, then* R *is a* w*-GD domain.*

Recall that R is a *PvMD* if $R_{\mathfrak{m}}$ is a valuation domain for any maximal w-ideal $\mathfrak{m}$ of R. Let S be w-linked over R. Then S is a *P*w_R*MD* if $S_{\mathfrak{m}}$ is a valuation domain for any maximal w_R-ideal $\mathfrak{m}$ of S.

Proposition 3.4. *The following statements are equivalent for a domain* R*.*

(1) $\mathrm{Na}(R,w)$ *is a GD domain.*

(2) R *is a* w*-GD domain and* R *is a UMT domain* (*i.e., every upper to zero in* $R[X]$ *is a maximal* w*-ideal*)*.*

(3) R *is a* w*-GD domain and* R^w *is a P*w_R*MD.*

Proof. (1) $\Leftrightarrow$ (2) This follows by [10, Theorem 2.6] and [3, Corollary 2.4].

(2) $\Leftrightarrow$ (3) This follows by the fact that R is a UMT domain if and only if R^w is a Pw_RMD ([18, Theorem 7.8.13]). $\square$

Corollary 3.5. *Let* S *be* w*-linked over* R*. Then the following statements are equivalent.*

(1) S *is a P*w_R*MD;*

(2) S *is integrally closed and* $\mathrm{Na}(S,w_R)$ *is a GD domain.*

(3) S *is integrally closed and* $\mathrm{Na}(S,w_R)$ *is a tree domain* (*i.e., no prime ideal of* $\mathrm{Na}(S,w_R)$ *contains incomparable prime ideals of* $\mathrm{Na}(S,w_R)$)*.*

(4) $\mathrm{Na}(S,w_R)$ *is an integrally closed GD domain.*

(5) $\mathrm{Na}(S,w_R)$ *is an integrally closed tree domain.*

Proof. This follows by [10, Corollary 2.8]. $\square$

Let S be w-linked over R. Obviously if S is a GD domain, then S is a w_R-GD domain. By Theorem 3.2, it is clear that if S is a w_R-GD domain, then S is a $w(S)$-GD domain. Note that valuation domains are GD domains. Then PvMDs are w-GD domains by Theorem 3.2. Pw_RMDs are w_R-GD domains again by Theorem 3.2.

Let S be w-linked over R. Then we get the following diagram.

$$\begin{array}{ccccc}
\text{Prüfer domain} & \Longrightarrow & \text{P}w_R\text{MD} & \Longrightarrow & \text{P}v\text{MD} \\
\Downarrow & & \Downarrow & & \Downarrow \\
\text{GD domain} & \Longrightarrow & w_R\text{-GD domain} & \Longrightarrow & w(S)\text{-GD domain.}
\end{array}$$

But the seven arrows are not reversible in general.

The following example shows that GD (resp., w-GD) domains may not be Prüfer domains (resp., PvMDs).

Example 3.6. Let $\mathbb{Z}$ denote the ring of integers and let $R = \mathbb{Z}[\sqrt{5}]$. Then R is a Noetherian domain of Krull dimension 1. Thus R is a GD domain, and so a w-GD. Note that R is not integrally closed because $\frac{1}{2}(1+\sqrt{5}) \notin R$ is integral over R. Then R is neither a Prüfer domain nor a PvMD.

The following example shows that w_R-GD domains may not be Pw_RMDs.

Example 3.7. Let $R = \mathbb{Z}$. Since $\text{GV}(R) = \{R\}$, it is clear that $S = \mathbb{Z}[\sqrt{5}]$ is w-linked over R. Obviously S is a w_R-GD domain. Note that Pw_RMDs are integrally closed ([18, Theorem 7.7.19]). Then S is not a Pw_RMD.

Let S be w-linked over R. Next we show that $w(S)$-GD domains are not w_R-GD domains and that w_R-GD domains are not GD domains in general. First we need the following theorem.

Theorem 3.8. *Let S be w-linked over R with quotient field F. Then the following statements hold.*

(1) *If S is a PvMD, not a Pw_RMD, then there exists $u \in F$ such that $S \subseteq (S[u])_w$ does not satisfy the w_R-GD property.*

(2) *If S is a Pw_RMD, not a Prüfer domain, then there exists $u \in F$ such that $S \subseteq S[u]$ does not satisfy the GD property.*

(3) *If R is a PvMD, not a Prüfer domain, then there exists an element u in its quotient field K such that $R \subseteq R[u]$ does not satisfy the GD property.*

Example 3.9. Let S be w-linked over R. By Theorem 3.8, we know that if S is a PvMD, not a Pw_RMD, then S is a $w(S)$-GD domain, not a w_R-GD domain. For example, let $R = k[Y, XY, X^2, X^3]$ and $S = k[X, Y]$, where k is a field. Then S is a PvMD, not a Pw_RMD [16, Example]. Similarly, if S is a Pw_RMD, not a Prüfer domain, then S is a w_R-GD domain, not a GD domain. If R is a PvMD, not a Prüfer domain, then R is a w-GD domain, not a GD domain.

