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GRADIENT ESTIMATES FOR ELLIPTIC EQUATIONS IN DIVERGENCE FORM WITH PARTIAL DINI MEAN OSCILLATION COEFFICIENTS

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Abstract. We provide detailed proofs for local gradient estimates for elliptic equations in divergence form with partial Dini mean oscillation coefficients in a ball and a half ball.

1. Introduction and main results

We consider second-order elliptic equations in divergence form

$$\operatorname{div}(\mathbf{A}Du) = \operatorname{div}\boldsymbol{f} \tag{1.1}$$

with coefficient $\mathbf{A}$ and data $\boldsymbol{f}$ which are irregular in one direction. The regularity theory for this type of equations has important applications in the problems of linearly elastic laminates and composite materials; see [7]. It is known that any weak solution u to (1.1) satisfies $Du \in L^p_{\mathrm{loc}}$ $(1 < p < \infty)$ provided that $\mathbf{A}$ is merely measurable in one direction and has small mean oscillation in the other directions, and that $\boldsymbol{f} \in L^p$; see, for instance, [3, 9, 10].

In a recent paper [5], the first named author and Dong studied L^∞-theory for stationary Stokes systems in divergence form. They proved that any weak solution to the Stokes system has bounded gradient provided that the coefficients and data satisfy *partial Dini mean oscillation condition.* We shall say that a locally integrable function is of partial Dini mean oscillation if its L^1-mean oscillation with respect to $x' = (x_2, \ldots, x_d)$ (after a rotation) satisfies the Dini condition; see Definition 1.1 for more precise definition. As remarked in [5], the corresponding regularity result can be established for elliptic equations other than Stokes systems.

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In this paper, we provide a detailed proof for both interior and boundary L^∞-regularity of weak solutions to the elliptic equation (1.1) with the coefficient and data which are of partial Dini mean oscillation. This may not be a surprising result to experts, but it still demands some effort and caution to deal with regularity theory for elliptic equations with irregular coefficients in one direction. Thus, we anticipate that our paper fills a gap in the literature and serves as a good reference to non-experts.

To state our main results more precisely, we introduce some notation and definitions. We use $x = (x_1, x')$ to denote a generic point in $\mathbb{R}^d$ ($d \geq 2$); it should be understood that $x_1 \in \mathbb{R}$ and $x' = (x_2, \ldots, x_d) \in \mathbb{R}^{d-1}$. We also write $y = (y_1, y')$ and $x_o = (x_{o1}, x'_o)$, etc. We denote

$$B_r(x) = \{y \in \mathbb{R}^d : |x - y| < r\},$$
$$B'_r(x') = \{y' \in \mathbb{R}^{d-1} : |x' - y'| < r\}.$$

In other words, $B_r(x)$ and $B'_r(x')$ are the usual Euclidean balls in $\mathbb{R}^d$ and $\mathbb{R}^{d-1}$, respectively. We also denote

$$B_r^+(x) = B_r(x) \cap \mathbb{R}^d_+, \quad \text{where } \mathbb{R}^d_+ = \{x = (x_1, x') \in \mathbb{R}^d : x_1 > 0\}.$$

We warn the readers that $B_r^+(x)$ is not necessarily a half ball. We shall often use the abbreviations B_r, B_r^+, and B'_r when the center is the origin. We write $D_{x'}u = (D_2 u, \ldots, D_d u)$ so that $Du = (D_1 u, D_{x'}u)$ and

$$(u)_\Omega = \fint_\Omega u \, dx = \frac{1}{|\Omega|} \int_\Omega u \, dx,$$

where $|\Omega|$ denotes the Lebesgue measure of a measurable set $\Omega \subset \mathbb{R}^d$.

Definition 1.1. (a) Let $f \in L^1(B_6)$. We say that f is of *partial Dini mean oscillation with respect to x' in B_4* if the function $\omega_f : (0,1] \to [0, \infty)$ defined by

$$\omega_f(r) := \sup_{x \in B_4} \fint_{B_r(x)} \left| f(y) - \fint_{B'_r(x')} f(y_1, z') \, dz' \right| dy$$

satisfies the Dini condition

$$\int_0^1 \frac{\omega_f(r)}{r} \, dr < \infty.$$

(b) Let $f \in L^1(B_6^+)$. We say that f is of *partial Dini mean oscillation with respect to x' in B_4^+* if the function $\omega_f : (0,1] \to [0, \infty)$ defined by

$$\omega_f^+(r) := \sup_{x \in B_4^+} \fint_{B_r^+(x)} \left| f(y) - \fint_{B'_r(x')} f(y_1, z') \, dz' \right| dy$$

satisfies the Dini condition

$$\int_0^1 \frac{\omega_f^+(r)}{r} \, dr < \infty.$$

The main results of the paper are as follows. Let $\mathcal{L}$ be a differential operator in divergence form

$$\mathcal{L}u = \operatorname{div}(\mathbf{A}Du) = D_i(a^{ij}D_ju),$$

where we use the Einstein summation convention on repeated indices. The coefficient $\mathbf{A} = (a^{ij})_{i,j=1}^d$ is a $d \times d$ matrix-valued function in $\mathbb{R}^d$, which satisfies the strong ellipticity condition, i.e., there is a constant $\lambda \in (0,1]$ such that for any $x \in \mathbb{R}^d$ and $\xi \in \mathbb{R}^d$, we have

$$|\mathbf{A}(x)| \le \lambda^{-1}, \quad a^{ij}(x)\xi_j\xi_i \ge \lambda|\xi|^2. \tag{1.2}$$

For $p \in [1,\infty)$, we say that $u \in W^{1,p}(\Omega)$ is a weak solution of $\mathcal{L}u = \operatorname{div} \boldsymbol{f} + g$ in a domain Ω if

$$\int_\Omega a^{ij}D_juD_i\phi\,dx = \int_\Omega (f_iD_i\phi - g\phi)\,dx$$

for any $\phi \in C_o^\infty(\Omega)$.

Theorem 1.2. *Let $p \in (1,\infty)$ and $u \in W^{1,p}(B_6)$ be a weak solution of*

$$\mathcal{L}u = \operatorname{div}\boldsymbol{f} \quad \text{in } B_6,$$

where $\boldsymbol{f} = (f_1,\dots,f_d) \in L^p(B_6)^d$ and $f_1 \in L^\infty(B_6)$. If $\mathbf{A}$ and $\boldsymbol{f}$ are of partial Dini mean oscillation with respect to x' in B_4, then we have

$$u \in W^{1,\infty}(B_1).$$

Moreover, $\hat{U} = a^{1j}D_ju - f_1$ and $D_{x'}u$ are continuous in $\overline{B}_1$.

Remark 1.3. One can extend the results in Theorem 1.2 to weak solutions of

$$\mathcal{L}u = \operatorname{div}\boldsymbol{f} + g \quad \text{in } B_6,$$

where $g \in L^q(B_6)$ with $q > d$. Indeed, by [6, Lemma 3.1], there exists $\boldsymbol{G} \in W^{1,q}(B_6)^d$ such that $\operatorname{div}\boldsymbol{G} = g$ in B_6, which implies that u satisfies

$$\mathcal{L}u = \operatorname{div}(\boldsymbol{f} + \boldsymbol{G}) \quad \text{in } B_6.$$

Moreover, by the Morrey inequality, we have that $\boldsymbol{G} \in C^\alpha(\overline{B}_6)^d$ with $\alpha = 1 - d/q$, and thus, $\boldsymbol{G}$ is of partial Dini mean oscillation.

Remark 1.4. Due to Theorem 3.2 with scaling and covering arguments, we see that Theorem 1.2 still holds for every $W^{1,1}$-weak solutions.

Theorem 1.5. *Let $p \in (1,\infty)$ and $u \in W^{1,p}(B_6^+)$ be a weak solution of*

$$\begin{cases} \mathcal{L}u = \operatorname{div}\boldsymbol{f} & \text{in } B_6^+,\\ \qquad u = 0 & \text{on } B_6 \cap \partial\mathbb{R}^d_+, \end{cases}$$

where $\boldsymbol{f} = (f_1,\dots,f_d) \in L^p(B_6^+)^d$ and $f_1 \in L^\infty(B_6^+)$. If $\mathbf{A}$ and $\boldsymbol{f}$ are of partial Dini mean oscillation with respect to x' in B_4^+, then we have

$$u \in W^{1,\infty}(B_1^+).$$

Moreover, $\hat{U} = a^{1j}D_ju - f_1$ and $D_{x'}u$ are continuous in $\overline{B}_1^+$.

Remark 1.6. By the same reasoning as in Remarks 1.3 and 1.4, one can extend the results in Theorem 1.5 to $W^{1,1}$-weak solutions of

$$\begin{cases} \mathcal{L}u = \operatorname{div} \boldsymbol{f} + g & \text{in } B_6^+, \\ u = 0 & \text{on } B_6 \cap \partial\mathbb{R}^d_+, \end{cases}$$

where $g \in L^q(B_6^+)$ with $q > d$.

Upper bounds of the L^∞-norm of Du and the modulus of continuity of $\hat{U}$ and $D_{x'}u$ can be found in Sections 2.2 and 2.3. By using the upper bounds and Remark 1.4 together with the fact that partially Hölder continuous functions are of partial Dini mean oscillation, one can obtain a partial Schauder estimate for $W^{1,1}$-weak solutions; see Remark 2.8 for more discussions. We note that the partial Schauder estimate for elliptic equations was studied long time ago by Fife [21]. See also [7, 12, 14, 17, 23, 24] and the references therein for some recent work in this direction.

We say that a function is of Dini mean oscillation if its "full" mean oscillation satisfies the Dini condition. It is recently shown in [13] that if the coefficients are of Dini mean oscillation, then solutions to divergence and nondivergence form elliptic equations satisfy interior C^1 and C^2 estimates, respectively. See also [8] for the corresponding regularity results up to the boundary. Note that any function that is of Dini mean oscillation is of partial Dini mean oscillation with respect to any directions. For further discussions about equations/systems with Dini mean oscillation coefficients, see [4, 15, 16]. We also refer the reader to [18, 19] for elliptic equations in divergence and nondivergence form with piecewise Dini mean oscillation coefficients.

The remainder of the paper is organized as follows. In Section 2, we provide the proofs of the main results, namely, Theorems 1.2 and 1.5. In Section 3, we give applications of the main theorems to weak type-$(1,1)$ estimates and $W^{1,p}$-estimates for $W^{1,1}$-weak solutions.

We finish this section with a remark that our results can be extended to strongly elliptic systems because their proofs do not use any scalar structure.

2. Proofs of main theorems

Throughout the paper, we use the following notation.

