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EXISTENCE, MULTIPLICITY AND REGULARITY OF SOLUTIONS FOR THE FRACTIONAL p-LAPLACIAN EQUATION

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Abstract. We are concerned with the following elliptic equations:

$$\begin{cases}(-\Delta)_p^s u = \lambda f(x,u) & \text{in } \Omega,\\ u = 0 & \text{on } \mathbb{R}^N \backslash \Omega,\end{cases}$$

where λ are real parameters, $(-\Delta)_p^s$ is the fractional p-Laplacian operator, $0 < s < 1 < p < +\infty$, $sp < N$, and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies a Carathéodory condition. By applying abstract critical point results, we establish an estimate of the positive interval of the parameters λ for which our problem admits at least one or two nontrivial weak solutions when the nonlinearity f has the subcritical growth condition. In addition, under adequate conditions, we establish an apriori estimate in $L^\infty(\Omega)$ of any possible weak solution by applying the bootstrap argument.

1. Introduction

In the present paper, we consider the existence of nontrivial weak solutions to the nonlinear elliptic equations involving the fractional p-Laplacian of the form

$$(P_\lambda) \qquad \begin{cases}(-\Delta)_p^s u = \lambda f(x,u) & \text{in } \Omega,\\ u = 0 & \text{on } \mathbb{R}^N \backslash \Omega,\end{cases}$$

where the fractional p-Laplacian operator $(-\Delta)_p^s$ is defined by

$$(-\Delta)_p^s u(x) = 2 \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^N \backslash B_\varepsilon(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}}\, dy, \quad x \in \mathbb{R}^N.$$

Here, λ are real parameters, $0 < s < 1 < p < +\infty$, $sp < N$, $B_\varepsilon(x) := \{y \in \mathbb{R}^N : |x-y| < \varepsilon\}$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies a Carathéodory condition.

In the last years the study of fractional and nonlocal problems of elliptic type has received a tremendous popularity because the interest in such operators has

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consistently increased within the framework of the mathematical theory to concrete some phenomena such as social sciences, fractional quantum mechanics, materials science, continuum mechanics, phase transition phenomena, image process, game theory and Lévy processes; see [6, 8, 15, 18, 21, 27, 28] and the references therein. Especially, in terms of fractional quantum mechanics, the nonlinear fractional Schrödinger equation was originally suggested by Laskin in [21,22] as an extension of the Feynman path integral, from the Brownian-like to the Lévy-like quantum mechanical paths. Fractional operators are closely related to financial mathematics, because Lévy processes with jumps revealed as more adequate models of stock pricing in comparison with the Brownian ones used in the celebrated Black–Scholes option pricing model. In these directions, many researchers have been extensively studied the fractional Laplacian type problems in various way; see [5, 7, 8, 14, 15, 19, 23, 31–33, 36] and the references therein. Especially, a mountain pass theorem and applications to Dirichlet problems involving non-local integro-differential operators of fractional Laplacian type are given in [33]; see also [32]. Iannizzotto et al. [19] have investigated the existence and multiplicity results for the fractional p-Laplacian type problems. One of key ingredients for obtaining these results is the Ambrosetti and Rabinowitz condition ((AR)-condition for short) in [2];

(AR) There exist positive constants C_1 and ζ such that $\zeta > p$ and

$$0 < \zeta F(x,t) \le f(x,t)t \quad \text{for} \quad x \in \Omega \quad \text{and} \quad |t| \ge C_1,$$

where $F(x,t) = \int_0^t f(x,s)\,ds$, and Ω is a bounded domain in $\mathbb{R}^N$.

As is known, the (AR)-condition is significant to ascertain the Palais-Smale condition of an energy functional which plays a momentous role in employing critical point theory originally introduced by the paper [2]. However this condition is very restrictive and gets rid of many nonlinearities. In this regard, Miyagaki and Souto [29] established the existence of a nontrivial solution for the superlinear problems without the (AR)-condition. Inspired by this paper, the existence and multiplicity of solutions for the p-Laplacian equation in a bounded domain $\Omega \subset \mathbb{R}^N$ were obtained by Liu-Li [25] (see also [24]) under the following assumption:

(LL) There exists $C_* > 0$ such that

$$\mathcal{F}(x,t) \le \mathcal{F}(x,\tau) + C_*$$

for each $x \in \Omega$, $0 < t < \tau$ or $\tau < t < 0$, where $\mathcal{F}(x,t) = f(x,t)t - pF(x,t)$.

Under this condition, Wei and Su [34] showed that the fractional Laplacian problem possesses infinitely many weak solutions. Recently, the authors in [4] investigated the existence of at least one nontrivial weak solutions of elliptic equations with variable exponent by utilizing an abstract nonsmooth critical point result provided in [13] (see [9]). In particular, they obtained this existence result without assuming the condition (LL) as well as the (AR)-condition.

In this respect, the first aim of this paper is to concretely provide an estimate of the positive interval of the parameters λ for which the problem (P_λ) possesses at least one nontrivial weak solution when the nonlinear term f fulfils the subcritical growth condition. In addition, under adequate conditions, we establish an apriori estimate in $L^\infty(\Omega)$ of any possible weak solution applying the bootstrap argument. As compared with the local case, the value of $(-\Delta)^s_p u(x)$ at any point $x \in \Omega$ relies not only on the values of u on the whole Ω, but actually on the whole space $\mathbb{R}^N$. Hence more complicated analysis than the papers [4, 11, 12] has to be carefully carried out when we investigate the accurate interval for the parameters for which problem (P_λ) possesses at least one nontrivial weak solution. As far as we are aware, there were no such existence results for fractional p-Laplacian problems in this situation although our result is motivated by the paper [4]. Also it is worth noticing that we obtain the existence of at least one nontrivial weak solution for our problem without using the facts that the energy functional related to (P_λ) fulfils the Palais-Smale condition and the mountain pass geometry that is crucial to take advantage of the mountain pass theorem in [2].

The other aim of this paper is to consider the existence and uniform estimates of two distinct solutions for our problem. To do this, we utilize an abstract critical point result in [3] which is a variant of the work of G. Bonanno [10]. Recently, G. Bonanno and A. Chinnì [12] investigated the existence of at least two distinct weak solutions to the $p(x)$-Laplacian problems by applying critical point theorem in [10]. Contrast to our first main result, (AR)-condition is needed to ensure the existence of two distinct weak solutions for nonlinear elliptic equations; see [12]. In that sense, we show that the problem (P_λ) has two distinct weak solutions provided that f fulfils a weaker condition than the (AR)-condition.

This paper is structured as follows. First we recall briefly some basic results for the fractional Sobolev spaces. Next, under certain conditions on the nonlinear term f, we obtain the existence of at least one or two nontrivial weak solutions for problem (P_λ) whenever the parameter λ belongs to a positive interval.

2. Preliminaries and main results

In this section, we introduce the definitions and some underlying properties of the fractional Sobolev spaces. We mention the reader to [1,17,18] for further references and for some of the proofs of results. From this, we establish the existence of a nontrivial weak solution for the problem (P_λ) when the nonlinearity f has the subcritical growth condition.

Let $s \in (0,1)$ and $p \in (1,+\infty)$. The fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ is defined as

$$W^{s,p}(\mathbb{R}^N) := \left\{ u \in L^p(\mathbb{R}^N) : \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\, dxdy < +\infty \right\}$$

endowed with the norm

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} := \left(\|u\|^p_{L^p(\mathbb{R}^N)} + |u|^p_{W^{s,p}(\mathbb{R}^N)}\right)^{\frac{1}{p}},$$

where

$$\|u\|^p_{L^p(\mathbb{R}^N)} := \int_{\mathbb{R}^N} |u|^p \, dx \quad \text{and} \quad |u|^p_{W^{s,p}(\mathbb{R}^N)} := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} \, dxdy.$$

Let $s \in (0,1)$ and $1 < p < +\infty$. Then $W^{s,p}(\mathbb{R}^N)$ is a separable and reflexive Banach space, and also the space $C_0^\infty(\mathbb{R}^N)$ is dense in $W^{s,p}(\mathbb{R}^N)$, that is $W_0^{s,p}(\mathbb{R}^N) = W^{s,p}(\mathbb{R}^N)$ (see e.g. [1, 17]).