In order to prove Theorem 3.8, now we give the following three lemmas.

Let M be a torsion-free R-module. Then M is said to be *of finite type* if there is a finitely generated R-module N contained in M such that $M_w = N_w$. Obviously a finitely generated R-module is of finite type.

Lemma 3.10. *Let R be an integrally closed domain with quotient field K and $u \in K \setminus \{0\}$. If the conductor of u to R, $(R : u) = \{r \in R \mid ru \in R\}$, is of finite type and $u(R : u) \subseteq \sqrt{(R : u)}$, then $u \in R$.*

Proof. Let $I = (R : u)$. Then I is a w-ideal of R and uI is a ideal of R. By assumption, there is a finitely generated ideal I_0 contained in I such that $I = (I_0)_w$, whence $uI = (uI_0)_w$. Since $uI \subseteq \sqrt{I}$, $uI_0 \subseteq \sqrt{I}$. Then there is a positive integer n such that $(uI_0)^n \subseteq I$. If $n = 1$, then $uI_0 \subseteq I$. Thus $uI = (uI_0)_w \subseteq I$. Hence u is w-integral over R. Note that R is integrally closed if and only if $R^w = R$. Thus $u \in R$. If $n > 1$, then $I_0(u^n(I_0)^{n-1}) \subseteq I$. Thus $(I_0)_w(u^n(I_0)^{n-1}) \subseteq I$. Therefore $u^n(I_0)^{n-1}$ is w-integral over R. Hence $u^n(I_0)^{n-1} \subseteq R$. So we have $u^{n-1}(I_0)^{n-1} \subseteq I$. Induction yields the result. $\square$

Let S and T be w-linked over R with $S \subseteq T$. If given a prime w_R-ideal P of S, there exists $Q \in w_R$-Spec(T) satisfying $Q \cap S = P$, we say that *w_R-LO* holds for the extension $S \subseteq T$. By Lemma 2.1, the definition of w_R-LO is equal to the statement: Let S and T be w-linked over R with $S \subseteq T$. Given a prime w_R-ideal P of S, there exists $Q \in \operatorname{Spec}(T)$ satisfying $Q \cap S = P$.

In [17], F. G. Wang proved that a domain R is a PvMD if and only if R is integrally closed and the conductor of u to R is of finite type for each nonzero element u in its quotient field K. By considering Lemma 3.10, we can get the following result.

Lemma 3.11. *Let S and T be w-linked over R and let F be the quotient filed of S with $S \subseteq T \subseteq F$. If S be a PvMD and $S \subseteq T$ satisfies w_R-LO, then $T = S$.*

Proof. Let $t \in T \setminus S$ and $I = (S : t)$. Then I is a w_R-ideal of S. For any prime w_R-ideal P of S containing I, there exists $Q \in \operatorname{Spec}(T)$ such that $Q \cap S = P$. Since $I \subseteq P \subseteq Q$, $tI \subseteq Q$. So we have $tI \subseteq Q \cap S = P$. Therefore any prime w_R-ideal of S containing I contains tI. Note that prime ideals of S minimal over I are w_R-ideals ([18, Theorem 7.2.12]). Thus $tI \subseteq \sqrt{I}$, which implies $t \in S$ by Lemma 3.10, a contradiction. Thus $T = S$. $\square$

Lemma 3.12. *Let S be w-linked over R with quotient field F. If $S \subseteq S[u]$ satisfies LO for $u \in F$, then $S \subseteq (S[u])_w$ satisfies w_R-LO.*

Proof. For $P \in w_R$-Spec(S), there exists some $Q \in \operatorname{Spec}(S[u])$ such that $Q \cap S = P$ by the LO property of $S \subseteq S[u]$. It is trivial to prove that $Q_w \cap S = P$. Now it suffices to show that $Q_w \in \operatorname{Spec}(S[u]_w)$. It is clear that Q_w is an ideal of $S[u]_w$. For $xy \in Q_w$, where $x, y \in S[u]_w$, there exist $J_1, J_2, J \in \operatorname{GV}(R)$ such that $xJ_1, yJ_2 \subseteq S[u]$ and $xyJ_1J_2J \subseteq Q$. Then either $xJ_1J \subseteq Q$ or $yJ_2J \subseteq Q$. Thus either $x \in Q_w$ or $y \in Q_w$. Hence $Q_w \in \operatorname{Spec}(S[u]_w)$, as desired. $\square$