Notation 2.1. For nonnegative (variable) quantities A and B, we denote $A \lesssim B$ if there exists a generic positive constant C such that $A \le CB$. We add subscript letters like $A \lesssim_{a,b} B$ to indicate the dependence of the implicit constant C on the parameters a and b.

2.1. Preliminary lemmas

In this subsection, we prove some preliminary results which will be used in the proof of Theorem 1.2. We define

$$\mathcal{L}_0 u = \operatorname{div}(\bar{\mathbf{A}} Du),$$

where $\bar{\mathbf{A}} = \bar{\mathbf{A}}(x_1) = (\bar{a}^{ij}(x_1))_{i,j=1}^d$ are functions of x_1 satisfying the strong ellipticity condition (1.2).

The following lemma is about Lipschitz estimates of u and linear combinations of $D_i u$ for $W^{1,2}$-weak solutions to $\mathcal{L}_0 u = 0$ in a ball. Such a regularity result is known to experts, and it can be proved by following the arguments used in deriving [11, Lemma 3.5], where the authors obtained Hölder estimates for linear combinations of derivatives of smooth solutions. See also [19, Lemma 2.7]. In this paper, for the sake of completeness, we provide a proof of the lemma.

Lemma 2.1. *If $u \in W^{1,2}(B_{2r})$ satisfies*

$$\mathcal{L}_0 u = 0 \quad \text{in } B_{2r},$$

then we have

$$\|Du\|_{L^\infty(B_r)} \lesssim_{d,\lambda} r^{-d/2} \|Du\|_{L^2(B_{2r})}, \tag{2.1}$$

$$[U]_{C^{0,1}(B_r)} + [D_{x'} u]_{C^{0,1}(B_r)} \lesssim_{d,\lambda} r^{-d/2-1} \|Du\|_{L^2(B_{2r})}, \tag{2.2}$$

where $U := \bar{a}^{1j} D_j u$ and

$$[U]_{C^{0,1}(B_r)} := \sup_{\substack{x,y\in B_r \\ x\neq y}} \frac{|U(x) - U(y)|}{|x-y|}.$$

Proof. We first prove (2.1). By scaling, it suffices to consider the case of $r = 1$. We let $1 \le \rho_1 < \rho_2 < 2$ and $i \in \{2, \ldots, d\}$. Since the coefficient of $\mathcal{L}_0$ is a function of only x_1, we see that

$$\mathcal{L}_0(D_i u) = 0 \quad \text{in } B_{\rho_2},$$

and thus by Caccioppoli's inequality,

$$\|DD_i u\|_{L^2(B_{\rho_1})} \lesssim_{d,\lambda,\rho_1,\rho_2} \|D_i u\|_{L^2(B_{\rho_2})}.$$

By repeating this process, we have that for $k \in \{0, 1, \ldots\}$,

$$\|D_{x'}^k u\|_{L^2(B_{\sqrt{2}})} + \|D_1 D_{x'}^k u\|_{L^2(B_{\sqrt{2}})} \lesssim_{d,\lambda,k} \|Du\|_{L^2(B_2)}, \tag{2.3}$$

where $D_{x'}^k$ denotes partial differentiation of order k with respect to x'. By the Sobolev imbedding theorem, $D_{x'} u(x_1, x')$, as a function of $x_1 \in (-1, 1)$, satisfies

$$\sup_{x_1 \in (-1,1)} |D_{x'} u(x_1, x')|^2 \lesssim \int_{-1}^{1} |D_{x'} u(y_1, x')|^2 + |D_1 D_{x'} u(y_1, x')|^2 \, dy_1.$$

On the other hand, there exists a positive integer k such that $D_{x'} u(y_1, x')$ and $D_1 D_{x'} u(y_1, x')$ as a function of $x' \in B_1'$, satisfy

$$\begin{aligned} &\sup_{x' \in B_1'} \left(|D_{x'} u(y_1, x')|^2 + |D_1 D_{x'} u(y_1, x')|^2 \right) \\ &\lesssim \|D_{x'} u(y_1, \cdot)\|_{W^{k,2}(B_1')}^2 + \|D_1 D_{x'} u(y_1, \cdot)\|_{W^{k,2}(B_1')}^2. \end{aligned}$$

Combining these together and using (2.3), we get

$$\|D_{x'}u\|_{L^\infty(B_1)} \le \|D_{x'}u\|_{L^\infty((-1,1)\times B_1')} \lesssim \|Du\|_{L^2(B_2)}. \tag{2.4}$$

Since $\bar{a}^{ij} = \bar{a}^{ij}(x_1)$, from the equation $\mathcal{L}_0 u = 0$, we have

$$D_1 U = -\sum_{i=2}^{d}\sum_{j=1}^{d} \bar{a}^{ij} D_{ij} u, \tag{2.5}$$

which together with (2.3) implies that $D_1 U$ has sufficiently many derivatives in x' with the estimates

$$\|D_{x'}^k U\|_{L^2(B_{\sqrt{2}})} + \|D_1 D_{x'}^k U\|_{L^2(B_{\sqrt{2}})} \lesssim_{d,\lambda,k} \|Du\|_{L^2(B_2)}$$

for any $k \in \{0,1,\ldots\}$. Thus, repeating the same argument as above, we have

$$\|U\|_{L^\infty(B_1)} \lesssim \|Du\|_{L^2(B_2)}. \tag{2.6}$$

Notice from the definition of U that

$$|D_1 u| = \frac{1}{\bar{a}^{11}}\left| U - \sum_{j=2}^{d} \bar{a}^{1j} D_j u \right| \lesssim_{d,\lambda} |U| + |D_{x'}u|. \tag{2.7}$$

Taking $\|\cdot\|_{L^\infty(B_1)}$ of both sides of the above inequality and using (2.4) and (2.6), we obtain that

$$\|Du\|_{L^\infty(B_1)} \lesssim \|Du\|_{L^2(B_2)}.$$

We have proved (2.1). We note that by the standard covering argument, we can replace B_2 by $B_{3/2}$ in the above.

We now turn to the proof of (2.2). Since $\mathcal{L}_0(D_{x'}u) = 0$ in B_2, we by the above to get

$$\|DD_{x'}u\|_{L^\infty(B_1)} \lesssim \|DD_{x'}u\|_{L^2(B_{3/2})} \lesssim \|Du\|_{L^2(B_2)},$$

where we used Caccioppoli's inequality in the second inequality. This together with (2.5) yields that

$$\|D_1 U\|_{L^\infty(B_1)} + \|D_{x'}U\|_{L^\infty(B_1)} \lesssim \|Du\|_{L^2(B_2)}.$$

The lemma is proved. □

Lemma 2.2. *Let $u \in W^{1,2}(B_R)$ satisfy $\operatorname{div}(\bar{\mathbf{A}} Du) = 0$ in B_R and set $U := \bar{a}^{1j} D_j u$. Fix any $q \in (0,2]$. Then, for any $r \in (0, \frac{1}{2}R]$ and $\boldsymbol{\theta} = (\theta_1, \theta') \in \mathbb{R}^d$, we have*

$$\left(\fint_{B_r} |U - (U)_{B_r}|^q + |D_{x'}u - (D_{x'}u)_{B_r}|^q \, dx\right)^{1/q}$$
$$\lesssim_{d,\lambda,q} \frac{r}{R}\left(\fint_{B_R} |U - \theta_1|^q + |D_{x'}u - \theta'|^q \, dx\right)^{1/q}.$$

Proof. We first claim that

$$\left(\fint_{B_r} |U-(U)_{B_r}|^q + |D_{x'}u-(D_{x'}u)_{B_r}|^q\,dx\right)^{1/q} \lesssim_{d,\lambda,q} \frac{r}{R}\left(\fint_{B_R}|U|^q+|D_{x'}u|^q\,dx\right)^{1/q}. \tag{2.8}$$

By (2.1) and a well-known covering argument (see e.g., [22, pp. 80–82]), we have

$$\|Du\|_{L^\infty(B_{2R/3})} \lesssim_{d,\lambda,q} R^{-d/q}\|Du\|_{L^q(B_R)}.$$

From this together with (2.2), it follows that

$$\begin{aligned}[U]_{C^{0,1}(B_{R/2})}+[D_{x'}u]_{C^{0,1}(B_{R/2})} &\lesssim R^{-d/2-1}\|Du\|_{L^2(B_{2R/3})}\\ &\lesssim R^{-d/2-1}\|Du\|_{L^\infty(B_{2R/3})}^{(2-q)/2}\|Du\|_{L^q(B_{2R/3})}^{q/2}\\ &\lesssim R^{-d/q-1}\|Du\|_{L^q(B_R)}.\end{aligned}$$

Thus we have (recall $r\le R/2$)

$$\left(\fint_{B_r}|U-(U)_{B_r}|^q+|D_{x'}u-(D_{x'}u)_{B_r}|^q\,dx\right)^{\frac{1}{q}} \lesssim \frac{r}{R}\left(\fint_{B_R}|Du|^q\,dx\right)^{1/q}.$$

Since it holds that (using (2.7))

$$|Du|\lesssim_{d,\lambda}|U|+|D_{x'}u|,$$

we conclude (2.8).

We are ready to prove the lemma. For any given $\boldsymbol{\theta}=(\theta_1,\ldots,\theta_d)\in\mathbb{R}^d$, we set

$$u_0=u_0(x_1)=\frac{1}{\bar a^{11}}\Big(\theta_1-\sum_{j=2}^d \bar a^{1j}\theta_j\Big).$$

A direct calculation shows that the function u_e given by

$$u_e=u_e(x_1,x')=u-\int_0^{x_1}u_0\,dy_1-\sum_{i=2}^d\theta_i x_i$$

satisfies $\mathcal{L}_0u_e=0$ in B_R. Therefore, applying (2.8) to u_e and using the fact that

$$U_e=\bar a^{1j}D_ju_e=U-\theta_1,\quad D_iu_e=D_iu-\theta_i,\quad i\in\{2,\ldots,d\},$$

we have

$$\left(\fint_{B_r}|U-(U)_{B_r}|^q+|D_{x'}u-(D_{x'}u)_{B_r}|^q\,dx\right)^{1/q} \lesssim\frac{r}{R}\left(\fint_{B_R}|U-\theta_1|^q+|D_{x'}u-\theta'|^q\,dx\right)^{1/q}.$$

The lemma is proved. $\square$

The following lemma is used to obtain weak type-$(1,1)$ estimates.