We consider the problem (P_λ) in the closed linear subspace defined by

$$X_s^p(\Omega) = \left\{u \in W^{s,p}(\mathbb{R}^N) : u(x) = 0 \text{ a.e. in } \mathbb{R}^N \backslash \Omega\right\}$$

with the norm

$$\|u\|_{X_s^p(\Omega)} = \left(|u|^p_{W^{s,p}(\mathbb{R}^N)} + \|u\|^p_{L^p(\Omega)}\right)^{\frac{1}{p}}.$$

Then $X_s^p(\Omega)$ is a uniformly convex Banach space; see Lemma 2.4 in [35]. Now we list some basic results which will be needed to obtain our main result.

Lemma 2.1 ([31])**.** *Let Ω be an bounded open set in $\mathbb{R}^N$, $s \in (0,1)$ and $p \in [1,+\infty)$. Then*

$$\|u\|^p_{L^p(\Omega)} \le \frac{sp\,|\Omega|^{\frac{sp}{N}}}{2\omega_N^{\frac{sp}{N}+1}} |u|^p_{W^{s,p}(\mathbb{R}^N)}$$

for any $u \in \tilde{W}^{s,p}(\mathbb{R}^N)$. Here $|\Omega|$ means the Lebesgue measure of Ω, ω_N denotes the volume of the N-dimensional unit ball and we denoted by $\tilde{W}^{s,p}(\mathbb{R}^N)$ the space of all $u \in X_s^p(\Omega)$ such that $\tilde{u} \in W^{s,p}(\mathbb{R}^N)$, where $\tilde{u}$ is the extension by zero of u.

Lemma 2.2 ([16])**.** *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz boundary, $s \in (0,1)$ and $p \in (1,+\infty)$. Then we have the following continuous embeddings*:

$$\begin{aligned} &W^{s,p}(\Omega) \hookrightarrow L^q(\Omega) && \textit{for all } q \in [1,p_s^*], && \textit{if } sp < N;\\ &W^{s,p}(\Omega) \hookrightarrow L^q(\Omega) && \textit{for every } q \in [1,\infty), && \textit{if } sp = N;\\ &W^{s,p}(\Omega) \hookrightarrow C_b^{0,\lambda}(\Omega) && \textit{for all } \lambda < s - N/p, && \textit{if } sp > N,\end{aligned}$$

where p_s^* is the fractional critical Sobolev exponent, that is

$$p_s^* := \begin{cases} \frac{Np}{N-sp} & \textit{if } sp < N,\\ +\infty & \textit{if } sp \ge N.\end{cases}$$

In particular, the space $W^{s,p}(\Omega)$ is compactly embedded in $L^q(\Omega)$ for any $q \in [p, p_s^*)$.

Lemma 2.3 ([26])**.** *Let $s \in (0,1)$ and $p \in (1,+\infty)$ be such that $sp < N$. Then, for all $u \in W^{s,p}(\mathbb{R}^N)$, there holds*

$$\|u\|^p_{L^{p_s^*}(\mathbb{R}^N)} \leq C_{p_s^*}\, |u|^p_{W^{s,p}(\mathbb{R}^N)},$$

where

$$C_{p_s^*} = \frac{(N+2p)^{3p} p^{p+2} 2^{(N+1)(N+2)} s(1-s)}{N^{\frac{p}{p_s^*}} |S^{N-1}|^{\frac{ps}{N}+1} (N-sp)^{p-1}}.$$

Here $\left|S^{N-1}\right|$ denotes the surface area of the $(N-1)$-dimensional unit sphere.

Thanks to the above lemma, it is possible to establish the estimate of a positive constant denoted by C_q which is crucial to obtain the positive interval of the parameters λ for which the problem (P_λ) possesses at least one or two nontrivial weak solutions.

Remark 2.4. Recall that for each $1 < hs < N$, putting $h_s^* = hN/(N-hs)$, from Lemma 2.2 one has $X_s^h(\Omega) \hookrightarrow L^{h_s^*}(\Omega)$ with continuous embedding. Precisely, according to Lemma 2.3, for each $u \in X_s^h(\Omega)$, it results

$$\|u\|_{L^{h_s^*}(\Omega)} \leq C_{h_s^*}^{\frac{1}{h}}\, |u|_{W^{s,h}(\mathbb{R}^N)}\,. \tag{2.1}$$

Let $q \in [1, h_s^*]$ be fixed. Set $\ell = h_s^*/q$ and $\ell' = h_s^*/(h_s^*-q)$. Since $|u|^q \in L^{\frac{h_s^*}{q}}(\Omega)$, the Hölder inequality implies that

$$\begin{aligned}\int_\Omega |u(x)|^q dx &= \|u^q\|_{L^1(\Omega)} \leq \|u^q\|_{L^\ell(\Omega)} \|1\|_{L^{\ell'}(\Omega)} \\ &= \Big(\int_\Omega |u(x)|^{h_s^*} dx\Big)^{\frac{q}{h_s^*}} |\Omega|^{\frac{1}{\ell'}} = \|u\|^q_{L^{h_s^*}(\Omega)} |\Omega|^{\frac{h_s^*-q}{h_s^*}}\end{aligned}$$

and so

$$\|u\|_{L^q(\Omega)} = (\|u^q\|_{L^\ell(\Omega)})^{\frac{1}{q}} \leq \|u\|_{L^{h_s^*}(\Omega)} |\Omega|^{\frac{h_s^*-q}{h_s^* q}}\,.$$

By (2.1) one has

$$\|u\|_{L^q(\Omega)} \leq \|u\|_{L^{h_s^*}(\Omega)} |\Omega|^{\frac{h_s^*-q}{h_s^* q}} \leq C_{h_s^*}^{\frac{1}{h}} |\Omega|^{\frac{h_s^*-q}{h_s^* q}}\, |u|_{W^{s,h}(\mathbb{R}^N)}\,. \tag{2.2}$$

Now, let $1 < q \leq p_s^*$. By applying (2.2) for $h = p$, it follows from Lemma 2.1 that

$$\|u\|_{L^q(\Omega)} \leq C_{p_s^*}^{\frac{1}{p}} |\Omega|^{\frac{p_s^*-q}{p_s^* q}}\, |u|_{W^{s,p}(\mathbb{R}^N)} \tag{2.3}$$

for each $u \in X_s^p(\Omega)$. Hence we will denote the positive constant C_q for which one has

$$\|u\|_{L^q(\Omega)} \leq C_q\, |u|_{W^{s,p}(\mathbb{R}^N)}\,, \tag{2.4}$$

where

$$C_q \leq C_{p_s^*}^{\frac{1}{p}} |\Omega|^{\frac{p_s^*-q}{p_s^* q}}\,.$$

Definition 2.5. Let $(X, \|\cdot\|_X)$ be a real Banach space If $\Phi, \Psi : X \to \mathbb{R}$ are two continuously Gâteaux differentiable functionals and $r \in \mathbb{R}$, we say that $J = \Phi - \Psi$ satisfies the Cerami condition cut off upper at r ($(C)^{[r]}$-condition for short) if any sequence $\{u_n\}_{n\in\mathbb{N}}$ in X such that

(C1) $\{J(u_n)\}_{n\in\mathbb{N}}$ is bounded;
(C2) $\|J'(u_n)\|_{X^*}(1 + \|u_n\|_X) \to 0$ as $n \to \infty$;
(C3) $0 < \Phi(u_n) < r$ for all $n \in \mathbb{N}$;

has a convergent subsequence.

