Bull. Korean Math. Soc. **57** (2020), No. 6, pp. 1335–1349
https://doi.org/10.4134/BKMS.b181086
pISSN: 1015-8634 / eISSN: 2234-3016

FINSLER METRICS WITH REVERSIBLE GEODESICS

Qiaoling Xia

Abstract. In this paper, we give an equivalent characterization for general (α, β)-metrics with reversible geodesics when the dimension of the manifold is greater than 2.

1. Introduction

Recall that a *Finsler metric* F on a smooth manifold M means a function $F : TM \to [0, \infty)$ such that $F_x = F|_{T_xM}$ is a Minkowski norm on T_xM at each point $x \in M$. In particular, if $F(x, y) = F(x, -y)$ for any $y \in T_xM \setminus \{0\}$, we say that F is *reversible*, or F is a *reversible Finsler metric*. Obviously, a Riemannian metric on M is reversible. However, a Finsler metric is not reversible in general. For example, Randers metrics $F = \alpha + \beta$, one of the simplest and most important non-Riemannian Finsler metrics, are not reversible, where α is a Riemannian metric and β is a nonzero 1-form on M. Because of the irreversibility of F, the distance function d induced by F might not be symmetric and the inverse of a geodesic is not necessarily a geodesic. We say that a Finsler metric F is said to be with *reversible geodesics* if, for any oriented geodesic curve, the same path traversed in the opposite sense is also a geodesic. Equivalently, its geodesic coefficients $G^i(x, y)$ are projectively equivalent to $G^i(x, -y)$, i.e.,

$$G^i(x, y) = G^i(x, -y) + Py^i, \tag{1}$$

where $P := P(x, y)$ is a scalar function on $TM \setminus \{0\}$ with $P(x, \lambda y) = \lambda P(x, y)$, where $\lambda > 0$ ([5], [7]). If $P = 0$, then F is said to be with *strictly reversible geodesics*. If $G^i(x, y) = Py^i$, F is said to be *locally projectively flat* ([4]). In recent years, there have been made some new progress on locally projectively flat Finsler metrics ([2], [4], [9], [6, 10, 11] etc. and the references therein).

A smooth manifold M equipped with a Finsler metric F is called a *Finsler space*. In particular, a Finsler space (M, F) is called a *reversible Finsler space* if F is reversible. On the other hand, a Finsler space (M, F) is said to be

Received November 13, 2018; Accepted October 16, 2019.
2010 *Mathematics Subject Classification.* Primary 53B40, 53C22, 58B20.
Key words and phrases. Finsler metric, general (α, β)-metric, reversible geodesic.
This author is supported by Zhejiang Provincial NSFC (No. LY19A010021) and NNSFC (No.11671352).

©2020 Korean Mathematical Society

geodesically reversible if F is the Finsler metric with reversible geodesics. A reversible Finsler space is geodesically reversible, but the converse might not be true (see examples in [2]). R. Bryant proved that a geodesically reversible Finsler metric of constant flag curvature on the 2-sphere is necessarily projectively flat ([3]). As an application, a reversible Finsler metric of constant flag curvature on the 2-sphere is necessarily a Riemannian metric of constant Gauss curvature. From this, it is important to study Finsler metrics with reversible geodesics since they are closely related with the projective properties of Finsler metrics. It is known that a Randers metric $F = \alpha + \beta$ is with reversible geodesics if and only if β is closed ([5]). In [7, 8], the authors gave the necessary and sufficient conditions for (α, β)-metrics $F = \alpha\phi(\beta/\alpha)$ with reversible geodesics and strictly reversible geodesics respectively, and obtained some new classes of (α, β)-metrics with reversible geodesics.

A more general class named general (α, β)-metrics were first introduced by Yu-Zhu in [12] in the following form

$$F = \alpha\phi(b^2, s), \quad s = \frac{\beta}{\alpha},$$

where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a nonzero 1-form on M with $b = \|\beta\|_\alpha$, and $\phi(b^2, s)$ is a smooth function with some restrictions (see (8) in the next section). It is easy to see that F is reversible if and only if $\phi(b^2, s) = \phi(b^2, -s)$, i.e., $\phi(b^2, s)$ is even in s. If $\phi(b^2, s)$ only depends on s and independent of b, then F is just an (α, β)-metric. In present paper, we give an equivalent characterization for general (α, β)-metrics with reversible geodesics. As mentioned before, a reversible Finsler space is geodesically reversible. So we only consider nonreversible general (α, β)-spaces with reversible geodesics.

Theorem 1.1. *Let $F = \alpha\phi\left(b^2, s\right)$, $s = \beta/\alpha$, be a nonreversible general (α, β)-metric on an $n(\geq 3)$-dimensional smooth manifold M. Then F is with reversible geodesics if and only if there is a local coordinate system, in which there are scalar functions $k(b^2) := -\frac{\phi_{12}(b^2,0)}{\phi_2(b^2,0)}$, $\sigma(x)$ and $k_i(x)(1 \leq i \leq 3)$ with $k_1 + b^2 k_2 = 1$ and $k_1 + b^2 k_3 = -1$ such that ϕ, α and β satisfy one of the following cases.*

(1) *$\phi(b^2, s) = \phi(b^2, -s) + 2cs$, where $c = \phi_2(b^2, 0) \neq 0$, and*

(i) *β is closed if c is a constant*;

(ii) *if c is not a constant, we have*

$$r_{ij} = \frac{\sigma}{b^2} b_i b_j + \frac{\tau}{b^2}(b_i s_j + b_j s_i) + \sum_{k,l \neq 1} r_{kl}\delta_i^k \delta_j^l, \quad s_{ij} = \frac{1}{b^2}(b_i s_j - b_j s_i) \tag{2}$$

with $\sum_{k\neq 1} r_{kl} b^k = 0 (l \neq 1)$, where $\tau := \frac{1-kb^2}{kb^2}$. In this case, $k \neq 0$, and β is closed when $s_i = 0$ for all $1 \leq i \leq n$. Otherwise, β is not closed.