Lemma 2.3. *Let T be a bounded linear operator from $L^2(B_{R_0})^k$ to $L^2(B_{R_0})^k$, where $R_0 \geq 4$ and $k \in \{1,2,\ldots\}$. Suppose that there exist positive constants $\mu < 1$, $c > 1$, and C such that for any $x_o \in B_1$, $0 < r < \mu$, and*

$$\boldsymbol{g} \in L^2(B_{R_0})^k \ \text{ with } \int_{B_{R_0}} \boldsymbol{g}\, dx = 0, \quad \operatorname{supp} \boldsymbol{g} \subset B_r(x_o) \cap B_1,$$

we have

$$\int_{B_1 \setminus B_{cr}(x_o)} |T\boldsymbol{g}|\, dx \leq C \int_{B_r(x_o)\cap B_1} |\boldsymbol{g}|\, dx. \tag{2.9}$$

Then the following hold true.

(a) *For any $\boldsymbol{f} \in L^2(B_{R_0})^k$ with $\operatorname{supp} f \subset B_1$ and $t > 0$, we have*

$$\left|\{x \in B_1 : |T\boldsymbol{f}(x)| > t\}\right| \lesssim_{d,k,\mu,c,C} \frac{1}{t} \int_{B_1} |\boldsymbol{f}|\, dx.$$

(b) *There exists a linear operator S from $L^1(B_1)^k$ to $L^1(B_1)^k$ such that for any $\boldsymbol{f} \in L^2(B_1)^k$,*

$$S\boldsymbol{f} = T\bar{\boldsymbol{f}} \quad \text{in } B_1,$$

where $\bar{\boldsymbol{f}}$ is the zero extension of $\boldsymbol{f}$ on $B_{R_0} \setminus B_1$. Moreover, for any $\boldsymbol{f} \in L^1(B_1)^k$ and $t > 0$, we have

$$\left|\{x \in B_1 : |S\boldsymbol{f}(x)| > t\}\right| \lesssim_{d,k,\mu,c,C} \frac{1}{t} \int_{B_1} |\boldsymbol{f}|\, dx. \tag{2.10}$$

In other words, T has an extension on the set

$$\{\boldsymbol{f} \in L^1(B_{R_0})^k : \operatorname{supp} \boldsymbol{f} \subset B_1\}$$

to the weak $L^1(B_1)$ space in such a way that for any $t > 0$, we have

$$\left|\{x \in B_1 : |T\boldsymbol{f}(x)| > t\}\right| \lesssim_{d,k,\mu,c,C} \frac{1}{t} \int_{B_1} |\boldsymbol{f}|\, dx.$$

Proof. The assertion (a) follows by applying [8, Lemma 4.1] with $L^2(B_1)^k$ in place of $L^2(\Omega)$ to the operator S on $L^2(B_1)^k$ given by

$$S\boldsymbol{f} = T(\boldsymbol{f}\chi_{B_1})\chi_{B_1}.$$

The assertion (b) is from [5, Lemma 3.3]. Indeed, for a given $\boldsymbol{f} \in L^1(B_1)^k$, one can find a sequence $\{\boldsymbol{f}_n\} \subset L^2(B_1)^k$ such that $\|\boldsymbol{f}_n - \boldsymbol{f}\|_{L^1(B_1)} \to 0$ as $n \to \infty$. Then by the assertion (a), the sequence $\{T(\boldsymbol{f}_n\chi_{B_1})\}$ is Cauchy in measure in B_1 and its limit, denoted by $S\boldsymbol{f}$, is measurable in B_1. By the hypothesis of the lemma, S also satisfies (2.9) for $\boldsymbol{g} \in L^1(B_1)^k$ with $\int_{B_1} \boldsymbol{g}\, dx = 0$ and $\operatorname{supp} \boldsymbol{g} \subset B_r(x_o) \cap B_1$, and thus, by following the proof of [8, Lemma 4.1] we see that S satisfies (2.10). $\square$

We finish this subsection by establishing a weak type-$(1,1)$ estimate for Du.

Lemma 2.4. *If $u \in W_0^{1,2}(B_4)$ satisfies $\mathcal{L}_0 u = \operatorname{div} \tilde{\boldsymbol{f}}$ in B_4, where $\tilde{\boldsymbol{f}} \in L^2(B_4)^d$ is supported in B_1, then for any $t > 0$, we have*

$$|\{x \in B_1 : |Du(x)| > t\}| \lesssim_{d,\lambda} \frac{1}{t} \int_{B_1} |\tilde{\boldsymbol{f}}| \, dx.$$

Proof. Consider a mapping $\tilde{\boldsymbol{f}} \mapsto \boldsymbol{f}$ given by

$$f_1 = (\bar{a}^{11})^{-1} \tilde{f}_1, \quad f_i = \tilde{f}_i - \bar{a}^{i1} f_1, \quad i \in \{2, \ldots, d\}.$$

We define a bounded linear operator T on $L^2(B_4)^d$ by setting $T\boldsymbol{f} = Du$, where $u \in W_0^{1,2}(B_4)$ is a unique weak solution of

$$\mathcal{L}_0 u = \operatorname{div} \tilde{\boldsymbol{f}}. \tag{2.11}$$

To prove the lemma, we only need to show that the operator T satisfies the hypothesis of Lemma 2.3 with $k = d$, $R_0 = 4$, $\mu = 1/2$, $c = 2$, and $C = C(d, \lambda) > 0$.

Fix $x_o \in B_1$ and $r \in (0, 1/2)$. Let $u \in W_0^{1,2}(B_4)$ satisfy (2.11) with $\boldsymbol{f} \in L^2(B_4)^d$ satisfying

$$\int_{B_4} \boldsymbol{f} \, dx = 0, \quad \operatorname{supp} \boldsymbol{f} \subset B_r(x_o) \cap B_1. \tag{2.12}$$

Note that (2.12) is the condition required in the hypothesis of Lemma 2.3, and it is used to get (2.16) below. Then for any $\phi \in W_0^{1,2}(B_4)$, it holds that

$$\int_{B_4} \bar{a}^{ij} D_j u D_i \phi \, dx = \int_{B_4} \bar{a}^{11} f_1 D_1 \phi \, dx + \sum_{i=2}^{d} \int_{B_4} (f_i + \bar{a}^{i1} f_1) D_i \phi \, dx. \tag{2.13}$$

Take $R \in [2r, 2)$ so that $B_1 \setminus B_R(x_o) \neq \emptyset$, and let $\boldsymbol{g} \in C_o^\infty(B_4)^d$ be supported in $(B_{2R}(x_o) \setminus B_R(x_o)) \cap B_1$. Then by the Lax-Milgram theorem, there exists a unique $v \in W_0^{1,2}(B_4)$ satisfying

$$\mathcal{L}_0^* v = \operatorname{div} \boldsymbol{g} \quad \text{in } B_4, \tag{2.14}$$

where $\mathcal{L}_0^*$ is the adjoint operator of $\mathcal{L}_0$; i.e., $\mathcal{L}_0^* v = D_i(\bar{a}^{ji} D_j v)$. Moreover, by the energy estimate we have

$$\|Dv\|_{L^2(B_4)} \lesssim_{d,\lambda} \|\boldsymbol{g}\|_{L^2((B_{2R}(x_o) \setminus B_R(x_o)) \cap B_1)}. \tag{2.15}$$

By setting $\phi = v$ in (2.13) and testing (2.14) with u, we have the identity

$$\int_{B_4} Du \cdot \boldsymbol{g} \, dx = \int_{B_4} f_1 V \, dx + \sum_{i=2}^{d} \int_{B_4} f_i D_i v \, dx,$$

where $V = \bar{a}^{j1} D_j v$. From this together with (2.12), it follows that

$$\begin{aligned} \int_{B_4} Du \cdot \boldsymbol{g} \, dx = & \int_{B_r(x_o)} f_1 \left(V - (V)_{B_r(x_o)}\right) \, dx \\ & + \sum_{i=2}^{d} \int_{B_r(x_o)} f_i \left(D_i v - (D_i v)_{B_r(x_o)}\right) \, dx. \end{aligned} \tag{2.16}$$

On the other hand, since v satisfies $\mathcal{L}_0^* v = 0$ in $B_R(x_o)$, (2.2) implies

$$[V]_{C^{0,1}(B_r(x_o))} + [D_{x'} v]_{C^{0,1}(B_r(x_o))} \lesssim R^{-d/2-1} \|Dv\|_{L^2(B_R(x_o))} \\ \lesssim R^{-d/2-1} \|\boldsymbol{g}\|_{L^2((B_{2R}(x_o) \setminus B_R(x_o)) \cap B_1)},$$

where we used (2.15) in the second inequality. Combining these together,

$$\left| \int_{(B_{2R}(x_o) \setminus B_R(x_o)) \cap B_1} Du \cdot \boldsymbol{g} \, dx \right| \\ \lesssim rR^{-d/2-1} \|\boldsymbol{f}\|_{L^1(B_r(x_o))} \|\boldsymbol{g}\|_{L^2((B_{2R}(x_o) \setminus B_R(x_o)) \cap B_1)}.$$

Thus by the duality and Hölder's inequality, we have

$$\int_{(B_{2R}(x_o) \setminus B_R(x_o)) \cap B_1} |Du| \, dx \lesssim rR^{-1} \|\boldsymbol{f}\|_{L^1(B_r(x_o))}.$$

For $i \in \{1, \ldots, N-1\}$, where N is the smallest positive integer such that $B_1 \subset B_{2^N r}(x_o)$, we set $R = 2^i r$. Then we obtain

$$\int_{B_1 \setminus B_{2r}(x_o)} |Du| \, dx \lesssim \sum_{i=1}^{N-1} 2^{-i} \|\boldsymbol{f}\|_{L^1(B_r(x_o))} \lesssim \|\boldsymbol{f}\|_{L^1(B_r(x_o))},$$

which implies that the operator T satisfies the hypothesis of Lemma 2.3. The lemma is proved. □

2.2. Proof of Theorem 1.2

In this subsection, we assume that the hypotheses in Theorem 1.2 hold. We shall derive a priori estimates for u under the assumption that the coefficient $\mathbf{A}$ and data $\boldsymbol{f}$ are sufficiently smooth so that $u \in C^1(\overline{B}_4)$. Then the general case follows from an approximation argument (see [7, pp. 134–135]) and $W_0^{1,p}$-solvability of elliptic equations (see Theorem 3.4).

Throughout the proof, we shall denote

$$\boldsymbol{U} = (\hat{U}, D_{x'} u), \quad \text{where } \hat{U} = a^{1j} D_j u - f_1$$

and define

$$\Phi(x_o, r) := \inf_{\boldsymbol{\theta} \in \mathbb{R}^d} \left(\fint_{B_r(x_o)} |\boldsymbol{U} - \boldsymbol{\theta}|^{\frac{1}{2}} \, dx \right)^2,$$

To prove Theorem 1.2, we will use the following decay estimate for $\Phi(x_o, r)$.