Now we recall the key lemma to get our main result. This assertion can be found in [20]; see [13] for the case of the Palais-Smale condition.

Lemma 2.6. *Let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that*

$$\inf_{u\in X} \Phi(u) = \Phi(0) = \Psi(0) = 0.$$

Assume that there are a positive constant μ and an element $\tilde{u}$ in X, with $0 < \Phi(\tilde{u}) < \mu$, such that

$$\frac{\sup_{\Phi(u)\le\mu} \Psi(u)}{\mu} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})} \tag{2.5}$$

holds and for each $\lambda \in \Lambda_\mu := \left(\frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{\mu}{\sup_{\Phi(u)\le\mu} \Psi(u)}\right)$, the functional $I_\lambda := \Phi - \lambda\Psi$ satisfies the $(C)^{[\mu]}$-condition. Then, for each $\lambda \in \Lambda_\mu$, the functional I_λ has a nontrivial point x_λ in $\Phi^{-1}((0, \mu))$ such that $I_\lambda(x_\lambda) \le I_\lambda(x)$ for all x in $\Phi^{-1}((0, \mu))$ and $I'_\lambda(x_\lambda) = 0$.

The following assertion is crucial to ensure that the problem (P_λ) has at least two distinct weak solutions.

Lemma 2.7 ([3])**.** *Let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that Gâteaux derivative of Ψ is compact and*

$$\inf_{u\in X} \Phi(u) = \Phi(0) = \Psi(0) = 0.$$

Assume that there are a constant $\mu > 0$ and an $\tilde{u}$ in X, with $0 < \Phi(\tilde{u}) < \mu$, such that

$$\frac{\sup_{\Phi(u)\le\mu} \Psi(u)}{\mu} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})}$$

holds and for every $\lambda \in \Lambda_\mu := \left(\frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{\mu}{\sup_{\Phi(u)\le\mu} \Psi(u)}\right)$, the functional $I_\lambda := \Phi - \lambda\Psi$ satisfies the (C)-condition and it is unbounded from below. Then, for each $\lambda \in \Lambda_\mu$ the functional I_λ has two distinct critical points u_0 and u_1 such that u_0 is a nontrivial local minimum satisfying $\Phi(u_0) < \mu$ and

$$I_\lambda(u_1) = \inf_{\psi\in\Upsilon} \max_{t\in[0,1]} I_\lambda(\psi(t)),$$

where Υ is the family of paths $\gamma : [0,1] \to (X, \|\cdot\|_X)$ with $\psi(0) = u_0$ and $\psi(1) = \bar{z}$, and $\bar{z}$ is such that $I_\lambda(\bar{z}) \le I_\lambda(u_0)$.

Definition 2.8. Let $0 < s < 1 < p < +\infty$. We say that $u \in X_s^p(\Omega)$ is a weak solution of the problem (P_λ) if

$$\begin{aligned} &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+ps}}\,dxdy \\ &= \lambda \int_\Omega f(x,u)v\,dx \end{aligned} \tag{2.6}$$

for all $v \in X_s^p(\Omega)$.

Let us define a functional $\Phi_{s,p} : X_s^p(\Omega) \to \mathbb{R}$ by

$$\Phi_{s,p}(u) = \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dxdy.$$

Then $\Phi_{s,p}$ is well defined on $X_s^p(\Omega)$, $\Phi_{s,p} \in C^1(X_s^p(\Omega), \mathbb{R})$ and its Fréchet derivative is given by

$$\langle \Phi'_{s,p}(u), v\rangle = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+ps}}\,dxdy$$

for any $v \in X_s^p(\Omega)$ where $\langle\cdot,\cdot\rangle$ denotes the pairing of $X_s^p(\Omega)$ and its dual $(X_s^p(\Omega))^*$; see [31].

Lemma 2.9 ([31])**.** *Let $0 < s < 1 < p < +\infty$. The functional $\Phi'_{s,p} : X_s^p(\Omega) \to (X_s^p(\Omega))^*$ is of type (S_+), i.e., if $u_n \rightharpoonup u$ in $X_s^p(\Omega)$ and*

$$\limsup_{n\to\infty} \left\langle \Phi'_{s,p}(u_n) - \Phi'_{s,p}(u), u_n - u\right\rangle \le 0,$$

then $u_n \to u$ in $X_s^p(\Omega)$ as $n \to \infty$.

We assume that for $1 < q < p_s^*$ and $x \in \Omega$,

(F1) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition and there exist nonnegative functions $\rho \in L^\infty(\Omega)$ and $\sigma \in L^\infty(\Omega)$ such that

$$|f(x,t)| \le \rho(x) + \sigma(x)\,|t|^{q-1}$$

for all $(x,t) \in \Omega \times \mathbb{R}$.

We denote with F the function defined by

$$F(x,\xi) := \int_0^\xi f(x,t)\,dt \quad \text{for } (x,\xi) \in \Omega \times \mathbb{R}.$$

Define the functional $\Psi : X_s^p(\Omega) \to \mathbb{R}$ by

$$\Psi(u) = \int_\Omega F(x,u)\,dx.$$

Next we define a functional $I_\lambda : X_s^p(\Omega) \to \mathbb{R}$ by

$$I_\lambda(u) = \Phi_{s,p}(u) - \lambda\Psi(u).$$

Theorem 2.10. *Suppose that* (F1) *holds and the following condition is verified*:

(F2)
$$\limsup_{t\to 0^+} \frac{\int_\Omega F(x,t)\,dx}{t^p} = +\infty.$$

Then, put

$$\lambda^* = \frac{1}{C_p(p)^{\frac{1}{p}}\,|\Omega|^{\frac{1}{p'}}\,\|\rho\|_{L^\infty(\Omega)} + q^{-1}(p)^{\frac{q}{p}} C_q^q \|\sigma\|_{L^\infty(\Omega)}}$$

for each $\lambda \in (0,\lambda^*)$, the problem (P_λ) has at least one nontrivial weak solution. Furthermore, if q in (F1) *satisfies $p \le q < p_s^*$, then any weak solution of (P_λ) belongs to the space $L^\infty(\Omega)$.*

Proof. It is clear that the functional $\Phi_{s,p}$ is in $C^1(X_s^p(\Omega),\mathbb{R})$. In addition, from the condition (F1) and Lemma 2.2, the functional Ψ is in $C^1(X_s^p(\Omega),\mathbb{R})$ and has compact derivative.