Proof of Theorem 3.8. (1) Assume the result is not true. Then we can get a contradiction. Note that S is a Pw_RMD if and only if $S_{\mathfrak{m}}$ is a valuation domain for any maximal w_R-ideal $\mathfrak{m}$ of S ([18, Theorem 7.7.19]). Since S is not a Pw_RMD, there exists some maximal w_R-ideal $\mathfrak{m}$ of S such that $S_\mathfrak{m}$ is not a valuation domain. Then there exists some $u \in F$ such that $u, u^{-1} \notin S_\mathfrak{m}$. Note that $\mathfrak{m}S_\mathfrak{m}[u] \neq S_\mathfrak{m}[u]$ or $\mathfrak{m}S_\mathfrak{m}[u^{-1}] \neq S_\mathfrak{m}[u^{-1}]$ ([12, Theorem 55]). Without loss of generality, we assume that $\mathfrak{m}S_\mathfrak{m}[u] \neq S_\mathfrak{m}[u]$. By Theorem 2.4, $S_\mathfrak{m} \subseteq ((S[u])_w)_\mathfrak{m} = S_\mathfrak{m}[u]$ satisfies the GD property. Then there is some prime ideal Q of $S_\mathfrak{m}[u]$ such that $Q \cap S_\mathfrak{m} = \mathfrak{m}S_\mathfrak{m}$. Note that $S_\mathfrak{m}$ is local. Then $S_\mathfrak{m} \subseteq S_\mathfrak{m}[u]$ satisfies LO. Obviously $S_\mathfrak{m}$ is w-linked over R. By Lemma 3.12, $S_\mathfrak{m} \subseteq (S_\mathfrak{m}[u])_w$ satisfies w_R-LO. By assumption, $S_\mathfrak{m}$ is a PvMD. Then $S_\mathfrak{m} = (S_\mathfrak{m}[u])_w$ by Lemma 3.11. Thus $u \in S_\mathfrak{m}$, contradicting $u \notin S_\mathfrak{m}$.

By the same way as the proof of (1), we can prove (2) and (3). □

Then by Theorem 3.8, we can get the following result.

Proposition 3.13. (1) *R is a Prüfer domain if and only if R is a PvMD and a GD domain.*

(2) *Let S be w-linked over R. Then S is a Prüfer domain if and only if S is a Pw_RMD and a GD domain.*

(3) *Let S be w-linked over R. Then S is a Pw_RMD if and only if S is a PvMD and a w_R-GD domain.*

4. A new characterization of Pw_RMDs

Now, we recall several concepts from [19]. Let S be w-linked over R. For S-modules M and N and for $f \in \mathrm{Hom}_S(M, N)$, we call f a *w_R-monomorphism* if $f_m : M_\mathfrak{m} \to N_\mathfrak{m}$ is a monomorphism for each maximal w_R-ideal $\mathfrak{m}$ of S. An S-module M is called a *w_R-flat module* if the induced map $1 \otimes f : M \otimes_S A \to M \otimes_S B$ is a w_R-monomorphism for any w_R-monomorphism $f : A \to B$. In particular, when $S = R$, we call M a *w-flat module* of R. It is known that an S-module M is a w_R-flat module if and only if $M_\mathfrak{m}$ is flat over $S_\mathfrak{m}$ for each maximal w_R-ideal $\mathfrak{m}$ of S [19, Proposition 3.1.8].

It is well known that R is a Prüfer domain if and only if each overring of R is flat, if and only if each overring of R is integrally closed. In [7], Dobbs *et al.* proved that R is a PvMD if and only if each t-linked overring of R is integrally closed. In [20], Xing and Wang proved that R is a PvMD if and only if each w-linked overring of R is w-flat. By the same way as the proof of [20, Theorem 2.5], we can get the following proposition.

Proposition 4.1. *Let S be w-linked over R. Then the following statements are equivalent.*

(1) *S is a Pw_RMD.*

(2) *Each w_R-linked overring of S is w_R-flat.*

(3) *Each w_R-linked overring of S is integrally closed.*

Here is a natural question. Let S be w-linked over R. If every w_R-linked overring of S that satisfies the w_R-GD property is w_R-flat over S, then is S precisely a Pw_RMD? The answer is negative.