Lemma 2.5. *Let $x_o \in B_3$, $r \in (0, \frac{1}{4}]$, and $\gamma \in (0,1)$. Then, for any $\rho \in (0, r]$, we have*

$$\Phi(x_o, \rho) \lesssim_{d,\lambda,\gamma} \left(\frac{\rho}{r} \right)^\gamma \Phi(x_o, r) + \|Du\|_{L^\infty(B_r(x_o))} \, \tilde{\omega}_{\mathbf{A}, B_3}(\rho) + \tilde{\omega}_{\boldsymbol{f}, B_3}(\rho),$$

where $\tilde{\omega}_{\bullet, B_3}$ is a function derived from $\bullet$ as formulated in (2.20)*; see Remark 2.6.*

Proof. Let $x_o = (x_{o1}, x_o') \in B_3$ and $r \in (0, \frac{1}{4}]$. Let the function u_e be given by

$$u_e = u - \int_0^{x_1} \frac{\bar{f}_1(y_1)}{\bar{a}^{11}(y_1)}\, dy_1, \quad \text{where } \bar{g}(x_1) := \fint_{B_r'(x_o')} g(x_1, y')\, dy'.$$

We set $\bar{\mathbf{A}} = \bar{\mathbf{A}}(x_1)$ and note that

$$\operatorname{div}(\bar{\mathbf{A}} D u_e) = \operatorname{div}(\bar{\mathbf{A}} D u) - D_1 f_1 = \operatorname{div}(\bar{\mathbf{A}} D u) - \operatorname{div} \bar{\boldsymbol{f}}$$

and thus by setting $\boldsymbol{F} := (\bar{\mathbf{A}} - \mathbf{A}) D u + \boldsymbol{f} - \bar{\boldsymbol{f}}$, we have

$$\operatorname{div}(\bar{\mathbf{A}} D u_e) = \operatorname{div} \boldsymbol{F} \quad \text{in } B_6.$$

We decompose $u_e = w + v$, where $w \in W_0^{1,2}(B_{4r}(x_o))$ is a unique weak solution of

$$\operatorname{div}(\bar{\mathbf{A}} D w) = \operatorname{div}(\chi_{B_r(x_o)} \boldsymbol{F}) \quad \text{in } B_{4r}(x_o).$$

Here, $\chi_{B_r(x_o)}$ is the characteristic function. Then by applying Lemma 2.4 with scaling and translation, we have

$$|\{x \in B_r(x_o) : |Dw(x)| > t\}| \lesssim_{d,\lambda} \frac{1}{t} \int_{B_r(x_o)} |\boldsymbol{F}|\, dx$$

for any $t > 0$. This implies that (c.f. [13, Eq. (2.11)])

$$\left(\fint_{B_r(x_o)} |Dw|^{\frac{1}{2}}\, dx \right)^2 \lesssim \fint_{B_r(x_o)} |\boldsymbol{F}|\, dx. \tag{2.17}$$

For the estimate of v, we apply Lemma 2.2 to the fact that

$$\operatorname{div}(\bar{\mathbf{A}} D v) = 0 \quad \text{in } B_r(x_o)$$

to get for any $\boldsymbol{\theta} = (\theta_1, \theta') \in \mathbb{R}^d$ that

$$\begin{aligned} &\left(\fint_{B_{\kappa r}(x_o)} |V - (V)_{B_{\kappa r}(x_o)}|^{\frac{1}{2}} + |D_{x'} v - (D_{x'} v)_{B_{\kappa r}(x_o)}|^{\frac{1}{2}}\, dx \right)^2 \\ &\lesssim \kappa \left(\fint_{B_r(x_o)} |V - \theta_1|^{\frac{1}{2}} + |D_{x'} v - \theta'|^{\frac{1}{2}}\, dx \right)^2 \end{aligned} \tag{2.18}$$

for any $\kappa \in (0, \frac{1}{2}]$, where $V := \bar{a}^{1j} D_j v$.

Now we set $U_e = \bar{a}^{1j} D_j u_e$, and observe that

$$D_{x'} u_e = D_{x'} u, \quad \hat{U} - U_e = \left(a^{1j} - \bar{a}^{1j}\right) D_j u - (f_1 - \bar{f}_1). \tag{2.19}$$

By (2.17), (2.18), and $u_e = w + v$, we have

$$\begin{aligned} &\left(\fint_{B_{\kappa r}(x_o)} |U_e - (V)_{B_{\kappa r}(x_o)}|^{\frac{1}{2}} + |D_{x'} u_e - (D_{x'} u_e)_{B_{\kappa r}(x_o)}|^{\frac{1}{2}}\, dx \right)^2 \\ &\lesssim \kappa \left(\fint_{B_r(x_o)} |V - \theta_1|^{\frac{1}{2}} + |D_{x'} v - \theta'|^{\frac{1}{2}}\, dx \right)^2 + \kappa^{-2d} \fint_{B_r(x_o)} |\boldsymbol{F}|\, dx \\ &\lesssim \kappa \left(\fint_{B_r(x_o)} |U_e - \theta_1|^{\frac{1}{2}} + |D_{x'} u_e - \theta'|^{\frac{1}{2}}\, dx \right)^2 + \kappa^{-2d} \fint_{B_r(x_o)} |\boldsymbol{F}|\, dx. \end{aligned}$$

From this together with (2.19), it follows that

$$\Phi(x_o,\kappa r) \le C_0\big(\kappa\Phi(x_o,r)+\kappa^{-2d}\|Du\|_{L^\infty(B_r(x_o))}\omega_{\mathbf{A},B_3}(r)+\kappa^{-2d}\omega_{\boldsymbol{f},B_3}(r)\big),$$

where $C_0=C_0(d,\lambda)$ is an absolute constant,

$$\omega_{\mathbf{A},B_3}(r) := \sup_{x\in B_3} \fint_{B_r(x)} \left| \mathbf{A}(y) - \fint_{B'_r(x')} \mathbf{A}(y_1,z')\,dz' \right| dy,$$

and $\omega_{\boldsymbol{f},B_3}(r)$ is defined in the same way.

We fix a $\kappa=\kappa(d,\lambda,\gamma)\in(0,\frac{1}{2}]$ so that $C_0\kappa^{1-\gamma}\le 1$. Then, we have

$$\Phi(x_o,\kappa r) \le \kappa^\gamma\Phi(x_o,r)+C\left(\|Du\|_{L^\infty(B_r(x_o))}\omega_{\mathbf{A},B_3}(r)+\omega_{\boldsymbol{f},B_3}(r)\right),$$

where $C=C(d,\lambda,\gamma)$. Let the function $\tilde\omega_{\bullet,B_3}$ be given by

$$\tilde\omega_{\bullet,B_3}(r) := \sum_{i=1}^\infty \kappa^{\gamma i}\left(\omega_{\bullet,B_3}(\kappa^{-i}r)[\kappa^{-i}r<1]+\omega_{\bullet,B_3}(1)[\kappa^{-i}r\ge 1]\right), \tag{2.20}$$

where we used Iverson bracket notation, i.e., $[P]=1$ if P is true and $[P]=0$ otherwise. By iterating and using the fact that

$$\sum_{i=1}^j \kappa^{\gamma(i-1)}\omega_{\bullet,B_3}(\kappa^{j-i}r) \le \kappa^{-\gamma}\tilde\omega_{\bullet,B_3}(\kappa^j r),$$

we obtain

$$\begin{aligned}&\Phi(x_o,\kappa^j r)\\ &\le \kappa^{\gamma j}\Phi(x_o,r)+C\left(\|Du\|_{L^\infty(B_r(x_o))}\tilde\omega_{\mathbf{A},B_3}(\kappa^j r)+\tilde\omega_{\boldsymbol{f},B_3}(\kappa^j r)\right).\end{aligned} \tag{2.21}$$

We note that the above inequality also obviously holds for $j=0$ so that it holds for all $j=0,1,2,\ldots$. Now, for $0<\rho\le r$, let j be the nonnegative integer satisfying $\kappa^{j+1}<\rho/r\le\kappa^j$. Then by (2.21) with ρ in place of $\kappa^j r$, we get

$$\begin{aligned}&\Phi(x_o,\rho)\\ &\le \kappa^{-\gamma}\left(\frac{\rho}{r}\right)^\gamma\Phi(x_o,\kappa^{-j}\rho)+C\left(\|Du\|_{L^\infty(B_{\kappa^{-j}\rho}(x_o))}\tilde\omega_{\mathbf{A},B_3}(\rho)+\tilde\omega_{\boldsymbol{f},B_3}(\rho)\right)\\ &\le \kappa^{-\gamma-2d}\left(\frac{\rho}{r}\right)^\gamma\Phi(x_o,r)+C\left(\|Du\|_{L^\infty(B_r(x_o))}\tilde\omega_{\mathbf{A},B_3}(\rho)+\tilde\omega_{\boldsymbol{f},B_3}(\rho)\right),\end{aligned}$$

where, we used that $\Phi(x_o,\kappa^{-j}\rho)\le\kappa^{-2d}\Phi(x_o,r)$. The lemma is proved. ☐

Remark 2.6. We note that the functions $\tilde\omega_{\mathbf{A},B_3}$ and $\tilde\omega_{\boldsymbol{f},B_3}$ in Lemma 2.5 satisfy

$$\sum_{j=0}^\infty \tilde\omega_{\bullet,B_3}(2^{-j}r) \lesssim \int_0^r \frac{\tilde\omega_\bullet(t)}{t}\,dt<\infty,$$

where we set

$$\tilde\omega_\bullet(r) := \sum_{i=1}^\infty \kappa^{\gamma i}\left(\omega_\bullet(\kappa^{-i}r)[\kappa^{-i}r<1]+\omega_\bullet(1)[\kappa^{-i}r\ge 1]\right).$$

We refer to [5, Lemma 8.1] for the proof.