We prove the existence of a nontrivial solution for $\lambda \in (0,\lambda^*)$. Fix $\lambda \in (0,\lambda^*)$, and choose $\mu = 1$ to satisfy the condition (2.5) of Lemma 2.6. From (F2), there exists

(2.7)
$$0 < \xi_\lambda < \min\left\{1, \left(\frac{p}{2\omega_N^2 d^{N-sp}\mathcal{M}}\right)^{\frac{1}{p}}\right\}$$

such that

(2.8)
$$\frac{p\ d^{sp}\operatorname{ess\,inf}_{x\in\Omega} F(x,\xi_\lambda)}{2^{N+1}\xi_\lambda^p \omega_N \mathcal{M}} > \frac{1}{\lambda},$$

where ω_N is given in Lemma 2.1 and

$$\mathcal{M} := \frac{2^{2p+N-sp-1}}{(p-sp)(N-sp+p)} + \frac{2}{2^{N-sp}sp(N+p-sp)} + \frac{1}{sp(N-sp)}.$$

First of all, we show that I_λ satisfies the $(C)^{[\mu]}$-condition. Let μ be a fixed positive number and let $\{u_n\}$ be a Cerami sequence in $X_s^p(\Omega)$ satisfying (C1), (C2), and (C3). Since $\Phi_{s,p}(u_n) < \mu$, it follows from Lemma 2.1 that

$$\begin{aligned}\|u_n\|^p_{X_s^p(\Omega)} &\le \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+sp}}\,dxdy + \frac{1}{p}\int_\Omega |u_n(x)|^p\,dx\\ &\le p\Phi_{s,p}(u_n) + \|u_n\|^p_{L^p(\Omega)} \le p\mu\left(1 + \frac{s\,|\Omega|^{\frac{sp}{N}}}{2\omega_N^{\frac{sp}{N}+1}}\right).\end{aligned}$$

Thus, $\{u_n\}$ is a bounded sequence in $X_s^p(\Omega)$ and we may suppose that $u_n \rightharpoonup u$ as $n\to\infty$ for some $u \in X_s^p(\Omega)$. Then taking into account Lemma 2.2 we have $u_n \to u$ as $n \to \infty$ in $L^p(\Omega)$. From (C2), there is a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ in $(0,+\infty)$, $\varepsilon_n \to 0^+$ such that

$$\langle \Phi'(u_n), u-u_n\rangle + \langle(-\lambda\Psi)'(u_n), u-u_n\rangle \ge \frac{-\varepsilon_n\|v-u_n\|_{X_s^p(\Omega)}}{1+\|u_n\|_{X_s^p(\Omega)}}$$

for each $n \in N$. Since $\varepsilon_n \to 0^+$, we deduce

$$\limsup_{n\to+\infty}\langle \Phi'(u_n), u_n - u\rangle \le \limsup_{n\to+\infty}\langle (-\lambda\Psi)'(u_n), u_n - u\rangle.$$

Invoking the compactness of $(-\lambda\Psi)'$, we obtain

$$\limsup_{n\to+\infty}\langle \Phi'(u_n), u_n - u\rangle \le 0.$$

Since $\Phi'_{s,p}$ is of type (S_+), we assert that $u_n \to u$ in $X_s^p(\Omega)$ as $n \to \infty$.

Next, to apply Lemma 2.6 with $\Phi = \Phi_{s,p}$, we provide that there is an element $\tilde{u} \in X_s^p(\Omega)$ satisfying $\Phi_{s,p}(\tilde{u}) < 1$ and the relation (2.5). Define

$$\tilde{u}(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R}^N \setminus B_N(x_0, d), \\ \xi_\lambda & \text{if } x \in B_N(x_0, \frac{d}{2}), \\ \frac{2\xi_\lambda}{d}(d - |x - x_0|) & \text{if } x \in B_N(x_0, d) \setminus B_N(x_0, \frac{d}{2}). \end{cases}$$

Then it is clear that $0 \le \tilde{u}(x) \le \xi_\lambda$ for all $x \in \mathbb{R}^N$, and so $\tilde{u} \in X_s^p(\Omega)$. Put $B_d = B_N(x_0, d)$. Then, it follows that

$$\begin{aligned}
\Phi_{s,p}(\tilde{u}) &= \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\tilde{u}(x) - \tilde{u}(y)|^p}{|x-y|^{N+sp}}\,dxdy \\
&= \frac{1}{p}\int_{B_d\setminus B_{\frac{d}{2}}}\int_{B_d\setminus B_{\frac{d}{2}}} \frac{|\tilde{u}(x) - \tilde{u}(y)|^p}{|x-y|^{N+sp}}\,dxdy \\
&\quad + \frac{2}{p}\int_{B_d\setminus B_{\frac{d}{2}}}\int_{\mathbb{R}^N\setminus B_d} \frac{|\tilde{u}(x) - \tilde{u}(y)|^p}{|x-y|^{N+sp}}\,dxdy \\
&\quad + \frac{2}{p}\int_{B_{\frac{d}{2}}}\int_{B_d\setminus B_{\frac{d}{2}}} \frac{|\tilde{u}(x) - \tilde{u}(y)|^p}{|x-y|^{N+sp}}\,dxdy \\
&\quad + \frac{2}{p}\int_{\mathbb{R}^N\setminus B_d}\int_{B_{\frac{d}{2}}} \frac{|\tilde{u}(x) - \tilde{u}(y)|^p}{|x-y|^{N+sp}}\,dxdy \\
&=: \frac{1}{p}(I_1 + 2I_2 + 2I_3 + 2I_4).
\end{aligned}$$

Next we estimate I_1–I_4, by the direct calculation, respectively:

- Estimate of I_1: For any positive constant ε small enough,

$$\begin{aligned}
I_1 &= \int_{B_d\setminus B_{\frac{d}{2}}}\int_{B_d\setminus B_{\frac{d}{2}}} \frac{|\tilde{u}(x) - \tilde{u}(y)|^p}{|x-y|^{N+sp}}\,dxdy \\
&\le \frac{2^p\xi_\lambda^p}{d^p}\int_{B_d\setminus B_{\frac{d}{2}}}\int_{B_d\setminus B_{\frac{d}{2}}} \frac{|x-y|^p}{|x-y|^{N+sp}}\,dxdy \\
&\le \frac{2^p\xi_\lambda^p\omega_N}{d^p}\int_{B_d\setminus B_{\frac{d}{2}}}\int_{\varepsilon}^{d+|y-x_0|} r^{p-sp-1}\,drdy
\end{aligned}$$

$$\leq \frac{2^p\xi_\lambda^p\omega_N}{d^p}\int_{B_d\setminus B_{\frac{d}{2}}}\frac{(d+|y-x_0|)^{p-sp}}{p-sp}\,dy$$
$$=\frac{2^p\xi_\lambda^p\omega_N^2}{(p-sp)d^p}\int_{\frac{3}{2}d}^{2d}r^{p+N-sp-1}\,dr$$
$$=\frac{2^p\xi_\lambda^p\omega_N^2d^{N-sp}}{(p-sp)(p+N-sp)}\big(2^{p+N-sp}-(\frac{3}{2})^{p+N-sp}\big).$$

• Estimate of I_2:

$$I_2=\int_{B_d\setminus B_{\frac{d}{2}}}\int_{\mathbb{R}^N\setminus B_d}\frac{|\tilde{u}(x)-\tilde{u}(y)|^p}{|x-y|^{N+sp}}\,dxdy$$
$$\leq\frac{2^p\xi_\lambda^p}{d^p}\int_{B_d\setminus B_{\frac{d}{2}}}\int_{\mathbb{R}^N\setminus B_d}\frac{|d-|y-x_0||^p}{|x-y|^{N+sp}}\,dxdy$$
$$=\frac{2^p\xi_\lambda^p\omega_N}{d^p}\int_{B_d\setminus B_{\frac{d}{2}}}\int_{d-|y-x_0|}^{\infty}\frac{|d-|y-x_0||^p}{r^{sp+1}}\,drdy$$
$$=\frac{2^p\xi_\lambda^p\omega_N}{d^psp}\int_{B_d\setminus B_{\frac{d}{2}}}|d-|y-x_0|\,|^{p-sp}\,dy$$
$$=\frac{2^p\xi_\lambda^p\omega_N^2}{d^psp}\int_0^{\frac{d}{2}}r^{N+p-sp-1}\,dr$$
$$=\frac{\xi_\lambda^pd^{N-sp}\omega_N^2}{2^{N-sp}sp(N+p-sp)}.$$