Example 4.2. Let $R = k[Y, XY, X^2, X^3], S = k[X, Y]$, where k is a field. By Example 3.9, S is not a Pw_RMD. Note that S is a Krull domain. Then for each $\mathfrak{m} \in w(S)$-Max(S), $S_\mathfrak{m}$ is a discrete valuation domain ([18, Theorem 7.9.3]). Thus $S_\mathfrak{m}$ is a Prüfer domain. Obviously each overring of $S_\mathfrak{m}$ is flat over $S_\mathfrak{m}$. If T is a w_R-linked overring of S that satisfies the w_R-GD property, then $T_\mathfrak{m}$ is an overring of $S_\mathfrak{m}$. Thus $T_\mathfrak{m}$ is flat over $S_\mathfrak{m}$. Hence T is w_R-flat over S. Then every w_R-linked overring of S that satisfies the w_R-GD property is w_R-flat over S.

Let S be w-linked over R. Indeed, we have a new characterization of Pw_RMDs: S is a Pw_RMD if and only if S is a w_R-GD domain and every w_R-linked overring of S that satisfies the w_R-GD property is w_R-flat over S. To get this result, we start with the following lemma.

Lemma 4.3. *Let S be w-linked over R and let F be the quotient field of S. Then S is a Pw_RMD if and only if $(S[u])_w$ is w_R-flat over S for each $u \in F$.*

Proof. By Proposition 4.1, the necessity is clear.

Conversely, it suffices to show that $S_\mathfrak{m}$ is a valuation ring for each $\mathfrak{m} \in w_R$-Max(S). If $\frac{x}{y} \notin S_\mathfrak{m}$, where $x, y \in S_\mathfrak{m}$, then $(y :_{S_\mathfrak{m}} x) \subseteq \mathfrak{m}S_\mathfrak{m}$. Since $(S[\frac{x}{y}])_w$ is w_R-flat over S, $S_\mathfrak{m}[\frac{x}{y}] = (S[\frac{x}{y}])_\mathfrak{m} = ((S[\frac{x}{y}])_w)_\mathfrak{m}$ is flat over $S_\mathfrak{m}$. Then $(y :_{S_\mathfrak{m}} x)S_\mathfrak{m}[\frac{x}{y}] = S_\mathfrak{m}[\frac{x}{y}]$ ([13, Proposition 4.12]). Thus $1 \in (y :_{S_\mathfrak{m}} x)S_\mathfrak{m}[\frac{x}{y}]$. Assume that

$$1 = \alpha_0 + \alpha_1 \frac{x}{y} + \cdots + \alpha_n \frac{x^n}{y^n},$$

where $\alpha_0, \alpha_1, \ldots, \alpha_n \in (y :_{S_\mathfrak{m}} x)$. Then

$$(1 - \alpha_0)(\frac{y}{x})^n - \alpha_1(\frac{y}{x})^{n-1} - \cdots - \alpha_{n-1}\frac{y}{x} - \alpha_n = 0.$$

Note that $\alpha_0 \in \mathfrak{m}S_\mathfrak{m}$. Then $1 - \alpha_0$ is a unit of $S_\mathfrak{m}$. So $\frac{y}{x}$ is integral over $S_\mathfrak{m}$. Hence $S_\mathfrak{m}[\frac{y}{x}]$ is integral over $S_\mathfrak{m}$. Then $S_m[\frac{y}{x}] = S_m$ by ([14, Proposition 2]). Thus $\frac{y}{x} \in S_m$, which implies that $S_\mathfrak{m}$ is a valuation ring. □

Theorem 4.4. *Let S be w-linked over R. Then S is a Pw_RMD if and only if S is a w_R-GD domain and every w_R-linked overring of S that satisfies the w_R-GD property is w_R-flat over S.*

Proof. Assume that S is a w_R-GD domain and every w_R-linked overring of S that satisfies the w_R-GD property is w_R-flat over S. Then $S \subseteq (S[u])_w$ satisfies the w_R-GD property for each $u \in F$ by Theorem 3.2, where F is the quotient field of S. Thus $(S[u])_w$ is w_R-flat over S. By Lemma 4.3, S is a Pw_RMD. The converse follows from Propositions 3.13(3) and 4.1. □

Corollary 4.5. *R is a PvMD if and only if R is a w-GD domain and every w-linked overring of R that satisfies the w-GD property is w-flat over R.*

By the same way as the proof of Theorem 4.4, we can also prove that R is a Prüfer domain if and only if R is a GD domain and every overring of R that satisfies the GD property is flat over R.

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