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. Let $\boldsymbol{\theta}_{x_o,r} \in \mathbb{R}^d$ be chosen so that

$$\Phi(x_o, r) = \left(\fint_{B_r(x_o)} |\boldsymbol{U} - \boldsymbol{\theta}_{x_o,r}|^{\frac{1}{2}} \, dx \right)^2 .$$

First, we derive L^∞-estimate for Du. Let $x_o \in B_3$ and $r \in (0, \frac{1}{4}]$. Recall that we assume $\mathbf{A}$ and $\boldsymbol{f}$ are sufficiently smooth so that $u \in C^1(\overline{B_4})$. Note that Lemma 2.5 particularly implies $\lim_{i\to\infty} \Phi(x_o, 2^{-i}r) = 0$ and thus, we have

$$\lim_{i\to\infty} \boldsymbol{\theta}_{x_o,2^{-i}r} = \boldsymbol{U}(x_o). \tag{2.22}$$

By averaging the obvious inequality

$$\left|\boldsymbol{\theta}_{x_o,\frac{1}{2}r} - \boldsymbol{\theta}_{x_o,r}\right|^{\frac{1}{2}} \lesssim \left|\boldsymbol{U} - \boldsymbol{\theta}_{x_o,\frac{1}{2}r}\right|^{\frac{1}{2}} + |\boldsymbol{U} - \boldsymbol{\theta}_{x_o,r}|^{\frac{1}{2}}$$

on $B_{\frac{1}{2}r}(x_o)$ and taking the square, we have

$$\left|\boldsymbol{\theta}_{x_o,\frac{1}{2}r} - \boldsymbol{\theta}_{x_o,r}\right| \lesssim_{d,\lambda,\gamma} \Phi(x_o, \tfrac{1}{2}r) + \Phi(x_o, r).$$

We apply the above inequality iteratively and use (2.22) to get

$$|\boldsymbol{U}(x_o) - \boldsymbol{\theta}_{x_o,r}| \lesssim \sum_{j=0}^{\infty} \Phi(x_o, 2^{-j}r). \tag{2.23}$$

Averaging the inequality $|\boldsymbol{\theta}_{x_o,r}|^{\frac{1}{2}} \lesssim |\boldsymbol{U} - \boldsymbol{\theta}_{x_o,r}|^{\frac{1}{2}} + |\boldsymbol{U}|^{\frac{1}{2}}$ on $B_r(x_o)$ and taking the square, we obtain

$$|\boldsymbol{\theta}_{x_o,r}| \lesssim \Phi(x_o, r) + r^{-d}\|\boldsymbol{U}\|_{L^1(B_r(x_o))} \lesssim r^{-d}\|\boldsymbol{U}\|_{L^1(B_r(x_o))},$$

Using the above inequality together with Lemma 2.5, Remark 2.6, and (2.23), we see that

$$\begin{aligned} |\boldsymbol{U}(x_o)| &\lesssim r^{-d}\|\boldsymbol{U}\|_{L^1(B_r(x_o))} \\ &\quad + \|Du\|_{L^\infty(B_r(x_o))} \int_0^r \frac{\tilde{\omega}_{\mathbf{A}}(t)}{t} \, dt + \int_0^r \frac{\tilde{\omega}_{\boldsymbol{f}}(t)}{t} \, dt. \end{aligned} \tag{2.24}$$

From the definition of $\hat{U}$, we have

$$D_1 u = \frac{1}{a^{11}} \left(\hat{U} - \sum_{j=2}^{d} a^{1j} D_j u + f_1 \right),$$

which implies

$$|Du| \le |D_1 u| + |D_{x'} u| \lesssim_{d,\lambda} |\boldsymbol{U}| + |f_1|, \tag{2.25}$$

and thus, we obtain by (2.24) that

$$|\boldsymbol{U}(x_o)| \le C_0 r^{-d} \|\boldsymbol{U}\|_{L^1(B_r(x_o))} + C_0 \|\boldsymbol{U}\|_{L^\infty(B_r(x_o))} \int_0^r \frac{\tilde{\omega}_{\mathbf{A}}(t)}{t}\,dt + C_0 \|f_1\|_{L^\infty(B_r(x_o))} \int_0^r \frac{\tilde{\omega}_{\mathbf{A}}(t)}{t}\,dt + C_0 \int_0^r \frac{\tilde{\omega}_{\boldsymbol{f}}(t)}{t}\,dt,$$

where $C_0 = C_0(d, \lambda, \gamma)$. Choose $r_0 \in (0, \frac{1}{4}]$ so that

$$C_0 \int_0^{r_0} \frac{\tilde{\omega}_{\mathbf{A}}(t)}{t}\,dt \le \frac{1}{3^d}. \tag{2.26}$$

Then for any $x_o \in B_3$ and $r \in (0, r_0]$, we have

$$|\boldsymbol{U}(x_o)| \le C_0 r^{-d} \|\boldsymbol{U}\|_{L^1(B_r(x_o))} + 3^{-d} \|\boldsymbol{U}\|_{L^\infty(B_r(x_o))} + 3^{-d} \|f_1\|_{L^\infty(B_r(x_o))} + C_0 \int_0^r \frac{\tilde{\omega}_{\boldsymbol{f}}(t)}{t}\,dt. \tag{2.27}$$

Here, the constant r_0 depends only on d, λ, γ, and $\omega_{\mathbf{A}}$.

For $k \in \{1, 2, \ldots\}$, we set $r_k = 3 - 2^{1-k}$. Then it holds that $B_{2^{-k}}(x_o) \subset B_{r_{k+1}}$ for any $x_o \in B_{r_k}$. We take k_0 sufficiently large such that $2^{-k_0} \le r_0$. Then for $k \ge k_0$, it follows from (2.27) with $r = 2^{-k}$ that

$$\|\boldsymbol{U}\|_{L^\infty(B_{r_k})} \le C_0 2^{dk} \|\boldsymbol{U}\|_{L^1(B_6)} + 3^{-d} \|\boldsymbol{U}\|_{L^\infty(B_{r_{k+1}})} + 3^{-d} \|f_1\|_{L^\infty(B_6)} + C_0 \int_0^1 \frac{\tilde{\omega}_{\boldsymbol{f}}(t)}{t}\,dt.$$

Multiplying both sides of the above inequality by 3^{-dk} and summing the terms with respect to $k = k_0, k_0 + 1, \ldots$, we have

$$\sum_{k=k_0}^{\infty} 3^{-dk} \|\boldsymbol{U}\|_{L^\infty(B_{r_k})} \le C \|\boldsymbol{U}\|_{L^1(B_6)} + \sum_{k=k_0+1}^{\infty} 3^{-dk} \|\boldsymbol{U}\|_{L^\infty(B_{r_k})} + C \|f_1\|_{L^\infty(B_6)} + C \int_0^1 \frac{\tilde{\omega}_{\boldsymbol{f}}(t)}{t}\,dt,$$

where $C = C(d, \lambda, \gamma)$. Since $r_{k_0} \ge 2$, we thus obtain

$$\|\boldsymbol{U}\|_{L^\infty(B_2)} \lesssim \|\boldsymbol{U}\|_{L^1(B_6)} + \|f_1\|_{L^\infty(B_6)} + \int_0^1 \frac{\tilde{\omega}_{\boldsymbol{f}}(t)}{t}\,dt.$$

Using this together with (2.25) and $\boldsymbol{U} \lesssim |Du| + |f_1|$, we get the following L^∞-estimate for Du:

$$\|Du\|_{L^\infty(B_2)} \lesssim_{d,\lambda,\gamma,\omega_{\mathbf{A}}} \|Du\|_{L^1(B_6)} + \|f_1\|_{L^\infty(B_6)} + \int_0^1 \frac{\tilde{\omega}_{\boldsymbol{f}}(t)}{t}\,dt. \tag{2.28}$$

Next, we estimate the modulus of continuity of $\boldsymbol{U}$. Let x, $y \in \overline{B}_1$ with $\rho := 2|x - y| \in \left(0, \frac{1}{4}\right]$. By quasi triangle inequalities, for $z \in B_\rho(x) \cap B_\rho(y)$, we have

$$|\boldsymbol{U}(x) - \boldsymbol{U}(y)|^{\frac{1}{2}} \lesssim |\boldsymbol{U}(x) - \boldsymbol{\theta}_{x,\rho}|^{\frac{1}{2}} + |\boldsymbol{U}(y) - \boldsymbol{\theta}_{y,\rho}|^{\frac{1}{2}} + |\boldsymbol{U}(z) - \boldsymbol{\theta}_{x,\rho}|^{\frac{1}{2}} + |\boldsymbol{U}(z) - \boldsymbol{\theta}_{y,\rho}|^{\frac{1}{2}}.$$

By averaging over $z \in B_\rho(x) \cap B_\rho(y)$, taking the square, and using (2.23), we obtain

$$
\begin{aligned}
|\boldsymbol{U}(x) - \boldsymbol{U}(y)| &\lesssim \sup_{x_o \in \overline{B}_1} |\boldsymbol{U}(x_o) - \boldsymbol{\theta}_{x_o,\rho}| + \Phi(x,\rho) + \Phi(y,\rho) \\
&\lesssim \sup_{x_o \in \overline{B}_1} \sum_{j=0}^{\infty} \Phi(x_o, 2^{-j}\rho).
\end{aligned}
$$

On the other hand, by Lemma 2.5 and Remark 2.6, we have for any $x_o \in \overline{B}_1$,

$$
\begin{aligned}
\sum_{j=0}^{\infty} \Phi(x_o, 2^{-j}\rho) &\lesssim \Phi(x_o,\rho) + \|Du\|_{L^\infty(B_2)} \int_0^\rho \frac{\tilde{\omega}_{\mathbf{A}}(t)}{t}\,dt + \int_0^\rho \frac{\tilde{\omega}_{\boldsymbol{f}}(t)}{t}\,dt \\
&\lesssim \rho^\gamma \Phi(x_o, \tfrac{1}{4}) + \|Du\|_{L^\infty(B_2)} \int_0^\rho \frac{\tilde{\omega}_{\mathbf{A}}(t)}{t}\,dt + \int_0^\rho \frac{\tilde{\omega}_{\boldsymbol{f}}(t)}{t}\,dt.
\end{aligned}
$$

By combining these together and using $\Phi(x_o, \frac{1}{4}) \lesssim \|\boldsymbol{U}\|_{L^1(B_6)}$, we get

$$
|\boldsymbol{U}(x) - \boldsymbol{U}(y)| \lesssim \rho^\gamma \|\boldsymbol{U}\|_{L^1(B_6)} + \|Du\|_{L^\infty(B_2)} \int_0^\rho \frac{\tilde{\omega}_{\mathbf{A}}(t)}{t}\,dt + \int_0^\rho \frac{\tilde{\omega}_{\boldsymbol{f}}(t)}{t}\,dt.
$$

Therefore, we obtain by (2.28) that

$$
\begin{aligned}
&|\boldsymbol{U}(x) - \boldsymbol{U}(y)| \qquad (2.29) \\
&\lesssim_{d,\lambda,\gamma,\omega_{\mathbf{A}}} |x-y|^\gamma \left(\|Du\|_{L^1(B_6)} + \|f_1\|_{L^1(B_6)} \right) + \int_0^{2|x-y|} \frac{\tilde{\omega}_{\boldsymbol{f}}(t)}{t}\,dt \\
&\qquad + \left(\|Du\|_{L^1(B_6)} + \|f_1\|_{L^\infty(B_6)} + \int_0^1 \frac{\tilde{\omega}_{\boldsymbol{f}}(t)}{t}\,dt \right) \int_0^{2|x-y|} \frac{\tilde{\omega}_{\mathbf{A}}(t)}{t}\,dt
\end{aligned}
$$

for any $x, y \in \overline{B}_1$ with $|x-y| \le \frac{1}{8}$. The theorem is proved. □

We close this subsection with a couple of remarks.