(2) *$\phi(b^2, s) \neq \phi(b^2, -s) + 2cs$, where $c = \phi_2(b^2, 0)$, and one of the following* (i)-(iv) *holds.*

(i) *β is parallel with respect to α.*

(ii) *ϕ satisfies* (32), *and* $r_{ij} = \sigma(k_1 a_{ij} + k_2 b_i b_j)(\sigma \neq 0)$. *In this case, β is closed.*

(iii)

$$r_{ij} = \sigma(k_1 a_{ij} + k_3 b_i b_j) + \frac{1}{b^2}(r_i b_j + r_j b_i), \tag{3}$$

where $r_i \neq 0 (i \neq 1)$, *and ϕ satisfies* (37)-(38), *where* $k_1 = k_1(b^2)$ *if* $\sigma \neq 0$. *In this case, c is a constant and β is closed.*

(iv)

$$r_{ij} = \sigma(k_1 a_{ij} + k_2 b_i b_j) + \frac{\tau}{b^2}(b_i s_j + b_j s_i), \ s_{ij} = \frac{1}{b^2}(b_i s_j - b_j s_i), \tag{4}$$

where $\tau := \frac{1-kb^2}{kb^2}$ *and* $s_i \neq 0(i \neq 1)$, *and ϕ satisfies* (26), (29) *and* (34). *In this case,* $k \neq 0$ *(i.e.,* $c \neq$ *const.) and β is not closed.*

The notations in Theorem 1.1, such as, Q, R, Ψ, Π, T and r_{ij}, s_{ij}, r_i, s_i etc. can be found in the next section. If β is parallel with respect to α, then F is a Berwald space with same geodesics as the underlying Riemannian space (M, α). If $F = \alpha\phi(b^2, s)$ is an (α, β)-metric, that is, ϕ is independent of b^2, then $R(s) = \Pi(s) = 0$ and $c = \phi_2(0)$ is a constant. From this, we know that (37) does not hold. Thus only (i) in (1) and (i)-(ii) in (2) might occur. For the case (ii) in (2), we have $k_1 t^2 + s^2 = 0$ by $\Psi(s) \neq \Psi(-s)$, $R(s) = \Pi(s) = 0$ and (32). This is impossible. Thus, the case (2)(ii) can not happen. Combining (1)(i) with (2)(i) gives the following result, which was due to Masca-Sabau-Shimada.

Corollary 1.2 ([7]). *Let $F = \alpha\phi\left(\frac{\beta}{\alpha}\right)$ be a nonreversible (α, β)-metric on an $n(\geq 3)$-dimensional manifold M. Then F is with reversible geodesics if and only if one of the following situations happens.*

(1) *β is parallel with respect to α.*

(2) *β is closed but not parallel with respect to α, and $\phi(s) = \phi(-s) + 2cs$, where c is a nonzero constant.*

It is worth mentioning that Corollary 1.2 is exactly Theorem 3.1 in [7], in which ϕ satisfies $\phi(s) = k_1\phi(-s) + k_2 s$, where $k_1 \neq 0$ and k_2 are constants. In fact, $k_1 = 1$ by letting $s = 0$ in the previous equation. In particular, when $\dim M \geq 3$, $F = \alpha + \beta$ is a Randers metric with reversible geodesics if and only if β is closed ([5]).

Corollary 1.3. *Let $F = \alpha\phi\left(b^2, s\right)$, $s = \beta/\alpha$, be a nonreversible general (α, β)-metric on an $n(\geq 3)$-dimensional manifold M. Assume that β is closed. Then F is with reversible geodesics if and only if there is a local coordinate system, in which there are scalar functions $k_1(x)$ and $\sigma(x)$, such that ϕ, α and β satisfy one of the following cases.*

(1) *β is parallel with respect to α.*

(2) $\phi(b^2, s) = \phi(b^2, -s) + 2cs$, *where c is a nonzero constant.*

(3) $\phi(b^2, s) = \phi(b^2, -s) + 2cs$, *where* c *is a nonconstant function of* b *and* $b = \|\beta\|_\alpha$ *is a constant along the direction perpendicular to* β *with respect to* α,

(4) ϕ *satisfies* (32) *and* $\phi(b^2, s) \neq \phi(b^2, -s) + 2cs$, *and* $r_{ij} = k_1\sigma a_{ij} + \frac{\sigma}{b^2}(1 - k_1)b_i b_j(\sigma \neq 0)$, *where* $c = c(b^2)$ *is a function of* b^2.

(5) ϕ *satisfies* (37)-(38) *and* $\phi(b^2, s) \neq \phi(b^2, -s) + 2cs$, *and* α, β *satisfy* $r_{ij} = k_1\sigma a_{ij} - \frac{\sigma}{b^2}(1+k_1)b_i b_j + \frac{1}{b^2}(r_i b_j + r_j b_i)$, *where* c *is a constant,* $r_i \neq 0 (i \neq 1)$, *and* $k_1 = k_1(b^2)$ *when* $\sigma \neq 0$.

The following example shows that there are many nonreversible Finsler metrics with reversible geodesics except for reversible Finsler metrics.

Example 1.4. Let

$$\phi(b^2, s) = s^{2n} + a_{2n-2}s^{2n-2} + \cdots + a_2 s^2 + a_1 s + a_0, \tag{5}$$

where $a_0 > 0$, $a_1 \neq 0$, $a_i = a_i(b^2)(0 \leq i \leq 2n-2)$ are smooth functions of one variable b^2 such that (8) is satisfied. For example, $\phi(b^2, s) = s^2 + a_1 s + kb^2$ satisfies (8), where $k \geq 1$ is a constant and $a_1 = a_1(b^2)$ is a nonzero function. Then ϕ defined by (5) satisfies $\phi(b^2, s) = \phi(b^2, -s) + 2a_1 s$. Moreover, for any $x, y \in \mathbb{R}^n$, let

$$\alpha = \frac{|y|}{2|x|}, \quad \beta = 2e^{-|x|^2}\langle x, y\rangle, \tag{6}$$

where $\langle\ ,\ \rangle$ is an Euclidean inner product in $\mathbb{R}$ and $|\cdot|$ means the length with respect to $\langle\ ,\ \rangle$. Thus, $b_i = 2e^{-|x|^2}x_i$ and $b^2 = 16|x|^4 e^{-2|x|^2}$. By a direct calculation, we have $b_{i|j} = \frac{1-|x|^2}{|x|^2}e^{|x|^2}b_i b_j$. In this case, β is closed. By Theorem 1.1 or Corollary 1.3, $F = \alpha\phi(b^2, \beta/\alpha)$ defined by (5)-(6) is a general (α, β)-metric with reversible geodesics.

2. Preliminaries

Let F be a Finsler metric on an n-dimensional smooth manifold M and $(x, y) = (x^i, y^i)$ the local coordinates on the tangent bundle TM. Let $g_y = g_{ij}(x, y)dx^i \otimes dx^j$ be a fundamental tensor of F, where $g_{ij} = \frac{1}{2}[F^2]_{y^i y^j}$, and

$$G^i(x, y) = \frac{1}{4}g^{il}\left\{[F^2]_{x^k y^l}y^k - [F^2]_{x^l}\right\}$$

are called the *geodesic coefficients* of F, where $(g^{ij}) := (g_{ij})^{-1}$.

Recall that a C^∞ curve $x : [a, b] \to M$ is called a *geodesic* of the Finsler metric F if it minimizes the Finslerian length for all piecewise C^∞ curves that keep their end points fixed ([1]). The geodesics on a Finsler manifold (M, F) are determined by the following ODE:

$$\ddot{x} + 2G^i(x, \dot{x}) = 0.$$

Two Finsler metrics F and $\tilde{F}$ are said to be *projectively equivalent* if they have the same positive geodesics as a point set. Equivalently, their geodesic coefficients G^i and $\tilde{G}^i$ are related by

$$G^i(x,y) = \tilde{G}^i(x,y) + Py^i, \tag{7}$$

where $P = P(x,y)$ is a positive y-homogeneous function of degree one. In particular, if $\tilde{F}$ is Euclidean, then F is *locally projectively flat*, namely, there is a local coordinate system (U, x^i) in M such that all geodesics on U are straight lines. In this case, $G^i = Py^i$ and P is called the *projective factor* of F ([4]).