• Estimate of I_3:

$$I_3=\int_{B_{\frac{d}{2}}}\int_{B_d\setminus B_{\frac{d}{2}}}\frac{|\tilde{u}(x)-\tilde{u}(y)|^p}{|x-y|^{N+sp}}\,dxdy$$
$$=\frac{2^p\xi_\lambda^p}{d^p}\int_{B_{\frac{d}{2}}}\int_{B_d\setminus B_{\frac{d}{2}}}\frac{|-\frac{d}{2}+|x-x_0||^p}{|x-y|^{N+sp}}\,dxdy$$
$$=\frac{2^p\xi_\lambda^p}{d^p}\int_{B_d\setminus B_{\frac{d}{2}}}\int_{B_{\frac{d}{2}}}\frac{|-\frac{d}{2}+|x-x_0||^p}{|x-y|^{N+sp}}\,dydx$$
$$=\frac{2^p\xi_\lambda^p\omega_N}{d^p}\int_{B_d\setminus B_{\frac{d}{2}}}|-\frac{d}{2}+|x-x_0|\,|^p\int_{|x-x_0|-\frac{d}{2}}^{|x-x_0|+\frac{d}{2}}\frac{1}{r^{sp+1}}\,drdx$$
$$\leq\frac{2^p\xi_\lambda^p\omega_N}{d^psp}\int_{B_d\setminus B_{\frac{d}{2}}}|-\frac{d}{2}+|x-x_0|\,|^{p-sp}\,dx$$

$$
\begin{aligned}
&= \frac{2^p \xi_\lambda^p \omega_N^2}{d^p sp} \int_0^{\frac{d}{2}} t^{N+p-sp-1}\, dt \\
&= \frac{d^{N-sp} \xi_\lambda^p \omega_N^2}{2^{N-sp} sp(N+p-sp)}.
\end{aligned}
$$

• Estimate of I_4:

$$
\begin{aligned}
I_4 = \int_{B_{\frac{d}{2}}} \int_{\mathbb{R}^N \setminus B_d} \frac{|\tilde{u}(x) - \tilde{u}(y)|^p}{|x-y|^{N+sp}}\, dxdy &= \xi_\lambda^p \int_{B_{\frac{d}{2}}} \int_{\mathbb{R}^N \setminus B_d} \frac{1}{|x-y|^{N+sp}}\, dxdy \\
&= \xi_\lambda^p \omega_N \int_{B_{\frac{d}{2}}} \int_{d-|y-x_0|}^{\infty} r^{-sp-1}\, drdy \\
&= \xi_\lambda^p \omega_N \int_{B_{\frac{d}{2}}} \frac{1}{sp(d-|y-x_0|)^{sp}}\, dy \\
&= \frac{\xi_\lambda^p \omega_N^2}{sp} \int_{\frac{d}{2}}^{d} r^{N-sp-1}\, dr \\
&= \frac{\xi_\lambda^p \omega_N^2 d^{N-sp}}{sp(N-sp)} \big(1 - \frac{1}{2^{N-sp}}\big) \\
&\le \frac{\xi_\lambda^p \omega_N^2 d^{N-sp}}{sp(N-sp)}.
\end{aligned}
$$

Hence it follows from the relation (2.7) that

$$
\Phi_{s,p}(\tilde{u}) \le \frac{2\xi_\lambda^p \omega_N^2 d^{N-sp} \mathcal{M}}{p} \le 1.
$$

Owing to the relation (2.8), we infer that

$$
\begin{aligned}
\Psi(\tilde{u}) &\ge \int_{B_{\frac{d}{2}}} F(x, \tilde{u})\, dx \\
&\ge \operatorname*{ess\,inf}_{x \in \Omega} F(x, \xi_\lambda) \left(\frac{\omega_N d^N}{2^N} \right)
\end{aligned}
$$

and thus

$$
\frac{\Psi(\tilde{u})}{\Phi_{s,p}(\tilde{u})} \ge \frac{p\ d^{sp} \operatorname{ess\,inf}_{x \in \Omega} F(x, \xi_\lambda)}{2^{N+1} \xi_\lambda^p \omega_N \mathcal{M}} > \frac{1}{\lambda}. \tag{2.9}
$$

Since $|u|_{W^{s,p}(\mathbb{R}^N)} = \Phi_{s,p}(u) \le p^{\frac{1}{p}}$, the condition (F1), Hölder's inequality and Remark 2.4 imply that, for each $u \in \Phi_{s,p}^{-1}((-\infty, 1])$, we get

$$
\begin{aligned}
\Psi(u) &= \int_\Omega F(x, u)\, dx \\
&\le \int_\Omega |\rho(x)|\, |u(x)|\ dx + \int_\Omega \frac{|\sigma(x)|}{q}\, |u(x)|^q\ dx
\end{aligned}
$$

$$\le \|\rho\|_{L^\infty(\Omega)} |\Omega|^{\frac{1}{p'}} \|u\|_{L^p(\Omega)} + q^{-1}\|\sigma\|_{L^\infty(\Omega)}\|u\|^q_{L^q(\Omega)}$$
$$\le C_p p^{\frac{1}{p}} |\Omega|^{\frac{1}{p'}} \|\rho\|_{L^\infty(\Omega)} + q^{-1}p^{\frac{q}{p}} C_q^q \|\sigma\|_{L^\infty(\Omega)},$$

and hence

$$\sup_{u\in\Phi_{s,p}^{-1}((-\infty,1])} \Psi(u) \le C_p p^{\frac{1}{p}} |\Omega|^{\frac{1}{p'}} \|\rho\|_{L^\infty(\Omega)} + q^{-1}p^{\frac{q}{p}} C_q^q \|\sigma\|_{L^\infty(\Omega)} = \frac{1}{\lambda^*} < \frac{1}{\lambda}. \tag{2.10}$$

Due to the inequalities (2.9) and (2.10), we have

$$\sup_{u\in\Phi_{s,p}^{-1}((-\infty,1])} \Psi(u) < \frac{1}{\lambda} < \frac{\Psi(\tilde{u})}{\Phi_{s,p}(\tilde{u})}.$$

Since $\lambda \in (\frac{\Phi_{s,p}(\tilde{u})}{\Psi(\tilde{u})}, \frac{1}{\sup_{\Phi_{s,p}(u)\le 1}\Psi(u)})$, Lemma 2.6 with $\mu = 1$ and $\Phi = \Phi_{s,p}$ guarantees that the problem (P_λ) has at least one nontrivial weak solution for each $\lambda \in (0, \lambda^*)$.

For the case that $p \le q < p_s^*$, we will show that a weak solution $u_{\lambda,0}$ of the problem (P_λ) belongs to the space $L^\infty(\Omega)$ for any $\lambda \in (0,\lambda^*)$. Suppose that $u_{\lambda,0}$ is non-negative. For $K > 0$, we define

$$\varrho_K(x) = \min\{u_{\lambda,0}(x), K\}$$

and choose $v = \varrho_K^{mp+1}$ $(m \ge 0)$ as a test function in (2.6). Then, $v \in W(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$, and it follows from (2.6) that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_{\lambda,0}(x)-u_{\lambda,0}(y)|^{p-2}(u_{\lambda,0}(x)-u_{\lambda,0}(y))(\varrho_K^{mp+1}(x)-\varrho_K^{mp+1}(y))}{|x-y|^{N+ps}}\,dxdy = \lambda\int_\Omega f(x,u_{\lambda,0})\varrho_K^{mp+1}\,dx. \tag{2.11}$$

The left-hand side of (2.11) can be estimated as follows:

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_{\lambda,0}(x)-u_{\lambda,0}(y)|^{p-2}(u_{\lambda,0}(x)-u_{\lambda,0}(y))(v_K^{mp+1}(x)-v_K^{mp+1}(y))}{|x-y|^{N+ps}}\,dxdy \\
&\ge \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_{\lambda,0}(x)-u_{\lambda,0}(y)|^{p-1}\left|\varrho_K^{mp+1}(x)-\varrho_K^{mp+1}(y)\right|}{|x-y|^{N+ps}}\,dxdy \\
&\ge C_5\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\varrho_K^{m+1}(x)-\varrho_K^{m+1}(y)|^p}{|x-y|^{N+ps}}\,dxdy \\
&\ge C_6\|\varrho_K^{m+1}\|^p_{W(\mathbb{R}^N)} \\
&\ge C_7\left(\int_\Omega |\varrho_K|^{(m+1)p_s^*}\,dx\right)^{\frac{p}{p_s^*}}
\end{aligned} \tag{2.12}$$

for some positive constants C_5, C_6, C_7.