Remark 2.7. Note that in the above proof, if $x, y \in \overline{B}_1$ with $|x-y| > \frac{1}{8}$, then by (2.28), we have

$$
\begin{aligned}
|\boldsymbol{U}(x) - \boldsymbol{U}(y)| &\le 2\|\boldsymbol{U}\|_{L^\infty(B_1)} \lesssim \|Du\|_{L^\infty(B_1)} + \|f_1\|_{L^\infty(B_1)} \\
&\lesssim |x-y|^\gamma \left(\|Du\|_{L^1(B_6)} + \|f_1\|_{L^\infty(B_6)} + \int_0^1 \frac{\tilde{\omega}_{\boldsymbol{f}}(t)}{t}\,dt \right).
\end{aligned}
$$

By combining it with (2.29), we have the following modulus of continuity estimate:

$$
\begin{aligned}
(2.30) \quad |\boldsymbol{U}(x) - \boldsymbol{U}(y)| \lesssim_{d,\lambda,\gamma,\omega_{\mathbf{A}}} &\int_0^{2|x-y|} \frac{\tilde{\omega}_{\boldsymbol{f}}(t)}{t}\,dt \\
&+ \left(\|Du\|_{L^1(B_6)} + \|f_1\|_{L^\infty(B_6)} + \int_0^1 \frac{\tilde{\omega}_{\boldsymbol{f}}(t)}{t} \right)
\end{aligned}
$$

$$\times\left(|x-y|^{\gamma}+\int_0^{2|x-y|}\frac{\tilde{\omega}_{\mathbf{A}}(t)}{t}\,dt\right)$$

for any $x,\, y\in\overline{B}_1$.

Remark 2.8. For $\gamma_0\in(0,1)$, we define the partial Hölder semi-norm with respect to x' by

$$[g]_{C^{\gamma_0}_{x'}(\Omega)}=\sup_{\substack{x,y\in\Omega\\ x_1=y_1,x'\neq y'}}\frac{|g(x)-g(y)|}{|x'-y'|^{\gamma_0}}.$$

Note that if $[g]_{C^{\gamma_0}_{x'}(B_6)}<\infty$ with $\gamma_0\in(0,\gamma)$, then for any $r\in(0,1]$, we have

$$\tilde{\omega}_{g,B_4}(r)+\int_0^r\frac{\tilde{\omega}_{g,B_4}(t)}{t}\,dt\lesssim_{\gamma_0,\gamma,\kappa}[g]_{C^{\gamma_0}_{x'}(B_6)}r^{\gamma_0}.$$

Therefore, by Remark 2.7, we recover the following partial Schauder estimate for $\boldsymbol{U}:=(\hat{U},D_{x'}u)$:

$$[\boldsymbol{U}]_{C^{\gamma_0}(B_1)}\lesssim_{d,\lambda,\gamma_0,[\mathbf{A}]_{C^{\gamma_0}_{x'}(B_6)}}\|Du\|_{L^1(B_6)}+\|f_1\|_{L^\infty(B_6)}+[\boldsymbol{f}]_{C^{\gamma_0}_{x'}(B_6)}$$

provided that $[\mathbf{A}]_{C^{\gamma_0}_{x'}(B_6)}+[\boldsymbol{f}]_{C^{\gamma_0}_{x'}(B_6)}<\infty$.

2.3. Proof of Theorem 1.5

In this subsection, we provide the proof and some related remarks of Theorem 1.5.

Proof of Theorem 1.5. The proof is based on odd/even extension technique. Set

$$\tilde{a}^{ij}(x_1,x')=\begin{cases}a^{ij}(|x_1|,x') & \text{for } i=j=1 \text{ or } i,j\in\{2,\dots,d\},\\ \operatorname{sgn}(x_1)a^{ij}(|x_1|,x') & \text{otherwise},\end{cases}$$

$$\tilde{f}_i(x_1,x')=\begin{cases}f_i(|x_1|,x') & \text{for } i=1,\\ \operatorname{sgn}(x_1)f_i(|x_1|,x') & \text{otherwise},\end{cases}$$

$$\tilde{u}(x_1,x')=\operatorname{sgn}(x_1)u(|x_1|,x').$$

Observe that $\tilde{u}\in W^{1,p}(B_6)$ satisfies

$$\operatorname{div}(\tilde{\mathbf{A}}Du)=\operatorname{div}\tilde{\boldsymbol{f}}\quad\text{in } B_6,$$

where $\tilde{\mathbf{A}}=(\tilde{a}^{ij})_{i,j=1}^d$ satisfies (1.2) with the same constant λ,

$$\tilde{\boldsymbol{f}}=(\tilde{f}_1,\dots,\tilde{f}_d)\in L^\infty(B_6)\times L^p(B_6)^{d-1},$$

and $\tilde{\mathbf{A}}$ and $\tilde{\boldsymbol{f}}$ are of partial Dini mean oscillation with respect to x' in B_4. Thus by applying Theorem 1.2, we see that $\tilde{u}\in W^{1,\infty}(B_1)$ and that $\tilde{a}^{1j}D_j\tilde{u}-\tilde{f}_1$ and $D_{x'}\tilde{u}$ are continuous in $\overline{B}_1$, which proves the theorem. Moreover, by using (2.28), (2.30), and the fact that

$$\omega_{\tilde{\mathbf{A}}}(r)\le 2\omega^+_{\mathbf{A}}(r),\quad \omega_{\tilde{\boldsymbol{f}}}(r)\le 2\omega^+_{\boldsymbol{f}}(r)\quad\text{for } r\in(0,1],$$

we have

$$\|Du\|_{L^\infty(B_2^+)} \le C \left(\|Du\|_{L^1(B_6^+)} + \|f_1\|_{L^\infty(B_6^+)} + \int_0^1 \frac{\tilde{\omega}_{\boldsymbol{f}}^+(t)}{t}\,dt \right) \tag{2.31}$$

and

$$\begin{aligned} |\boldsymbol{U}(x) - \boldsymbol{U}(y)| \le C \int_0^{2|x-y|} \frac{\tilde{\omega}_{\boldsymbol{f}}^+(t)}{t}\,dt \\ + C \left(\|Du\|_{L^1(B_6^+)} + \|f_1\|_{L^\infty(B_6^+)} + \int_0^1 \frac{\tilde{\omega}_{\boldsymbol{f}}^+(t)}{t} \right) \\ \times \left(|x-y|^\gamma + \int_0^{2|x-y|} \frac{\tilde{\omega}_{\mathbf{A}}^+(t)}{t}\,dt \right) \end{aligned} \tag{2.32}$$

for any $x, y \in \overline{B_1^+}$, where

$$\boldsymbol{U} = (\hat{U}, D_{x'}u), \quad \tilde{\omega}_\bullet^+(r) := \sum_{i=1}^\infty \kappa^{\gamma i} \left(\omega_\bullet^+(\kappa^{-i}r)[\kappa^{-i}r < 1] + \omega_\bullet^+(1)[\kappa^{-i}r \ge 1] \right),$$

and C is a constant depending only on d, λ, γ, and $\omega_{\tilde{\mathbf{A}}}$. Here, one can replace the parameter $\omega_{\tilde{\mathbf{A}}}$ by $\omega_{\mathbf{A}}^+$, by taking $r_0 \in \left(0, \frac{1}{4}\right]$ so that

$$C_0 \int_0^{r_0} \frac{\tilde{\omega}_{\tilde{\mathbf{A}}}(t)}{t}\,dt \le 2C_0 \int_0^{r_0} \frac{\tilde{\omega}_{\mathbf{A}}^+(t)}{t}\,dt \le \frac{1}{3^d}$$

in (2.26). □

Remark 2.9. As demonstrated in Remark 2.8, we recover the following partial Schauder estimate for $\boldsymbol{U} = (\hat{U}, D_{x'}u)$:

$$[\boldsymbol{U}]_{C^{\gamma_0}(B_1^+)} \lesssim_{d,\lambda,\gamma_0,[\mathbf{A}]_{C_{x'}^{\gamma_0}(B_6^+)}} \|Du\|_{L^1(B_6^+)} + \|f_1\|_{L^\infty(B_6^+)} + [\boldsymbol{f}]_{C_{x'}^{\gamma_0}(B_6^+)}$$

provided that $[\mathbf{A}]_{C_{x'}^{\gamma_0}(B_6^+)} + [\boldsymbol{f}]_{C_{x'}^{\gamma_0}(B_6^+)} < \infty$.

Remark 2.10. By using the extensions

$$\tilde{a}^{ij}(x_1, x') = \begin{cases} a^{ij}(|x_1|, x') & \text{for } i = j = 1 \text{ or } i, j \in \{2, \dots, d\}, \\ \operatorname{sgn}(x_1) a^{ij}(|x_1|, x') & \text{otherwise,} \end{cases}$$

$$\tilde{f}_i(x_1, x') = \begin{cases} \operatorname{sgn}(x_1) f_i(|x_1|, x') & \text{for } i = 1, \\ f_i(|x_1|, x') & \text{otherwise,} \end{cases}$$

$$\tilde{u}(x_1, x') = u(|x_1|, x'),$$

and following the steps in the proof of Theorem 1.5, one can obtain the same estimates (2.31) and (2.32) for weak solutions to the equation with the conormal derivative condition

$$\begin{cases} \mathcal{L}u = \operatorname{div} \boldsymbol{f} & \text{in } B_6^+, \\ a^{ij} D_j u n_i = f_i n_i & \text{on } B_6 \cap \partial\mathbb{R}_+^d, \end{cases}$$

where $n = (n_1, \dots, n_d)$ is the outward unit normal.

3. Applications

In this section, we give applications of our main theorems.

3.1. Weak type-$(1,1)$ estimate

In this subsection, we prove local weak type-$(1,1)$ estimates under the condition that, for example, in the interior case,

$$\omega_{\mathbf{A}}(r) \le C_0 \left(\ln \frac{r}{4} \right)^{-2}, \quad \forall r \in (0,1], \tag{3.1}$$

which is stronger than the partial Dini mean oscillation condition. Such a condition on the L^1-mean oscillation in *all* the directions was introduced in [13, Section 3] for the interior weak type-$(1,1)$ estimates of $W^{1,2}$-weak solutions. See also [8] for boundary estimates and [19] for weighted estimates.