General (α,β)-metrics form a more abroad class of Finsler metrics, which can be expressed in the form

$$F = \alpha\phi(b^2, s), \quad s = \frac{\beta}{\alpha},$$

where α and β are a Riemannian metric and a 1-form on M respectively, and $\phi = \phi(b^2, s)$ is a positive smooth function of two variables $b^2 = \|\beta\|_\alpha^2$ and s. Then $F = \phi(b^2, s)$ is a regular Finsler metric if and only if $\phi(b^2, s)$ satisfies

$$\phi - s\phi_2 > 0, \quad \phi - s\phi_2 + (b^2 - s^2)\phi_{22} > 0, \qquad |s| \le b < b_0 \tag{8}$$

when $n \ge 3$ ([12]), where ϕ_2 and ϕ_{22} mean the first derivative and the second derivative of ϕ with respect to the second variable s. Similarly, we shall use ϕ_1, ϕ_{11} to denote the first and second derivatives of ϕ with respect to the first variable b^2, and ϕ_{12} to denote the second mixed derivative of ϕ. By (8), it is easy to see that $\phi(b^2, s)$ can not be an odd function in s for general (α,β)-metrics.

Let $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ and $\beta = b_i(x)y^i$. Denote by $b_{i|j}$ the coefficients of the covariant derivative of β with respect to α, and let

$$r_{ij} = \frac{1}{2}(b_{i|j} + b_{j|i}),\ s_{ij} = \frac{1}{2}(b_{i|j} - b_{j|i}),\ r_{00} = r_{ij}y^iy^j,\ s^i{}_0 = a^{ij}s_{jk}y^k,$$

$$r_i = b^jr_{ji},\ s_i = b^js_{ji},\ r_0 = r_iy^i,\ s_0 = s_iy^i,\ r^i = a^{ij}r_j,\ s^i = a^{ij}s_j,\ r = b^ir_i.$$

It is easy to see that β is closed if and only if $s_{ij} = 0$.

According to [12], the spray coefficients G^i of a general (α,β)-metric $F = \alpha\phi(b^2, s)$ are related to the spray coefficients ${}^\alpha G^i$ of α and given by

$$\begin{aligned} G^i = G^i_\alpha &+ \alpha Q s^i{}_0 + \left\{\Theta(-2\alpha s_0 Q + r_{00} + 2r\alpha^2 R) + \alpha(r_0 + s_0)\Omega\right\}\frac{y^i}{\alpha} \\ &+ \left\{\Psi(-2\alpha s_0 Q + r_{00} + 2r\alpha^2 R) + \alpha\Pi(r_0 + s_0)\right\}b^i - \alpha^2 R(r^i + s^i), \end{aligned} \tag{9}$$

where

$$Q = \frac{\phi_2}{\phi - s\phi_2}, \qquad R = \frac{\phi_1}{\phi - s\phi_2},$$

$$\Theta = \frac{(\phi - s\phi_2)\phi_2 - s\phi\phi_{22}}{2\phi\big(\phi - s\phi_2 + (b^2 - s^2)\phi_{22}\big)}, \qquad \Psi = \frac{\phi_{22}}{2\big(\phi - s\phi_2 + (b^2 - s^2)\phi_{22}\big)},$$

$$\Pi = \frac{(\phi - s\phi_2)\phi_{12} - s\phi_1\phi_{22}}{(\phi - s\phi_2)\big(\phi - s\phi_2 + (b^2 - s^2)\phi_{22}\big)}, \quad \Omega = \frac{2\phi_1}{\phi} - \frac{s\phi + (b^2 - s^2)\phi_2}{\phi}\Pi.$$

For the sake of simplicity, let $t^2 := b^2 - s^2$ and

$$T(b^2, s) := \frac{\phi_{12}}{\phi - s\phi_2 + t^2\phi_{22}}.$$

Then $\Pi = T - 2sR\Psi$. Also, we shall simply use $\phi = \phi(s)$, $Q = Q(s)$, $R = R(s)$, $\Psi(s)$ and $\Pi(s)$ instead of $\phi(b^2, s)$, $Q(b^2, s)$, $R(b^2, s)$, $\Psi(b^2, s)$, $\Pi(b^2, s)$ respectively throughout the paper. Moreover, let

$$\begin{aligned}
L_0 :=& \, Q(s) + Q(-s),\\
L_1 :=& -2s_0\alpha\left[Q(s)\Theta(s) - Q(-s)\Theta(-s)\right] + r_{00}\left[\Theta(s) + \Theta(-s)\right]\\
&+ 2r\alpha^2\left[R(s)\Theta(s) + R(-s)\Theta(-s)\right] + \alpha(r_0 + s_0)\left[\Omega(s) - \Omega(-s)\right],\\
L_2 :=& -2s_0\alpha\left[Q(s)\Psi(s) + Q(-s)\Psi(-s)\right] + r_{00}\left[\Psi(s) - \Psi(-s)\right]\\
&+ 2r\alpha^2\left[R(s)\Psi(s) - R(-s)\Psi(-s)\right] + \alpha(r_0 + s_0)\left[\Pi(s) + \Pi(-s)\right],\\
L_3 :=& \, \alpha^2\left[R(s) - R(-s)\right].
\end{aligned}$$

It follows from (1) and (9) that F is a Finsler metric with reversible geodesics if and only if

$$\alpha L_0 s^i{}_0 + \alpha^{-1}L_1 y^i + L_2 b^i - L_3(r^i + s^i) = Py^i \tag{10}$$

for some homogeneous function $P = P(x, y)$ of degree one in y.

Assume that F is with reversible geodesics. Then we have (10). Contracting this with $y_i := a_{ij}y^j$ yields

$$P = \alpha^{-2}\left[\alpha L_1 + \beta L_2 - (r_0 + s_0)L_3\right]. \tag{11}$$

Plugging this into (10) gives

$$\alpha^3 L_0 s^i{}_0 + L_2(\alpha^2 b^i - \beta y^i) + L_3\left[(r_0 + s_0)y^i - \alpha^2(r^i + s^i)\right] = 0. \tag{12}$$

By contracting this with $b_i := a_{ij}b^j$, one obtains

$$L_2 = C_1 L_3 + C_2 L_0, \tag{13}$$

where

$$C_1 := \frac{r - s\alpha^{-1}(r_0 + s_0)}{t^2}, \qquad C_2 := -\frac{\alpha s_0}{t^2}.$$

Inserting (13) into (12) yields

$$\begin{aligned}
&L_3\left[C_1(\beta y^i - \alpha^2 b^i) - (r_0 + s_0)y^i + \alpha^2(r^i + s^i)\right]\\
&= L_0\left[\alpha^3 s^i{}_0 + C_2(\alpha^2 b^i - \beta y^i)\right].
\end{aligned} \tag{14}$$