Now, two cases arise, i.e., $p < q < p_s^*$ or $q = p$.

Case I: For $p < q < p_s^*$, due to the assumption (F1) and the Hölder inequality, we get the following estimation for the right-hand side of (2.11) as follows:

(2.13)

$$\begin{aligned}
&\lambda \int_\Omega f(x, u_{\lambda,0})\varrho_K^{mp+1}\,dx \\
&\le \lambda \int_\Omega \rho(x)|u_{\lambda,0}|^{mp+1} + \sigma(x)|u_{\lambda,0}|^{mp+p}\,dx \\
&\le \lambda \int_\Omega \rho(x)(|u_{\lambda,0}|^{mp+p} + |u_{\lambda,0}|^{m+1})\,dx \\
&\quad + \lambda\|\sigma\|_{L^\infty(\Omega)}\,|\Omega|^{\frac{1}{\tau_1}}\left(\int_\Omega |u_{\lambda,0}|^{(m+1)p\tau_1'}|u_{\lambda,0}|^{(q-p)\tau_1'}\,dx\right)^{\frac{1}{\tau_1'}} \\
&\le \lambda\|\rho\|_{L^\infty(\Omega)}\int_\Omega |u_{\lambda,0}|^{(m+1)p}\,dx + \lambda\|\rho\|_{L^\infty(\Omega)}\,|\Omega|^{\frac{1}{p'}}\left(\int_\Omega |u_{\lambda,0}|^{(m+1)p}\,dx\right)^{\frac{1}{p}} \\
&\quad + \lambda\|\sigma\|_{L^\infty(\Omega)}\,|\Omega|^{\frac{1}{\tau_1}}\left(\int_\Omega |u_{\lambda,0}|^{(m+1)\zeta}\,dx\right)^{\frac{p}{\zeta}}\left(\int_{\mathbb{R}^N} |u_{\lambda,0}|^{(q-p)\tau_1'\frac{\zeta}{\zeta-p\tau_1'}}\,dx\right)^{\frac{\zeta-p\tau_1'}{\zeta\tau_1'}},
\end{aligned}$$

where $\tau_1 = \frac{p_s^*}{p_s^*-q}$, and $\zeta = \frac{pp_s^*\tau_1'}{p_s^*-(q-p)\tau_1'}$. Obviously $\zeta \le p_s^*$, $1 < \frac{\zeta}{p\tau_1'}$, and $\frac{(q-p)\tau_1'\zeta}{\zeta-p\tau_1'} = p_s^*$, and hence (2.13) yields

(2.14)

$$\begin{aligned}
&\lambda \int_\Omega f(x, u_{\lambda,0})\varrho_K^{mp+1}\,dx \\
&\le \lambda\|\rho\|_{L^\infty(\mathbb{R}^N)}\int_\Omega |u_{\lambda,0}|^{(m+1)p}\,dx + \lambda\|\rho\|_{L^\infty(\Omega)}\,|\Omega|^{\frac{1}{p'}}\left(\int_\Omega |u_{\lambda,0}|^{(m+1)p}\,dx\right)^{\frac{1}{p}} \\
&\quad + \lambda\|\sigma\|_{L^\infty(\Omega)}\,|\Omega|^{\frac{1}{\tau_1'}}\left(\int_\Omega |u_{\lambda,0}|^{p_s^*}\,dx\right)^{\frac{\zeta-p\tau_1'}{\zeta\tau_1'}}\left(\int_{\mathbb{R}^N} |u_{\lambda,0}|^{(m+1)\zeta}\,dx\right)^{\frac{p}{\zeta}}.
\end{aligned}$$

It follows from (2.11), (2.12), (2.14), and the Sobolev inequality that there exist positive constants C_7, C_8 and C_9 (independent of K and $m > 0$) such that

$$\|\varrho_K\|_{L^{(m+1)p_s^*}(\Omega)}^{(m+1)p} \le C_7\|u_{\lambda,0}\|_{L^{(m+1)p}(\Omega)}^{(m+1)p} + C_8\|u_{\lambda,0}\|_{L^{(m+1)p}(\Omega)}^{m+1} + C_9\|u_{\lambda,0}\|_{L^{(m+1)\zeta}(\Omega)}^{(m+1)p}.$$

Hence, for any positive constant K there are positive constants C_{10} and C_{11} such that

$$\|\varrho_K\|_{L^{(m+1)p_s^*}(\Omega)} \le \begin{cases} C_{10}^{\frac{1}{(m+1)p}}\|u_{\lambda,0}\|_{L^{(m+1)t}(\Omega)} & \text{if} \quad \|u_{\lambda,0}\|_{L^{(m+1)p}(\Omega)} \ge 1, \\ C_{11}^{\frac{1}{(m+1)p}}\|u_{\lambda,0}\|_{L^{(m+1)\zeta}(\Omega)} & \text{if} \quad \|u_{\lambda,0}\|_{L^{(m+1)p}(\Omega)} < 1, \end{cases}$$

where t is either p or ζ. This estimation is a starting point for a bootstrap technique. Since $u_{\lambda,0}(x) = \lim_{K\to\infty} \varrho_K(x)$ for almost every $x \in \Omega$, obvious modifications of the proof of Proposition 1 in [20] yields that $\|u_{\lambda,0}\|_{L^\infty(\Omega)} \le C_{14}$ for some positive constant C_{14}.

Case II: For $q = p$, as in the relation (2.13), the right-hand side of (2.11) can be estimated by using the assumption (F2) and the Hölder inequality:

$$\begin{aligned}
&\lambda \int_\Omega f(x,u_{\lambda,0})\varrho_K^{mp+1}\,dx \\
&\le \lambda\|\rho\|_{L^\infty(\Omega)}\|u_{\lambda,0}\|^{(m+1)p}_{L^{(m+1)p}(\Omega)} + \lambda\|\rho\|_{L^\infty(\Omega)}\,|\Omega|^{\frac{1}{p'}}\,\|u_{\lambda,0}\|^{m+1}_{L^{(m+1)p}(\Omega)} \\
&\quad + \lambda\int_\Omega \sigma(x)|u_{\lambda,0}|^{(m+1)p}\,dx \\
&\le \lambda(\|\rho\|_{L^\infty(\Omega)} + \|\sigma\|_{L^\infty(\Omega)})\|u_{\lambda,0}\|^{(m+1)p}_{L^{(m+1)p}(\Omega)} + \lambda\|\rho\|_{L^\infty(\Omega)}\,|\Omega|^{\frac{1}{p'}}\,\|u_{\lambda,0}\|^{m+1}_{L^{(m+1)p}(\Omega)}.
\end{aligned}$$

This together with (2.11) and (2.12) yields that

$$\|\varrho_K\|^{(m+1)p}_{L^{(m+1)p_s^*}(\Omega)} \le C_{15}\|u_{\lambda,0}\|^{(m+1)p}_{L^{(m+1)p}(\Omega)} + C_{16}\|u_{\lambda,0}\|^{m+1}_{L^{(m+1)p}(\Omega)}$$

for any positive constant K and for some positive constants C_{15} and C_{16}. Using the bootstrap argument (similarly as in the proof of Proposition 1 in [20]), we conclude that $u_{\lambda,0}$ belongs to $L^\infty(\Omega)$.