Theorem 3.1. (a) *Let T_0 be a bounded linear operator on $L^2(B_6)^d$ defined by*

$$T_0 \boldsymbol{f} = Du,$$

where $u \in W_0^{1,2}(B_6)$ is a unique weak solution of

$$\begin{cases} \operatorname{div}(\mathbf{A}Du) = \operatorname{div} \boldsymbol{f} & \text{in } B_6, \\ u = 0 & \text{on } \partial B_6. \end{cases}$$

If $\mathbf{A}$ satisfies (3.1), then T_0 has an extension on the set

$$\{ \boldsymbol{f} \in L^1(B_6)^d : \operatorname{supp} \boldsymbol{f} \subset B_1 \}$$

in such a way that for any $t > 0$, we have

$$\big| \{ x \in B_1 : |T_0 \boldsymbol{f}(x)| > t \} \big| \le \frac{C}{t} \int_{B_1} |\boldsymbol{f}| \, dx,$$

where $C = C(d, \lambda, \omega_{\mathbf{A}}, C_0)$.

(b) *Let T_0^+ be a bounded linear operator on $L^2(B_6^+)^d$ defined by*

$$T_0^+ \boldsymbol{f} = Du,$$

where $u \in W_0^{1,2}(B_6^+)$ is a unique weak solution of

$$\begin{cases} \operatorname{div}(\mathbf{A}Du) = \operatorname{div} \boldsymbol{f} & \text{in } B_6^+, \\ u = 0 & \text{on } \partial B_6^+. \end{cases}$$

If $\mathbf{A}$ satisfies (3.1) with $\omega_{\mathbf{A}}^+$ in place of $\omega_{\mathbf{A}}$, then T_0^+ has an extension on the set

$$\{ \boldsymbol{f} \in L^1(B_6^+)^d : \operatorname{supp} \boldsymbol{f} \subset B_1^+ \}$$

in such a way that for any $t > 0$, we have

$$\big| \{ x \in B_1^+ : |T_0^+ \boldsymbol{f}(x)| > t \} \big| \le \frac{C}{t} \int_{B_1^+} |\boldsymbol{f}| \, dx,$$

where $C = C(d, \lambda, \omega_{\mathbf{A}}^{+}, C_0)$.

Proof. We first prove part (a). We denote by $\mathcal{L}^*$ the adjoint operator, i.e.,

$$\mathcal{L}^* u = \operatorname{div}(\mathbf{A}^{\mathsf{T}} Du), \quad \mathbf{A}^{\mathsf{T}} = (a^{ji})_{i,j=1}^{d}.$$

We note that $\mathbf{A}^{\mathsf{T}}$ is also of partial Dini mean oscillation with respect to x' in B_4 satisfying $\omega_{\mathbf{A}^{\mathsf{T}}} = \omega_{\mathbf{A}}$. Moreover, it follows that (see [13, Eq. (3.5)])

$$\tilde{\omega}_{\mathbf{A}^{\mathsf{T}}}(r) \lesssim_{d,\lambda,C_0} \left(\ln \frac{r}{4}\right)^{-2}, \quad \forall r \in (0,1],$$

which implies

$$\int_0^r \frac{\tilde{\omega}_{\mathbf{A}^{\mathsf{T}}}(t)}{t}\,dt \lesssim \left(\ln \frac{4}{r}\right)^{-1}, \quad \forall r \in (0,1]. \tag{3.2}$$

Consider a mapping $\boldsymbol{f} \mapsto \hat{\boldsymbol{f}}$ given by

$$\hat{f}_1 = (a^{11})^{-1} f_1, \quad \hat{f}_i = f_i - a^{i1}\hat{f}_1, \quad i \in \{2, \ldots, d\}.$$

We define a bounded linear operator T on $L^2(B_6)^d$ by setting $T\hat{\boldsymbol{f}} := T_0 \boldsymbol{f}$. It suffices to show that T satisfies the hypothesis of Lemma 2.3 with $k = d$, $R_0 = 6$, $\mu = \frac{1}{3}$, $c = 6$, and $C = C(d, \lambda, \omega_{\mathbf{A}}, C_0) > 0$. Fix $x_o \in B_1$ and $r \in (0, \frac{1}{3})$. Let $\hat{\boldsymbol{f}} \in L^2(B_6)^d$ be a function satisfying

$$\int_{B_6} \hat{\boldsymbol{f}}\,dx = 0 \quad \text{and} \quad \operatorname{supp} \hat{\boldsymbol{f}} \subset B_r(x_o) \cap B_1.$$

Let $T\hat{\boldsymbol{f}} = Du$, i.e., $u \in W_0^{1,2}(B_6^+)$ is a unique weak solution of

$$\operatorname{div}(\mathbf{A} Du) = \operatorname{div} \boldsymbol{f} \quad \text{in } B_6. \tag{3.3}$$

Take $R \in [6r, 2)$ so that $B_1^+ \setminus B_R(x_o) \neq \emptyset$, and let $\boldsymbol{g} \in C_o^\infty(B_6)^d$ be a function supported in $(B_{2R}(x_o) \setminus B_R(x_o)) \cap B_1$. By the Lax-Milgram theorem, there exists a unique $v \in W_0^{1,2}(B_6)$ satisfying

$$\mathcal{L}^* v = \operatorname{div} \boldsymbol{g} \quad \text{in } B_6. \tag{3.4}$$

Set $\boldsymbol{V} := (a^{j1} D_j v, D_{x'} v)$. Since v satisfies $\mathcal{L}^* v = 0$ in $B_R(x_o)$, by a similar calculation that lead to (2.30), we obtain

$$\begin{aligned} &|\boldsymbol{V}(x) - \boldsymbol{V}(y)| \\ &\lesssim_{d,\lambda,\gamma,\omega_{\mathbf{A}}} R^{-d/2} \|Dv\|_{L^2(B_R(x_o))} \left(\left(\frac{|x-y|}{R} \right)^{\gamma} + \int_0^{2|x-y|} \frac{\tilde{\omega}_{\mathbf{A}^{\mathsf{T}}}(t)}{t}\,dt \right) \end{aligned}$$

for any $x, y \in B_r(x_o) \subset B_{R/6}(x_o)$. From this together with (3.2), we get

$$\begin{aligned} &\|\boldsymbol{V} - (\boldsymbol{V})_{B_r(x_o)}\|_{L^\infty(B_r(x_o))} \\ &\lesssim_{d,\lambda,\gamma,\omega_{\mathbf{A}},C_0} R^{-d/2} \|Dv\|_{L^2(B_R(x_o))} \left(\left(\frac{r}{R} \right)^{\gamma} + \left(\ln \frac{1}{r} \right)^{-1} \right). \end{aligned} \tag{3.5}$$

Testing (3.3) and (3.4) with v and u, respectively, one can obtain that

$$\int_{(B_{2R}(x_o)\setminus B_R(x_o))\cap B_1} Du\cdot \boldsymbol{g}\,dx = \int_{B_r(x_o)} \hat{\boldsymbol{f}}\cdot\big(\boldsymbol{V}-(\boldsymbol{V})_{B_r(x_o)}\big)\,dx.$$

Thus by Hölder's inequality, duality, (3.5), and the L^2-estimate

$$\|Dv\|_{L^2(B_6)} \lesssim \|\boldsymbol{g}\|_{L^2((B_{2R}(x_o)\setminus B_R(x_o))\cap B_1)},$$

we have

$$\begin{aligned}\int_{(B_{2R}(x_o)\setminus B_R(x_o))\cap B_1} |Du|\,dx &\lesssim R^{d/2}\left(\int_{(B_{2R}(x_o)\setminus B_R(x_o))\cap B_1}|Du|^2\,dx\right)^{1/2}\\ &\lesssim \left(\left(\frac{r}{R}\right)^{\gamma}+\left(\ln\frac{1}{r}\right)^{-1}\right)\|\hat{\boldsymbol{f}}\|_{L^1(B_r(x_o))}.\end{aligned}$$

Let N be the smallest positive integer such that $B_1\subset B_{2^N\cdot 3\cdot r}(x_o)$. Then by taking $R=2^i\cdot 3\cdot r$, $i\in\{1,\dots,N-1\}$, and using the fact that $N-1\lesssim \ln(1/r)$, we obtain

$$\int_{B_1^+\setminus B_{6r}(x_o)}|Du|\,dx \lesssim \sum_{i=1}^{N-1}\big(2^{-i\gamma}+(\ln(1/r))^{-1}\big)\|\hat{\boldsymbol{f}}\|_{L^1(B_r^+(x_o))}\lesssim \|\hat{\boldsymbol{f}}\|_{L^1(B_r^+(x_o))},$$

which implies that the operator T satisfies the hypothesis of Lemma 2.3.

To prove part (b) one may use a similar argument as above; see the proof of [5, Theorem 5.6]. However, in this case, it is easier to directly extend the operator T_0^+ by

$$T_0^+\boldsymbol{f} = T_0\tilde{\boldsymbol{f}}\big|_{B_1^+}$$

on the set $\{\boldsymbol{f}\in L^1(B_6^+)^d : \operatorname{supp}\boldsymbol{f}\subset B_1^+\}$, where T_0 is the operator from part (a) with $\tilde{\mathbf{A}}$ in place of $\mathbf{A}$. Here, $\tilde{\mathbf{A}}$ and $\tilde{\boldsymbol{f}}$ are odd or even extensions of $\mathbf{A}$ and $\boldsymbol{f}$ as in the proof of Theorem 1.5. Then the extension is well-defined. Moreover, since $\tilde{\mathbf{A}}$ also satisfies (3.1), by the result in part (a), we obtain the desired estimate. The theorem is proved. □

3.2. L^p-estimates for $W^{1,1}$-weak solutions

In [2] (see also [1, Appendix]), Brezis proved $W^{1,p}$-regularity for $W^{1,1}$-weak solutions to divergence form elliptic equations with Dini continuous coefficients. This regularity result was extended in [13,19] to the equations with (piecewise) Dini mean oscillation coefficients. We also refer the reader to [13, 18, 20] for similar results on nondivergence type equations. The proofs in those papers are based on duality and bootstrap arguments combined with regularity theories, in particular, the boundedness of the gradient of solutions.

In the same manner, by using our results in Theorems 1.2 and 1.5, we prove the following $W^{1,p}$-regularity for $W^{1,1}$-weak solutions when the coefficient $\mathbf{A}$ is of partial Dini mean oscillation.