Conversely, assume that (13) and (14) hold. Then we have

$$\begin{aligned}
L_2 b^i + \alpha s^i{}_0 L_0 &= C_1 L_3 b^i + C_2 L_0 b^i + \alpha s^i{}_0 L_0\\
&= \alpha^{-2}\left\{C_1 L_3 \beta y^i - L_3(r_0 + s_0)y^i + \alpha^2 L_3(r^i + s^i) + C_2 L_0 \beta y^i\right\}.
\end{aligned}$$

Consequently, by (9), we get

$$G^i(x, y) - G^i(x, -y) = \alpha s^i{}_0 L_0 + \alpha^{-1}L_1 y^i + L_2 b^i - L_3(r^i + s^i)$$

$$= \alpha^{-2}\left\{C_1L_3\beta - L_3(r_0+s_0) + C_2L_0\beta + \alpha L_1\right\}y^i.$$

Consequently, there exists a homogeneous function

$$P = \alpha^{-2}\left\{C_1L_3\beta - L_3(r_0+s_0) + C_2L_0\beta + \alpha L_1\right\}$$

of degree one in y such that (1) holds, which means that F is with reversible geodesics. This proves the following:

Proposition 2.1. *Let $F = \alpha\phi(b^2, s)$ be a general (α, β)-metric on a manifold M. Then F is with reversible geodesics if and only if both* (13) *and* (14) *hold.*

3. Some lemmas

In this section, we first give some lemmas to be used in the next section.

Lemma 3.1. *$Q(b^2, s) + Q(b^2, -s) = 0$ if and only if $\phi(b^2, s)$ is even in s. In this case, $\phi_2(b^2, 0) = 0$.*

Proof. The sufficiency is obvious. Conversely, assume that $Q(s) + Q(-s) = 0$, which is equivalent to

$$\phi_2(s)\phi(-s) + \phi(s)\phi_2(-s) = 0.$$

Consequently, there exists a function $c = c(b^2)$ only depending on b^2 such that $\phi(s) = c(b^2)\phi(-s)$. Letting $s = 0$ yields $c(b^2) = 1$. Thus, $\phi(s) = \phi(-s)$. □

It is easy to obtain the following corollary from Lemma 3.1.

Corollary 3.2. *If $Q(b^2, s)+Q(b^2, -s) = 0$, then $\Psi(b^2, s) = \Psi(b^2, -s)$, $R(b^2, s) = R(b^2, -s)$ and $\Pi(b^2, s) = -\Pi(b^2, -s)$.*

Lemma 3.3. *$\Psi(b^2, s) - \Psi(b^2, -s) = 0$ if and only if $\phi(b^2, s) = \phi(b^2, -s) + 2s\phi_2(b^2, 0)$.*

Proof. Assume that $\Psi(s) = \Psi(-s)$, which is equivalent to

$$\phi_{22}(s)[\phi(-s) + s\phi_2(-s)] - \phi_{22}(-s)(\phi - s\phi_2)(s) = 0.$$

Note that $\frac{d}{ds}[\phi(s) - s\phi_2(s)] = -s\phi_{22}(s)$. The above equation implies $\phi - s\phi_2 = c(b^2)[\phi(-s) + s\phi_2(-s)]$, where $c(b^2)$ is a function independent of s. Let $s = 0$, we have $c(b^2) = 1$ since $\phi(0) > 0$. Thus, $\phi - \phi(-s) = s(\phi_2 + \phi_2(-s))$, which means that there is a function $\bar{c}(b^2)$ independent of s such that $\phi(s) - \phi(-s) = 2s\bar{c}(b^2)$. Differentiating this with respect to s and letting $s = 0$ leads to $\bar{c}(b^2) = \phi_2(0)$. Thus, $\phi(s) = \phi(-s) + 2s\phi_2(0)$. Conversely, a direct calculation gives the conclusion. □

Corollary 3.4. *If $\Psi(b^2, s) = \Psi(b^2, -s)$, then*

$$Q(b^2, s) + Q(b^2, -s) = \frac{2\phi_2(b^2, 0)}{\phi - s\phi_2},\quad R(b^2, s) - R(b^2, -s) = \frac{2s\phi_{12}(b^2, 0)}{\phi - s\phi_2}, \tag{15}$$

$$R(b^2, s) - R(b^2, -s) - s[\Pi(b^2, s) + \Pi(b^2, -s)] = \frac{4sb^2\phi_{12}(b^2, 0)}{\phi - s\phi_2}\Psi(b^2, s). \tag{16}$$

Proof. Since $\Psi(s) = \Psi(-s)$, we have $\phi(s) = \phi(-s) + 2s\phi_2(0)$ by Lemma 3.3. From this, we have

$$\phi_2(s) = -\phi_2(-s) + 2\phi_2(0), \quad \phi_{22}(s) = \phi_{22}(-s),$$
$$\phi_1(s) = \phi_1(-s) + 2s\phi_{12}(0), \quad \phi_{12}(s) = -\phi_{12}(-s) + 2\phi_{12}(0).$$

Thus,

$$\begin{aligned}\phi - s\phi_2 &= \phi(-s) + 2s\phi_2(0) - s[-\phi_2(-s) + 2\phi_2(0)]\\ &= \phi(-s) + s\phi_2(-s),\end{aligned}$$

and

$$\phi - s\phi_2 + t^2\phi_{22} = \phi(-s) + s\phi_2(-s) + t^2\phi_{22}(-s).$$

From these, one obtains (15). Since

$$\Pi(s) = \frac{\phi_{12}}{\phi - s\phi_2 + t^2\phi_{22}} - 2sR\Psi,$$

we have

$$\begin{aligned}\Pi(s) + \Pi(-s) &= \frac{\phi_{12} + \phi_{12}(-s)}{\phi - s\phi_2 + t^2\phi_{22}} - 2sR\Psi + 2sR(-s)\Psi(-s)\\ &= \frac{2\phi_{12}(0)}{\phi - s\phi_2 + t^2\phi_{22}} - 2s\Psi[R - R(-s)].\end{aligned}$$

Consequently,

$$\begin{aligned}&R(s) - R(-s) - s[\Pi(s) + \Pi(-s)]\\ &= \frac{-2s\phi_{12}(0)}{\phi - s\phi_2 + t^2\phi_{22}} + (1 + 2s^2\Psi)[(R - R(-s)]\\ &= \frac{-2s\phi_{12}(0)}{\phi - s\phi_2 + t^2\phi_{22}} + \frac{\phi - s\phi_2 + b^2\phi_{22}}{\phi - s\phi_2 + t^2\phi_{22}} \cdot \frac{2s\phi_{12}(0)}{\phi - s\phi_2}\\ &= \frac{2sb^2\phi_{12}(0)\phi_{22}}{(\phi - s\phi_2)(\phi - s\phi_2 + t^2\phi_{22})} = \frac{4sb^2\phi_{12}(0)}{\phi - s\phi_2}\Psi.\end{aligned}$$

□

Lemma 3.5. *Let* $R(s) = R(-s)$. *Then*

$$\Pi(s) + \Pi(-s) = T(s) + T(-s) - 2sR[\Psi(s) - \Psi(-s)].$$

Proof. It directly follows from $\Pi(s) = T(s) - 2sR(s)\Psi(s)$ and $R(s) = R(-s)$. □

4. Proof of Theorem 1.1

4.1. Necessity

In this subsection, we try to find the necessary conditions for general (α, β)-metrics with reversible geodesics.