Similarly we can handle the case $u_{\lambda,0}(x) < 0$ for almost all $x \in \Omega$ if we replace $u_{\lambda,0}$ by $-u_{\lambda,0}$ in the above arguments. Consequently, we have an apriori estimate in $L^\infty(\Omega)$ of any possible weak solution of (P_λ). □

Theorem 2.11. *Let f be satisfied* (F1)–(F2). *Assume furthermore that the following conditions are verified*:

(F3) $\lim_{|t|\to\infty} \frac{f(x,t)}{|t|^{p-1}} = \infty$ *uniformly for almost all $x \in \Omega$.*

(F4) *There exist a constant $\nu \ge 1$ and a positive constant C_* such that*

$$\nu\mathcal{F}(x,t) \ge \mathcal{F}(x,st) - C_*$$

for $(x,t) \in \Omega\times\mathbb{R}$ and $s \in [0,1]$, where $\mathcal{F}(x,t) = f(x,t)t - pF(x,t)$.

Furthermore, if q in (F1) *satisfies $p < q < p_s^*$ and λ_* is given in Theorem 2.10, then, for each $\lambda \in (0,\lambda^*]$, the problem (P_λ) has at least two nontrivial weak solutions which belong to $L^\infty(\Omega)$.*

Proof. With the analogous arguments as Theorem 1.6 in [30], the functional I_λ satisfies the (C)-condition. By the condition (F3), for any $\mathcal{C} > 0$, there exists a constant $\delta > 0$ such that

$$F(x,t) \ge \mathcal{C}\,|t|^p \tag{2.15}$$

for $|t| > \delta$ and for almost all $x \in \mathbb{R}^N$. Take $v \in X_s^p(\Omega)\setminus\{0\}$. Then the relation (2.15) implies that

$$I_\lambda(tv) = \Phi_{s,p}(tv) - \lambda\Psi(tv) \le t^p\left(\frac{1}{p}\|v\|^p_{X_s^p(\Omega)} - \lambda\mathcal{C}\int_\Omega |v(x)|^p\,dx\right)$$

for large enough $t > 1$. If $\mathcal{C}$ is sufficiently large, then we know that $I_\lambda(tv) \to -\infty$ as $t \to \infty$ and thus I_λ is unbounded from below. If we follow the basic lines of the proof in Theorem 2.10, Lemma 2.7 with $\mu = 1$ implies the existence of at least two distinct nontrivial weak solutions to the problem (P_λ) for each $\lambda \in (0, \lambda^*)$.

To finish the proof, we will prove that (P_{λ^*}) has at least two nontrivial solutions. The proof is quite close to that of [4, Theorem 3.1] (see also [3]). From Lemma 2.6 in the case $\mu = 1$, for each $\lambda \in (0, \lambda^*)$, we obtain the existence of a nontrivial solution u_λ for the problem (P_λ) in $\Phi_{s,p}^{-1}((0,1))$ such that $I_\lambda(u_\lambda) \le I_\lambda(v)$ for all $v \in \Phi_{s,p}^{-1}((0,1))$. From this, we drive a sequence $\{v_n\} \subset \Phi_{s,p}^{-1}((0,1))$ such that $\|v_n\|_{X_s^p(\Omega)} \to 0$ as $n \to \infty$ and

$$I_\lambda(u_\lambda) \le I_\lambda(v_n).$$

Combining this with the continuity of I_λ, we deduce

$$I_\lambda(u_\lambda) \le 0 \quad \text{for all } \lambda \in (0, \lambda^*). \tag{2.16}$$

Fix $\lambda_0 \in (0, \lambda^*)$. Choose a sequence $\{\lambda_n\}$ such that $0 < \lambda_0 < \lambda_n < \lambda^*$ and $\lambda_n \to \lambda^*$ as $n \to \infty$. Then there is a corresponding sequence $\{u_{\lambda_n}\}$ in $\Phi_{s,p}^{-1}((0,1))$ such that u_{λ_n} is a minimal point for the problem (P_{λ_n}). From Definition 2.8, we arrive at

$$\langle \Phi'_{s,p}(u_{\lambda_n}), v\rangle = \lambda_n \int_\Omega f(x, u_{\lambda_n}) v \, dx \tag{2.17}$$

for any $v \in X_s^p(\Omega)$. Since $\{u_{\lambda_n}\}$ is bounded, we may suppose that $u_{\lambda_n} \rightharpoonup u_{\lambda^*}$ in $X_s^p(\Omega)$ and $u_{\lambda_n} \to u_{\lambda^*}$ in $L^q(\Omega)$ as $n \to \infty$ for any $1 < q < p_s^*$, due to Lemma 2.2. It follows from (2.17) with $v = u_{\lambda_n} - u^*$ that we get

$$\limsup_{n\to\infty} \langle \Phi'_{s,p}(u_{\lambda_n}), u_{\lambda_n} - u_{\lambda^*}\rangle = \limsup_{n\to\infty} \lambda_n \int_\Omega f(x, u_{\lambda_n})(u_{\lambda_n} - u_{\lambda^*}) \, dx = 0.$$

Since Φ is of type (S_+) by Lemma 2.9, we conclude that $u_{\lambda_n} \to u_{\lambda^*}$ in $X_s^p(\Omega)$ and so $I_{\lambda^*}(u_{\lambda^*}) \le I_{\lambda^*}(v)$ for all $v \in \Phi_{s,p}^{-1}((0,1))$, because $\lambda_n \to \lambda^*$ as $n \to \infty$. Letting $n \to \infty$ in (2.17) gives

$$\langle \Phi'_{s,p}(u_{\lambda^*}), v\rangle = \lambda^* \int_\Omega f(x, u_{\lambda^*}) v \, dx$$

for all $v \in X_s^p(\Omega)$. Hence u_{λ^*} is a critical point for I_{λ^*} and so is a weak solution for (P_{λ^*}). Also since $u_{\lambda_n} \to u_{\lambda^*}$ in $X_s^p(\Omega)$ and $\lambda_n \to \lambda^*$ as $n \to \infty$, one has $I_{\lambda^*}(u_{\lambda^*}) \le I_{\lambda^*}(v)$ for all $v \in \Phi_{s,p}^{-1}((0,1))$.

Now, we claim that $u_{\lambda^*} \ne 0$. Note that every u_λ is a minimal point for I_λ and $I_\lambda(u_\lambda) \le I_\lambda(v)$ for all $v \in \Phi_{s,p}^{-1}((0,1))$ and all $\lambda \in (0, \lambda^*)$. Since $u_\lambda \in \Phi_{s,p}^{-1}((0,1))$ for every $\lambda \in (0, \lambda^*)$, we see that

$$I_{\lambda_n}(u_{\lambda_n}) \le I_{\lambda_n}(u_{\lambda_0}) \quad \text{and} \quad I_{\lambda_0}(u_{\lambda_0}) \le I_{\lambda_0}(u_{\lambda_n}) \quad \text{for all } n \in \mathbb{N}.$$