Theorem 3.2. *Let $p\in(1,\infty)$.*

(a) *Let $u \in W^{1,1}(B_6)$ be a weak solution of*

$$\operatorname{div}(\mathbf{A}Du) = \operatorname{div} \boldsymbol{f} \quad \text{in } B_6, \tag{3.6}$$

where $\boldsymbol{f} \in L^p(B_6)^d$. If $\mathbf{A}$ is of partial Dini mean oscillation with respect to x' in B_4, then we have $u \in W^{1,p}(B_1)$ with the estimate

$$\|u\|_{W^{1,p}(B_1)} \le C \left(\|u\|_{W^{1,1}(B_6)} + \|\boldsymbol{f}\|_{L^p(B_6)}\right),$$

where $C = C(d, \lambda, \omega_{\mathbf{A}}, p)$.

(b) *Let $u \in W^{1,1}(B_6^+)$ be a weak solution of*

$$\begin{cases} \operatorname{div}(\mathbf{A}Du) = \operatorname{div} \boldsymbol{f} & \text{in } B_6^+, \\ u = 0 & \text{on } B_6 \cap \mathbb{R}^d_+, \end{cases}$$

where $\boldsymbol{f} \in L^p(B_6^+)^d$. If $\mathbf{A}$ is of partial Dini mean oscillation with respect to x' in B_4^+, then we have $u \in W^{1,p}(B_1^+)$ with the estimate

$$\|u\|_{W^{1,p}(B_1^+)} \le C \left(\|u\|_{W^{1,1}(B_6^+)} + \|\boldsymbol{f}\|_{L^p(B_6^+)}\right),$$

where $C = C(d, \lambda, \omega^+_{\mathbf{A}}, p)$.

To prove Theorem 3.2, we utilize $W^{1,p}$-solvability result, which can be found in, for instance, [10, Theorem 8.6], where the authors considered higher order elliptic systems with lower order terms and leading coefficients which are partially BMO in Reifenberg flat domain. For reader's convenience, we state the theorem on the second-order equations without lower order terms; see Theorem 3.4 below.

Assumption 3.3 (γ)**.** There exists $R_0 \in (0, 1]$ such that the following hold.

(i) For $x_o \in \Omega$ and $0 < R \le \min\{R_0, \operatorname{dist}(x_o, \partial\Omega)\}$, there exists a coordinate system depending on x_o and R such that in this new coordinate system, we have that

$$\fint_{B_R(x_o)} \left| \mathbf{A}(x_1, y') - \fint_{B'_R(x'_o)} \mathbf{A}(x_1, y')\, dy' \right| dx \le \gamma \tag{3.7}$$

(ii) For any $x_o \in \partial\Omega$ and $0 < R \le R_0$, there is a coordinate system depending on x_o and R such that in the new coordinate system we have that (3.7) holds, and

$$\{y : x_{o1} + \gamma R < y_1\} \cap B_R(x_o) \subset \Omega_R(x_o) \subset \{y : x_{o1} - \gamma R < y_1\} \cap B_R(x_o),$$

where x_{o1} is the first coordinate of x_o in the new coordinate system.

Theorem 3.4. *Let Ω be a bounded domain in $\mathbb{R}^d$ and $p \in (1, \infty)$. Then there exists a constant $\gamma = \gamma(d, \lambda, p) > 0$ such that, under Assumption 3.3 (γ), the following holds: for $f \in L^p(\Omega)^d$, there exists a unique $u \in W_0^{1,p}(\Omega)$ satisfying*

$$\operatorname{div}(\mathbf{A}Du) = \operatorname{div} \boldsymbol{f} \quad \text{in } \Omega$$

and

$$\|Du\|_{L^p(\Omega)} \lesssim_{d,\lambda,p,R_0,|\Omega|} \|\boldsymbol{f}\|_{L^p(\Omega)}.$$

We are ready to prove Theorem 3.2.

Proof of Theorem 3.2. We only prove the first assertion because the second is an easy consequence of the extension technique as in the proof of Theorem 1.5. Let η be a smooth function in $\mathbb{R}^d$ satisfying

$$0 \le \eta \le 1, \quad \eta \equiv 1 \text{ on } B_{5/2}, \quad \operatorname{supp} \eta \subset B_3, \quad |D\eta| \lesssim_d 1.$$

We define operators $\tilde{\mathcal{L}}$ and $\tilde{\mathcal{L}}^*$ by

$$\tilde{\mathcal{L}}u = \operatorname{div}(\tilde{\mathbf{A}}Du), \quad \tilde{\mathcal{L}}^*u = \operatorname{div}(\tilde{\mathbf{A}}^{\mathsf{T}}Du),$$

where $\tilde{\mathbf{A}} = \eta\mathbf{A} + \lambda(1-\eta)\mathbf{I}$. Here, λ is the ellipticity constant from (1.2) and $\mathbf{I}$ is the $d \times d$ identity matrix. Then one can check that $\tilde{\mathbf{A}}$ satisfies the strong ellipticity condition (1.2), and that

$$\omega_{\tilde{\mathbf{A}}}(r) \lesssim_{d,\lambda} \omega^+_{\mathbf{A}}(r) + r.$$

Thus by [5, Lemma 8.1 (c)], for any $\gamma > 0$, there exists

$$k_0 = k_0(d, \omega_{\tilde{\mathbf{A}}}, \gamma) = k_0(d, \lambda, \omega_{\mathbf{A}}, \gamma) \in (0,1)$$

such that

$$\sup_{r \in (0,k_0)} \omega_{\tilde{\mathbf{A}}}(r) < \gamma.$$

From this we see that the following holds:

- For any $\gamma > 0$, there exists $R_0 \in (0,1]$, depending only on d, λ, $\omega_{\mathbf{A}}$, and γ, such that $\tilde{\mathbf{A}}$ and $\Omega = B_6$ satisfy Assumption 3.3 (γ).

Obviously, the same results hold for $\tilde{\mathbf{A}}^{\mathsf{T}}$. Therefore, Theorem 3.4 is available for $\tilde{\mathcal{L}}$ and $\tilde{\mathcal{L}}^*$ in $\Omega = B_6$.

Now let $u \in W^{1,1}(B_6)$ be a weak solution of (3.6) with $\boldsymbol{f} \in L^p(B_6)^d$. Then for any $\phi \in C^\infty_o(B_6)$, by testing (3.6) with $\eta\phi$, we have

$$\begin{aligned}\int_{B_6} \tilde{\mathbf{A}}Du \cdot D\phi\, dx &= \lambda \int_{B_6} (1-\eta) Du \cdot D\phi\, dx \\ &\quad - \int_{B_6} \mathbf{A}Du \cdot D\eta\phi\, dx + \int_{B_6^+} \boldsymbol{f} \cdot D(\eta\phi)\, dx.\end{aligned} \tag{3.8}$$

We consider the following two cases:

$$1 < p < \frac{d}{d-1}, \quad \frac{d}{d-1} \le p < \infty.$$

i. $1 < p < \frac{d}{d-1}$: Set $p' = \frac{p}{p-1} > d$, and let $\boldsymbol{\psi} \in C^\infty_o(B_6)^d$ with $\operatorname{supp} \boldsymbol{\psi} \subset B_2$.

By Theorem 3.4, there exists a unique $v \in W^{1,p'}_0(B_6)$ satisfying

$$\tilde{\mathcal{L}}^* v = \operatorname{div} \boldsymbol{\psi} \quad \text{in } B_6 \tag{3.9}$$

and

$$\|Dv\|_{L^{p'}(B_6)} \le C\|\boldsymbol{\psi}\|_{L^{p'}(B_2)}, \tag{3.10}$$

where $C = C(d, \lambda, p', R_0) = C(d, \lambda, \omega_{\mathbf{A}}, p)$. Since $\tilde{\mathbf{A}}^{\mathsf{T}}$ is of partial Dini mean oscillation, by using Theorem 1.2 with scaling and covering argument, we see that Dv is bounded in B_3. Thus, as a test function to (3.9), we can apply $\zeta u \in W_0^{1,1}(B_{5/2})$, where ζ is a smooth function satisfying

$$0 \le \zeta \le 1, \quad \zeta \equiv 1 \text{ in } B_2, \quad \operatorname{supp} \zeta \subset B_{5/2}, \quad |D\zeta| \lesssim_d 1.$$

By testing (3.9) with ζu and setting $\phi = \zeta v$ in (3.8), we have

$$\int_{B_6} D(\zeta u) \cdot \boldsymbol{\psi}\, dx = \int_{B_6} u\tilde{\mathbf{A}} D\zeta \cdot Dv - v\tilde{\mathbf{A}} Du \cdot D\zeta\, dx + \int_{B_6} \boldsymbol{f} \cdot D(\zeta v)\, dx.$$

Since ψ are supported in B_2 and $\zeta \equiv 1$ in B_2, the left-hand side of the above identity is equal to

$$\int_{B_2} Du \cdot \boldsymbol{\psi}\, dx.$$

Hence by using Hölder's inequality, the Sobolev inequality, and (3.10), we get

$$\left| \int_{B_2} Du \cdot \boldsymbol{\psi}\, dx \right| \lesssim \Big(\|u\|_{W^{1,1}(B_{5/2})} + \|\boldsymbol{f}\|_{L^p(B_{5/2})} \Big) \|\boldsymbol{\psi}\|_{L^{p'}(B_2)}.$$

Therefore, by duality and the Sobolev inequality, we have

$$\|u\|_{W^{1,p}(B_2)} \lesssim \|u\|_{W^{1,1}(B_{5/2})} + \|\boldsymbol{f}\|_{L^p(B_{5/2})}. \tag{3.11}$$

ii. $\frac{d}{d-1} \le p < \infty$: Following the same argument used deriving (3.11), for $1 \le r \le R \le 2$ and $\frac{d}{d-1} \le q < \infty$, we see that

$$\|u\|_{W^{1,q}(B_r)} \le C\big(\|u\|_{W^{1,q^*}(B_R)} + \|\boldsymbol{f}\|_{L^q(B_R)} \big), \tag{3.12}$$

where $C = C(d, \lambda, \omega_{\mathbf{A}}, q, r, R)$ provided that $u \in W^{1,q^*}(B_R)$. Here, q^* is any number in $(1, q)$ if $q = \frac{d}{d-1}$ and $q^* = \frac{dq}{d+q}$ if $q > \frac{d}{d-1}$. Let k be the smallest positive integer such that

$$k > \frac{dp - p - d}{p}.$$

We set

$$p_i = \frac{dp}{d + pi}, \quad r_i = 1 + \frac{i}{k}, \quad i \in \{0, 1, \dots, k\}.$$

By applying (3.12) iteratively, we have

$$\|u\|_{W^{1,p}(B_1)} \lesssim \|u\|_{W^{1,p_k}(B_2)} + \|\boldsymbol{f}\|_{L^p(B_2)}.$$

Since $p_k < \frac{d}{d-1}$, using (3.11) with $p = p_k$, we get the desired estimate. The theorem is proved. □

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