Proposition 4.1. *Under the same assumptions as in Theorem 1.1, if* $F = \alpha\phi(b^2, s)$ *is with reversible geodesics, then there is a local coordinate system, in which there are scalar functions* $k(b^2) := -\frac{\phi_{12}(b^2,0)}{\phi_2(b^2,0)}$, $\sigma(x)$ *and* $k_i(x)(1 \le i \le 3)$ *with* $k_1 + b^2k_2 = 1$ *and* $k_1 + b^2k_3 = -1$, *such that* ϕ, α *and* β *satisfy one of the cases from* (1)(i) *to* (2)(iv) *in Theorem 1.1.*

Proof. For any point $p \in M$, we take an orthogonal basis on a neighbourhood $U \subset M$ of p such that

$$\alpha = \sqrt{\sum_{i=1}^{n}(y^i)^2}, \quad \beta = by^1.$$

Denote $\bar{\alpha} = \sqrt{\sum_{a=2}^{n}(y^a)^2}$ and $t = \sqrt{b^2 - s^2}$ as before. Thus we have $y^1 = s\bar{\alpha}/t$. In the following, we use the indices conventions as follows unless the otherwise specified.

$$1 \le i, j, k, l, \ldots \le n, \quad 2 \le a, b, c, d, \ldots \le n.$$

Hence we can choose the coordinate transformation: $(s, y^a) \to (y^i)$ on U defined by

$$y^1 = \frac{s}{t}\bar{\alpha}, \quad y^a = y^a.$$

In the new coordinates system $\{(s, y^a)\}$, we have

$$\alpha = \frac{b}{t}\bar{\alpha}, \quad \beta = \frac{bs}{t}\bar{\alpha}.$$

Moreover, $r_1 = br_{11}, r_a = br_{1a}, r = b^2r_{11}$ and $s_a = bs_{1a}$. Let

$$\bar{r}_{10} = r_{1a}y^a, \qquad \bar{r}_{00} = r_{ab}y^ay^b, \qquad \bar{r}_0 = r_ay^a,$$
$$\bar{s}_{10} = s_{1a}y^a, \qquad \bar{s}_0 = s_ay^a, \qquad \bar{s}^i{}_0 = y^as^i{}_a.$$

Then

$$r_{00} = \frac{s^2}{t^2}r_{11}\bar{\alpha}^2 + \frac{2s}{t}\bar{r}_{10}\bar{\alpha} + \bar{r}_{00},$$
$$r_{10} = \frac{s}{t}r_{11}\bar{\alpha} + \bar{r}_{10}, \quad r_0 = \frac{bs}{t}r_{11}\bar{\alpha} + b\bar{r}_{10},$$
$$s_{a0} = \frac{s}{t}s_{a1}\bar{\alpha} + \bar{s}_{a0}, \quad s_{10} = \bar{s}_{10}, \quad s_0 = \bar{s}_0.$$

Since F is with reversible geodesics, we have (13) and (14) by Proposition 2.1. In local coordinates $\{(s, y^a)\}$, (14) becomes

$$\begin{aligned} &[R(s) - R(-s)]\{tb^2\bar{\alpha}^2(r^i + s^i) - tb^2(\bar{r}_0 + \bar{s}_0)y^i - b\bar{\alpha}[tr_1\bar{\alpha} - s(\bar{r}_0 + \bar{s}_0)]b^i\} \\ &= [Q(s) + Q(-s)]\{st\bar{s}_0y^i - b\bar{\alpha}\bar{s}_0b^i + tsb\bar{\alpha}^2s^i{}_1 + t^2b\bar{\alpha}\bar{s}^i{}_0\}. \end{aligned} \tag{17}$$

Observe that (17) holds identically for $i = 1$. Now we consider the case when $i = a(2 \le a \le n)$. In this case, (17) can be rewritten as

$$[R(s) - R(-s)][b^2\bar{\alpha}^2(r^a + s^a) - b^2(\bar{r}_0 + \bar{s}_0)y^a] \tag{18}$$

$$= [Q(s)+Q(-s)][s\bar{s}_0 y^a + sb\bar{\alpha}^2 s^a{}_1 + tb\bar{\alpha}\bar{s}^a{}_0].$$

On the other hand, from the definitions of L_0, L_2, L_3, (13) is rewritten as

$$\begin{aligned} & -2s_0\alpha\left[Q(s)\Psi(s)+Q(-s)\Psi(-s)\right]+r_{00}\left[\Psi(s)-\Psi(-s)\right] \\ & +2r\alpha^2\left[R(s)\Psi(s)-R(-s)\Psi(-s)\right]+\alpha(r_0+s_0)\left[\Pi(s)+\Pi(-s)\right] \\ = {} & C_1\alpha^2\left[R(s)-R(-s)\right]+C_2\left[Q(s)+Q(-s)\right]. \end{aligned}$$

In local coordinates $\{(s, y^a)\}$, the above equation becomes

$$\begin{aligned} & -2bt^2\bar{s}_0\left[Q(s)\Psi(s)+Q(-s)\Psi(-s)\right]\bar{\alpha} \\ & +t\left[\Psi(s)-\Psi(-s)\right]\left[s^2 r_{11}\bar{\alpha}^2+2st\bar{r}_{10}\bar{\alpha}+t^2\bar{r}_{00}\right] \\ & +2tb^4 r_{11}\left[R(s)\Psi(s)-R(-s)\Psi(-s)\right]\bar{\alpha}^2 \\ & +t\left[\Pi(s)+\Pi(-s)\right]\left[sb^2 r_{11}\bar{\alpha}^2+bt(\bar{r}_0+\bar{s}_0)\bar{\alpha}\right] \\ = {} & [R(s)-R(-s)][tr_{11}b^2\bar{\alpha}^2-bs(\bar{r}_0+\bar{s}_0)\bar{\alpha}]-b\bar{s}_0[Q(s)+Q(-s)]\bar{\alpha}. \end{aligned}$$

Note that α is irrational. The above equation can be decomposed as

$$\begin{aligned} & -2t^2b\bar{s}_0\left[\Psi(s)Q(s)+\Psi(-s)Q(-s)\right]+2st^2\bar{r}_{10}\left[\Psi(s)-\Psi(-s)\right] \\ & +t^2b(\bar{r}_0+\bar{s}_0)[\Pi(s)+\Pi(-s)] \\ = {} & -bs(\bar{r}_0+\bar{s}_0)[R(s)-R(-s)]-b\bar{s}_0[Q(s)+Q(-s)], \end{aligned} \tag{19}$$

and

$$\begin{aligned} & t^2\left[\Psi(s)-\Psi(-s)\right]\bar{r}_{00} \\ = {} & -r_{11}\Big\{s^2[\Psi(s)-\Psi(-s)]+2b^4\left[R(s)\Psi(s)-R(-s)\Psi(-s)\right] \\ & +sb^2\left[\Pi(s)+\Pi(-s)\right]-b^2\left[R(s)-R(-s)\right]\Big\}\bar{\alpha}^2. \end{aligned} \tag{20}$$

Next we discuss the equations (18), (19) and (20) according to the different cases.