According to $\lambda_0 < \lambda_n$ for all $n \in \mathbb{N}$, these inequality imply that

$$\Psi(u_{\lambda_0}) \le \Psi(u_{\lambda_n}) \quad \text{for all } n \in \mathbb{N},$$

and thus

$$\Psi(u_{\lambda_0}) \le \Psi(u_{\lambda^*}). \tag{2.18}$$

In fact, if we let $u_{\lambda^*} = 0$ in (2.18), then we get $\Psi(u_{\lambda_0}) \le 0$ and so $I_{\lambda_0}(u_{\lambda_0}) > 0$ which contradicts (2.16). Since I_{λ^*} is unbounded from below and u_{λ^*} is a nontrivial local minimum, it is not strictly global. This together with Lemma 2.7 guarantees that the given problem has a nontrivial weak solution which is different from u_{λ^*}. As in the analogous arguments in Theorem 2.10, we conclude that any possible weak solution of (P_λ) belongs to the space $L^\infty(\Omega)$. □

3. Appendix

In this section, our main results continue to hold when (P_λ) is replaced by the equations driven by a non-local integro-differential operator of elliptic type as follows:

$$(P_{\mathcal{K}}) \qquad \begin{cases} -\mathfrak{L}_{\mathcal{K}} u = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 \quad \text{on } \mathbb{R}^N \backslash \Omega. \end{cases}$$

Here $\mathfrak{L}_{\mathcal{K}}$ is a non-local operator $\mathfrak{L}_{\mathcal{K}}$ defined pointwise as

$$\mathfrak{L}_{\mathcal{K}} u(x) = 2\int_{\mathbb{R}^N} |u(x)-u(y)|^{p-2}(u(x)-u(y))\mathcal{K}(x-y)dy \quad \text{for all } x \in \mathbb{R}^N,$$

where $\mathcal{K} : \mathbb{R}^N \setminus \{0\} \to (0,+\infty)$ is a kernel function with the following properties

- ($\mathcal{K}$1) $m\mathcal{K} \in L^1(\mathbb{R}^N)$, where $m(x) = \min\{|x|^p, 1\}$;
- ($\mathcal{K}$2) there exist positive constants θ_0, θ_1 such that $\theta_0 \le \mathcal{K}(x)|x|^{N+ps} \le \theta_1$ for almost all $x \in \mathbb{R}^N \setminus \{0\}$;
- ($\mathcal{K}$3) $\mathcal{K}(x) = \mathcal{K}(-x)$ for all $x \in \mathbb{R}^N \setminus \{0\}$.

By the condition ($\mathcal{K}$1), the function

$$(x,y) \mapsto (u(x)-u(y))\mathcal{K}(x-y)^{\frac{1}{p}} \in L^p(\mathbb{R}^{2N})$$

for all $u \in C_0^\infty(\mathbb{R}^N)$. Let us denote with $W_{\mathcal{K}}^{s,p}(\mathbb{R}^N)$ the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{W_{\mathcal{K}}^{s,p}(\mathbb{R}^N)} := \left(\|u\|^p_{L^p(\mathbb{R}^N)} + |u|^p_{W_{\mathcal{K}}^{s,p}(\mathbb{R}^N)} \right)^{\frac{1}{p}},$$

where

$$|u|^p_{W_{\mathcal{K}}^{s,p}(\mathbb{R}^N)} := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u(x)-u(y)|^p \mathcal{K}(x-y)\,dxdy.$$

Lemma 3.1 ([35])**.** *Let $\mathcal{K} : \mathbb{R}^N \setminus \{0\} \to (0,\infty)$ be a function satisfying the conditions ($\mathcal{K}$1)–($\mathcal{K}$3). Then if $u \in W_{\mathcal{K}}^{s,p}(\mathbb{R}^N)$, then $u \in W^{s,p}(\mathbb{R}^N)$. Moreover*

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} \le \max\{1, \theta_0^{-\frac{1}{p}}\}\|u\|_{W_{\mathcal{K}}^{s,p}(\mathbb{R}^N)};$$

From Lemmas 2.1 and 3.1, we can obtain the following assertion immediately.

Lemma 3.2 ([35])**.** *Let $\mathcal{K} : \mathbb{R}^N \setminus \{0\} \to (0, \infty)$ satisfy the conditions ($\mathcal{K}$1)–($\mathcal{K}$3). Then there exists a positive constant $\mathcal{C}_0 = \mathcal{C}_0(N, p, s)$ such that for any $u \in W^{s,p}_{\mathcal{K}}(\mathbb{R}^N)$ and $1 \le q \le p^*_s$*

$$\begin{aligned}\|u\|^p_{L^q(\Omega)} &\le \mathcal{C}_0 \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy \\ &\le \frac{\mathcal{C}_0}{\theta_0}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u(x)-u(y)|^p \mathcal{K}(x-y)\, dxdy;\end{aligned}$$

In this section, the basic space is the closed linear subspace defined as

$$X_{\mathcal{K}}(\Omega) = \left\{u \in W^{s,p}_{\mathcal{K}}(\mathbb{R}^N) : u(x) = 0 \text{ a.e. in } \mathbb{R}^N\backslash\Omega\right\}$$

with the norm

$$\|u\|_{X_{\mathcal{K}}(\Omega)} := \left(|u|^p_{W^{s,p}_{\mathcal{K}}(\mathbb{R}^N)} + \|u\|^p_{L^p(\Omega)}\right)^{\frac{1}{p}}.$$

Then $X_{\mathcal{K}}(\Omega)$ is a uniformly convex Banach space; see Lemma 2.4 in [35].

Definition 3.3. Let $0 < s < 1 < p < +\infty$ with $ps < N$ and conditions ($\mathcal{K}$1)-($\mathcal{K}$3) are satisfied. We say that $u \in X_{\mathcal{K}}(\Omega)$ is a weak solution of the problem ($P_{\mathcal{K}}$) if

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))\mathcal{K}(x-y)\, dxdy \\ &= \lambda \int_{\Omega} f(x,u)v\, dx\end{aligned}$$

for all $v \in X_{\mathcal{K}}(\Omega)$.

Let us define a functional $\Phi_{p,K} : X_{\mathcal{K}}(\Omega) \to \mathbb{R}$ by

$$\Phi_{p,K}(u) = \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u(x)-u(y)|^p \mathcal{K}(x-y)\, dxdy.$$

Then the functional $\Phi_{p,K}$ is well defined on $X_{\mathcal{K}}(\Omega)$, $\Phi_{p,K} \in C^1(X_0, \mathbb{R})$ and its Fréchet derivative is given by

$$\begin{aligned}&\langle \Phi'_{p,K}(u), v\rangle \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))\mathcal{K}(x-y)\, dxdy\end{aligned}$$

for any $v \in X_{\mathcal{K}}(\Omega)$; see [31].

With the help of Lemma 3.2, if we replace $X^p_s(\Omega)$ and $\Phi_{s,p}$ by $X_{\mathcal{K}}(\Omega)$ and $\Phi_{p,K}$, respectively, then the proofs of the following consequences are almost identical to those of Theorems 2.10 and 2.11. Hence we skip their proofs.

Theorem 3.4. *Suppose that conditions ($\mathcal{K}$1)–($\mathcal{K}$3) are satisfied. Let* (F1)–(F2) *hold. Then there exists $\lambda^* > 0$ such that the problem ($P_{\mathcal{K}}$) has at least one nontrivial weak solution for each $\lambda \in (0, \lambda^*)$. Furthermore, if q in* (F1) *satisfies $p \le q < p^*_s$, then any weak solution of ($P_{\mathcal{K}}$) belongs to the space $L^\infty(\Omega)$.*

Theorem 3.5. *Suppose that conditions* ($\mathcal{K}1$)–($\mathcal{K}3$) *are satisfied. Assume that* (F1)–(F4) *hold. Furthermore, if* q *in* (F1) *satisfies* $p < q < p_s^*$ *and* λ_* *is given in Theorem 3.4, then, for each* $\lambda \in (0, \lambda^*]$, *the problem* $(P_{\mathcal{K}})$ *has at least two nontrivial weak solutions which belong to* $L^\infty(\Omega)$.

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