If $Q(s)+Q(-s)=0$, then $\phi(b^2,s)=\phi(b^2,-s)$ by Lemma 3.1. From this and Corollary 3.2, the equations (18), (19) and (20) hold identically regardless of the choices of α and β. In this case, F is reversible, which is excluded.

Now we consider the case when $Q(s)+Q(-s)\neq 0$. In this case, $\phi(b^2,s)$ is not even in s and $s^a{}_b = s_{ab} = 0$ from (18) because of the irrationality of α. Thus, (18) is simplified as

$$\begin{aligned} & b^2[R(s)-R(-s)][\bar{\alpha}^2(r_a+s_a)-(\bar{r}_0+\bar{s}_0)y^a] \\ = {} & s[Q(s)+Q(-s)](\bar{s}_0 y^a - s_a\bar{\alpha}^2). \end{aligned} \tag{21}$$

Differentiating this with respect to y^b, y^c respectively yields

$$\begin{aligned} & b^2[R(s)-R(-s)][2\delta_{bc}(r_a+s_a)-\delta^a_c(r_b+s_b)-(r_c+s_c)\delta^a_b] \\ = {} & s[Q(s)+Q(-s)][s_b\delta^a_c+s_c\delta^a_b-2\delta_{bc}s_a], \end{aligned}$$

which implies that

$$b^2[R(s)-R(-s)](r_a+s_a)=-s[Q(s)+Q(-s)]s_a \tag{22}$$

by letting $b=c\neq a$, here we used the assumption that $n\geq 3$. Hence there exists a function $k(x)$ such that

$$s_a=b^2k(x)(r_a+s_a), \tag{23}$$

and

$$\{R(s)-R(-s)+sk[Q(s)+Q(-s)]\}(r_a+s_a)=0. \tag{24}$$

Case I. $\Psi(s)=\Psi(-s)$.

In this case, $\phi(b^2,s)=\phi(b^2,-s)+2cs$ by Lemma 3.3, where $c=\phi_2(b^2,0)\neq 0$ since $\phi(b^2,s)$ is not even in s. Hence (20) holds identically by Corollary 3.4. Consequently, r_{ab} are arbitrary functions. Moreover, (19) is reduced to

$$\begin{aligned}\Big\{b^2k\left[1-2t^2\Psi(s)\right][Q(s)+Q(-s)]&+s[R(s)-R(-s)]\\ &+t^2[\Pi(s)+\Pi(-s)]\Big\}(\bar{r}_0+\bar{s}_0)=0,\end{aligned} \tag{25}$$

where we used (23). Let $\sigma:=r_{11}(x)$ in the following.

If $r_a+s_a=0$, then $s_{1a}=r_{1a}=b_{1|a}=0$ because of (23), which implies that β is closed and $b=\|\beta\|_\alpha$ is a constant in the direction perpendicular to β with respect to α. Obviously, $r_j=\sigma b_j$ and $r=\sigma b^2$. In this case, (24)-(25) hold automatically.

If $r_a+s_a\neq 0$, then we get

$$R(s)-R(-s)=-sk[Q(s)+Q(-s)], \tag{26}$$

$$k\left[1-2b^2\Psi(s)\right][Q(s)+Q(-s)]+\Pi(s)+\Pi(-s)=0 \tag{27}$$

from (24)-(25). Differentiating (26) with respect to s and letting $s=0$ yields

$$k=k(b^2)=-\frac{\phi_{12}(b^2,0)}{\phi_2(b^2,0)}.$$

Further, when $k=0$, we have $\phi_{12}(b^2,0)=0$, which implies that c is a nonzero constant. By Corollary 3.4, (26)-(27) are always satisfied. Besides this, we have $s_a=0$ from (23) and hence $r_a\neq 0$. Thus, β is closed but $r_{1a}\neq 0$. When $k\neq 0$, equivalently, c is not a constant, we get $r_a=\tau s_a$ and $r_{1a}=\frac{\tau}{b}s_a$ from (23), where $\tau(b^2):=\frac{1-kb^2}{kb^2}$. Obviously, $s_a\neq 0$. It is easy to check that (26)-(27) are satisfied automatically from Corollary 3.4 and the definitions of k,Ψ.

Together with the above two cases on r_a+s_a, we always have $s_{11}=s_{ab}=0$, $s_{1a}=\frac{1}{b}s_a$, $r_{11}=\sigma$ and $r=\sigma b^2$. Moreover, either $s_a=0$ when c is a nonzero constant, or $r_a=\tau s_a$ when c is not a constant. In the latter case, back to the original coordinate system, we have

$$r_{ij}=\frac{\sigma}{b^2}b_ib_j+\frac{\tau}{b^2}(b_is_j+b_js_i)+\sum_{k,l\neq 1}r_{kl}\delta_i^k\delta_j^l,\quad s_{ij}=\frac{1}{b^2}(b_is_j-b_js_i) \tag{28}$$

with $\sum_{k\neq 1} r_{kl}b^k = 0 (l \neq 1)$. Consequently, one obtains the case (1)(i) or (1)(ii) in Theorem 1.1.

Case 2. $\Psi(s) \neq \Psi(-s)$.

In this case, $\phi(b^2, s) \neq \phi(b^2, -s) + 2cs$ by Corollary 3.4, where $c = \phi_2(b^2, 0)$. Moreover, by (20), there exists a function $k_1(x)$ such that $\bar{r}_{00} = k_1(x)\sigma\bar{\alpha}^2$ and hence $r_{ab} = k_1\sigma\delta_{ab}$. Inserting this into (20) yields

$$(29) \qquad \sigma\Big\{(k_1t^2 + s^2)[\Psi(s) - \Psi(-s)] + 2b^4[R(s)\Psi(s) - R(-s)\Psi(-s)] - b^2\,[R(s) - R(-s)] + sb^2\,[\Pi(s) + \Pi(-s)]\Big\} = 0.$$

If $r_a + s_a = 0$, then $s_a = r_a = r_{1a} = s_{1a} = 0$ by (23), which implies that β is closed and $b = \|\beta\|_\alpha$ is a constant along the direction perpendicular to β with respect to α. Back to the original coordinate system, we have

$$(30) \qquad r_{ij} = k_1\sigma a_{ij} + \frac{\sigma}{b^2}(1 - k_1)b_ib_j.$$

Note that (18)-(19) hold identically. Thus ϕ only satisfies (29). In particular, when $\sigma = 0$, β is parallel with respect to α regardless of the choice of ϕ except $\Psi(s) \neq \Psi(-s)$. When $\sigma \neq 0$, (α, β) satisfies (30) and ϕ satisfies

$$(31) \qquad (k_1t^2 + s^2)[\Psi(s) - \Psi(-s)] + 2b^4[R(s)\Psi(s) - R(-s)\Psi(-s)] - b^2\,[R(s) - R(-s)] + sb^2\,[\Pi(s) + \Pi(-s)] = 0,$$

which is equivalent to

$$(32) \qquad (k_1t^2 + s^2)[\Psi(s) - \Psi(-s)] + 2b^2t^2[R(s)\Psi(s) - R(-s)\Psi(-s)] - b^2\,[R(s) - R(-s)] + sb^2\,[T(s) + T(-s)] = 0,$$

where we used $\Pi = T - 2sR\Psi$.

If $r_a + s_a \neq 0$, then we get (26) from (24), where $k = k(b^2) = -\frac{\phi_{12}(b^2,0)}{\phi_2(b^2,0)}$. By (23), we have $\bar{s}_0 = kb^2(\bar{r}_0 + \bar{s}_0)$. From this, we have

$$(33) \qquad \bar{r}_0 = (1 - kb^2)(\bar{r}_0 + \bar{s}_0).$$

Plugging these into (19) and using (26), we get

$$(34) \qquad -2b^4k[\Psi(s)Q(s) + \Psi(-s)Q(-s)] + 2s(1 - kb^2)[\Psi(s) - \Psi(-s)] + b^2[\Pi(s) + \Pi(-s)] + kb^2[Q(s) + Q(-s)] = 0.$$

Further, when $k = 0$, i.e., c is a nonzero constant, we have $s_a = s_{1a} = 0$ and $r_a \neq 0$. By (26), we have $R(s) = R(-s)$. From (34) and (29), one obtains

$$(35) \qquad 2s[\Psi(s) - \Psi(-s)] + b^2[\Pi(s) + \Pi(-s)] = 0,$$

$$(36) \qquad \sigma\Big\{\left(k_1t^2 + s^2 + 2b^4R\right)[\Psi(s) - \Psi(-s)] + sb^2[\Pi(s) + \Pi(-s)]\Big\} = 0,$$

which are equivalent to

$$(37) \qquad 2s(1 - b^2R)[\Psi(s) - \Psi(-s)] + b^2[T(s) + T(-s)] = 0,$$

$$(38) \qquad \sigma\left(k_1t^2 - s^2 + 2b^4R\right) = 0$$

by Lemma 3.5. Note that (37) implies $R(s) = R(-s)$, and $k_1 = k_1(b^2)$ by letting $s = 0$ in (38) when $\sigma \neq 0$. By now, we have $s_{ij} = 0$, $r_{11}(x) = \sigma(x)$, $r_{1a} = \frac{1}{b} r_a \neq 0$, $r_{ab} = k_1 \sigma \delta_{ab}$ and $r = \sigma b^2$. Back to the original coordinate system, we have

$$r_{ij} = k_1 \sigma a_{ij} - \frac{\sigma}{b^2}(1 + k_1) b_i b_j + \frac{1}{b^2}(r_i b_j + r_j b_i), \tag{39}$$

where r_i and $k_1(x)$ are smooth functions with $r_i \neq 0 (i \neq 1)$. Moreover, ϕ satisfies (37) and (38), where $k_1 = k_1(b^2)$ when $\sigma \neq 0$.

When $k \neq 0$, i.e., c is not a constant, we have $r_a = \tau s_a$ by (23), where $\tau = \tau(b^2) := \frac{1 - kb^2}{kb^2}$. Obviously, $s_a \neq 0$. Similarly, in the original coordinate system, we have

$$r_{ij} = k_1 \sigma a_{ij} + \frac{\sigma(1 - k_1)}{b^2} b_i b_j + \frac{\tau}{b^2}(b_i s_j + b_j s_i), \quad s_{ij} = \frac{1}{b^2}(b_i s_j - b_j s_i), \tag{40}$$

where $s_i \neq 0 (i \neq 1)$. In this case, β is not closed and ϕ satisfies (26), (29) and (34).

Let $k_2 := b^{-2}(1 - k_1)$ and $k_3 := -b^{-2}(1 + k_1)$. Then $k_1 + b^2 k_2 = 1$ and $k_1 + b^2 k_3 = -1$. From the above arguments, one obtains the cases from (2)(i) to (2)(iv) in Theorem 1.1. This finishes the proof. □

4.2. Sufficiency

In this subsection, we prove the necessary conditions in Proposition 4.1 are also sufficient. Note that these conditions are valid in dimension two. Thus we obtain the proof of Theorem 1.1.

Proposition 4.2. *Let $F = \alpha \phi(b^2, s)$ be a general (α, β)-metrics on an $n(\geq 2)$-dimensional manifold M. Suppose that ϕ, α and β satisfy one of the cases from* (1)(i) *to* (2)(iv) *in Theorem 1.1. Then F is a Finsler metric with reversible geodesics.*

Proof. It suffices to check that the equation (10) holds according to each case.

Case 1. Assume that ϕ, α and β satisfy (1) in Theorem 1.1. Then $Q(s) \neq Q(-s)$ and $\Psi(s) = \Psi(-s)$ by Lemmas 3.1-3.3. Further,

(i) by the assumption, we have $s_0 = s^i{}_0 = 0$ and $\phi_{12}(b^2, 0) = 0$, which implies that $R(s) = R(-s)$ and $\Pi(s) + \Pi(-s) = 0$ by Corollary 3.2. Thus $L_2 = L_3 = 0$. Consequently, (10) holds, where $P = \alpha^{-1} L_1$. This means that F is with reversible geodesics.

(ii) since $c = \phi_2(b^2, 0)$ is not a constant, we have $\phi_{12}(b^2, 0) \neq 0$ and hence $k \neq 0$. It follows from Corollary 3.4 that

$$\begin{aligned} s[\Pi(s) + \Pi(-s)] &= (1 - 2b^2 \Psi)[R(s) - R(-s)], \\ R(s) - R(-s) &= -sk[Q(s) + Q(-s)]. \end{aligned}$$

Moreover, by (2), we have

$$r_j = \sigma b_j + \tau s_j, \quad r = \sigma b^2, \quad s^i{}_0 = \frac{1}{b^2}(s_0 b^i - s \alpha s^i).$$

With these, one obtains

$$L_2 b^i = -\left(sk\sigma\alpha^2 + b^{-2}s_0\alpha\right) b^i[Q(s)+Q(-s)],$$
$$-L_3(r^i+s^i) = \left[sk\sigma\alpha^2 b^i + sb^{-2}s^i\right)\alpha^2[Q(s)+Q(-s)].$$

Thus $L_2b^i - L_3(r^i+s^i) = -\alpha L_0 s^i{}_0$, which implies that (10) is true, where $P = \alpha^{-1}L_1$. So F is with reversible geodesics.

Case 2. Assume that ϕ, α and β satisfy (2) in Theorem 1.1. Then $Q(s) \neq Q(-s)$ and $\Psi(s) \neq \Psi(-s)$ by Lemmas 3.1 and 3.3. Further,

(i) if β is parallel with respect to α, then $r_{ij} = s_{ij} = 0$. It is obvious that (10) is true. So, F is with reversible geodesics.

(ii) if ϕ satisfies (32) and (α,β) satisfies $r_{ij} = \sigma(k_1a_{ij} + k_2b_ib_j)(\sigma \neq 0)$, then ϕ satisfies (31) and

$$r_{00} = \frac{\sigma}{b^2}(k_1t^2+s^2)\alpha^2, \quad r_j = \sigma b_j, \quad r = \sigma b^2, \quad s_{ij} = s^i{}_0 = s_0 = 0.$$

By the definition of L_2 and L_3, one obtains

$$\begin{aligned} & L_2 - \sigma L_3 \\ = {} & \frac{\sigma}{b^2}\alpha^2\Big\{(k_1t^2+s^2)[\Psi(s)-\Psi(-s)] + 2b^4[R(s)\Psi(s) - R(-s)\Psi(-s)] \\ & - b^2[R(s)-R(-s)] + sb^2[\Pi(s)+\Pi(-s)]\Big\} = 0 \end{aligned}$$

by (31). Thus (10) holds and F is with reversible geodesics.

(iii) under the assumption of (iii), we have $r_{00} = \frac{\sigma}{b^2}(k_1t^2 - 2s^2)\alpha^2 + \frac{2s}{b^2}r_0\alpha$, $r = \sigma b^2$ and $s^i{}_0 = s_0 = 0$. Moreover, since $\Pi = T - 2sR\Psi$, ϕ satisfies

$$2s[\Psi(s)-\Psi(-s)] + b^2[\Pi(s)+\Pi(-s)] = 0,$$
$$R(s) = R(-s), \quad \sigma(k_1t^2 - s^2 + 2b^4R) = 0.$$

With these, we have $L_3 = 0$, and

$$L_2 = \frac{\sigma}{b^2}(k_1t^2 - s^2 + 2b^4R)[\Psi(s)-\Psi(-s)]\alpha^2 = 0.$$

Thus (10) also holds and hence F is with reversible geodesics.

(iv) if c is not a constant, then $k \neq 0$. Assume that (4) holds, and ϕ satisfies (26), (29) and (34). Then $r_j = \sigma b_j + \tau s_j$, $r = \sigma b^2$ and

$$r_{00} = \frac{\sigma}{b^2}(k_1t^2+s^2)\alpha^2 + \frac{2s\tau}{b^2}s_0\alpha, \quad s^i{}_0 = \frac{1}{b^2}(s_0b^i - s\alpha s^i),$$

where $s_0 \neq 0$. From these, we have

$$\begin{aligned} & \alpha L_0 s^i{}_0 + L_3 b^i - L_3(r^i+s^i) \\ = {} & -\frac{\alpha^2}{kb^2}\left[ks(Q(s)+Q(-s)) + R(s) - R(-s)\right]s^i \\ & + \frac{s_0\alpha}{kb^4}\Big\{kb^2[Q(s)+Q(-s)] - 2kb^4[Q(s)\Psi(s) + Q(-s)\Psi(-s)] \\ & + 2s(1-kb^2)[\Psi(s)-\Psi(-s)] + b^2[\Pi(s)+\Pi(-s)]\Big\}b^i \end{aligned}$$

$$+\frac{\sigma\alpha^2}{b^2}\Big\{(k_1t^2+s^2)[\Psi(s)-\Psi(-s)]+2b^4[R(s)\Psi(s)-R(-s)\Psi(-s)]$$
$$+sb^2[\Pi(s)+\Pi(-s)]-b^2[R(s)-R(-s)]\Big\}b^i=0.$$

Consequently, (10) holds and F is with reversible geodesics. This ends the proof. □

Proof of Theorem 1.1. It directly follows from Propositions 4.1-4.2. □

References

[1] D. Bao, S.-S. Chern, and Z. Shen, *An introduction to Riemann-Finsler geometry*, Graduate Texts in Mathematics, **200**, Springer-Verlag, New York, 2000. https://doi.org/10.1007/978-1-4612-1268-3

[2] R. L. Bryant, *Projectively flat Finsler 2-spheres of constant curvature*, Selecta Math. (N.S.) **3** (1997), no. 2, 161–203. https://doi.org/10.1007/s000290050009

[3] ———, *Geodesically reversible Finsler 2-spheres of constant curvature*, in Inspired by S. S. Chern, 95–111, Nankai Tracts Math., 11, World Sci. Publ., Hackensack, NJ, 2006. https://doi.org/10.1142/9789812772688_0004

[4] S.-S. Chern and Z. Shen, *Riemann-Finsler geometry*, Nankai Tracts in Mathematics, **6**, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2005. https://doi.org/10.1142/5263

[5] M. Crampin, *Randers spaces with reversible geodesics*, Publ. Math. Debrecen **67** (2005), no. 3-4, 401–409.

[6] Y. Feng and Q. Xia, *On a class of locally projectively flat Finsler metrics (II)*, Differential Geom. Appl. **62** (2019), 39–59. https://doi.org/10.1016/j.difgeo.2018.09.004

[7] I. M. Masca, V. S. Sabau, and H. Shimada, *Reversible geodesics for (α,β)-metrics*, Internat. J. Math. **21** (2010), no. 8, 1071–1094. https://doi.org/10.1142/S0129167X10006355

[8] ———, *Two dimensional (α,β)-metrics with reversible geodesics*, Publ. Math. Debrecen **82** (2013), no. 2, 485–501. https://doi.org/10.5486/PMD.2013.5494

[9] X. Mo and H. Zhu, *On a class of locally projectively flat general (α,β)-metrics*, Bull. Korean Math. Soc. **54** (2017), no. 4, 1293–1307. https://doi.org/10.4134/BKMS.b160551

[10] Q. Xia, *On a class of projectively flat Finsler metrics*, Differential Geom. Appl. **44** (2016), 1–16. https://doi.org/10.1016/j.difgeo.2015.10.002

[11] ———, *On a class of Finsler metrics of scalar flag curvature*, Results Math. **71** (2017), no. 1-2, 483–507. https://doi.org/10.1007/s00025-016-0539-6

[12] C. Yu and H. Zhu, *On a new class of Finsler metrics*, Differential Geom. Appl. **29** (2011), no. 2, 244–254. https://doi.org/10.1016/j.difgeo.2010.12.009

Qiaoling Xia
Department of Mathematics, School of Sciences
Hangzhou Dianzi University
Hangzhou, Zhejiang Province, 310018, P. R. China
Email address: xiaqiaoling@hdu.edu.cn