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CONJUGACY INVARIANTS OF QUATERNION MATRICES

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Abstract. In this paper, we find new conjugacy invariants of $Sl(3,\mathbb{H})$. This result is a generalization of Foreman's result for $Sl(2,\mathbb{H})$.

1. Introduction

It is well known from elementary linear algebra that the trace and the determinant are conjugacy invariants when entries of the matrices are real or complex numbers. However, for quaternion matrices, we cannot apply the same arguments because of the non-commutativity of quaternions. In Theorem 4.1 of [2], Foreman finds some conjugacy invariants of $Sl(2,\mathbb{H})$. There is an error in his computation and it is corrected in [7]. Those conjugacy invariants are used to classify quaternionic Möbius transformations ([2], [7]). Furthermore, traces play an important role in the study of quaternionic hyperbolic Kleinian groups in dimension two ([3–5]). For the study of quaternionic Möbius transformations and quaternionic hyperbolic Kleinian groups in higher dimensions, it is a crucial step to find conjugacy invariants.

In this paper, we generalise Foremann's result to $Sl(3,\mathbb{H})$. Roughly speaking, $Sl(3,\mathbb{H})$ is the collection of all 3×3 quaternionic matrices whose complex adjoint matrix is of determinant 1 (see the following Section for the details). Our main theorem is the following.

Theorem 1.1. *Let*

$$M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & l \end{pmatrix}$$

be an arbitrary matrix in $Sl(3,\mathbb{H})$. *Then the following functions on* $Sl(3,\mathbb{H})$ *are conjugacy invariants*:

$$\begin{aligned}\mathcal{F}_1(M) = {} & 2|b|^2 Re(d\bar{g}\bar{f}) + 2|c|^2 Re(g\overline{d}\overline{h}) + 2|d|^2 Re(b\bar{h}\bar{c}) + 2|f|^2 Re(h\bar{b}\bar{g}) \\ & + 2|g|^2 Re(c\overline{f}\overline{b}) + 2|h|^2 Re(f\bar{c}\bar{d}) + 2|a|^2 Re(e\bar{h}\bar{f} + l\overline{f}\overline{h}) \\ & - 2Re(a)\big(|e|^2|l|^2 + |f|^2|h|^2 - 2Re(e\bar{h}l\bar{f})\big) + 2|e|^2 Re(a\bar{g}\bar{c} + l\bar{c}\bar{g})\end{aligned}$$

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$$
\begin{aligned}
&- 2Re(e)\big(|a|^2|l|^2 + |c|^2|g|^2 - 2Re(a\bar{g}l\bar{c})\big) + 2|l|^2 Re(a\overline{d}\overline{b} + e\bar{b}\bar{d})\\
&- 2Re(l)\big(|a|^2|e|^2 + |b|^2|d|^2 - 2Re(a\bar{d}e\bar{b})\big)\\
&- 2Re(\bar{a}c\bar{l}hd + \bar{g}l\bar{c}bd + \bar{e}f\bar{l}gb + \bar{h}l\bar{f}db)\\
&- 2Re(\bar{d}e\bar{b}cg + \bar{a}b\bar{e}fg + \bar{f}e\bar{h}gc + \bar{l}h\bar{e}dc)\\
&- 2Re(\bar{b}a\bar{d}fh + \bar{e}d\bar{a}ch + \bar{c}a\bar{g}hf + \bar{l}g\bar{a}bf),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{F}_2(M) = {}& 2Re\big(g\overline{d}\overline{b}c + \bar{a}bfg + \bar{a}chd + b\bar{e}fg + \bar{e}dch + \bar{b}\bar{d}fh + f\bar{c}\bar{g}h + \overline{f}\overline{b}\bar{g}l + \bar{c}\overline{d}\overline{h}l\big)\\
&+ \big(4Re(e)Re(l) - 2Re(fh)\big)|a|^2 - 4Re(e)Re(a\bar{g}\bar{c}) - 4Re(l)Re(a\overline{d}\overline{b})\\
&+ \big(4Re(a)Re(l) - 2Re(cg)\big)|e|^2 - 4Re(a)Re(e\bar{h}\bar{f}) - 4Re(l)Re(e\bar{b}\bar{d})\\
&+ \big(4Re(a)Re(e) - 2Re(bd)\big)|l|^2 - 4Re(e)Re(l\bar{c}\bar{g}) - 4Re(a)Re(l\overline{f}\overline{h})\\
&+ |a|^2|e|^2 + |b|^2|d|^2 - 2Re(a\bar{d}e\bar{b}) + |a|^2|l|^2 + |c|^2|g|^2\\
&- 2Re(a\bar{g}l\bar{c}) + |e|^2|l|^2 + |h|^2|f|^2 - 2Re(e\bar{h}l\bar{f}),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{F}_3(M) = {}& - 2Re(a)|e|^2 - 2Re(a)|l|^2 - 2Re(e)|a|^2 - 2Re(e)|l|^2 - 2Re(l)|a|^2\\
&- 2Re(l)|e|^2 + 4Re(e)Re(cg) + 4Re(a)Re(fh) + 4Re(l)Re(bd)\\
&- 8Re(a)Re(e)Re(l)\\
&+ 2Re(a\overline{d}\overline{b} + a\bar{g}\bar{c} + e\bar{b}\bar{d} + e\bar{h}\bar{f} + l\bar{c}\bar{g} + l\overline{f}\overline{h} - bfg - dch),\\
\mathcal{F}_4(M) = {}& |a|^2 + |e|^2 + |l|^2 + 4Re(a)Re(e) + 4Re(a)Re(l) + 4Re(e)Re(l)\\
&- 2Re(bd) - 2Re(cg) - 2Re(fh),\\
\mathcal{F}_5(M) = {}& - 2Re(a + e + l).
\end{aligned}
$$

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2. Preliminaries

In this section, we briefly present basic definitions and properties of quaternions and quaternion matrices. We recommend [1, 6, 8] for more details.

2.1. Quaternions

The skew field $\mathbb{H}$ is a real vector space with the basis $1, i, j, k$. Multiplication is defined by the following principles:

$$i^2 = j^2 = k^2 = -1, \; ij = k = -ji.$$

For an arbitrary element $a = a_1 + a_2 i + a_3 j + a_4 k \in \mathbb{H}$ with $a_1, a_2, a_3, a_4 \in \mathbb{R}$, its *real part* $Re(a)$ is a_1 and its *vectorial part* (or *imaginary part*) $\upsilon(a)$

is $a_2i + a_3j + a_4k$. The *conjugate* and the *norm* of a are defined by $\bar{a} = Re(a) - \upsilon(a) = a_1 - a_2i - a_3j - a_4k$ and $|a| = \sqrt{a_1^2 + a_2^2 + a_3^3 + a_4^2}$ respectively. The inverse of a is $a^{-1} = \bar{a}/|a|^2$.

2.2. Quaternion matrices

Let $M_{m\times n}(\mathbb{H})$ denote the collection of all $m \times n$ matrices with quaternion entries. For $A = (a_{st}) \in M_{m\times n}(\mathbb{H})$ and $q \in \mathbb{H}$, we define $qA = (qa_{st})$ and $Aq = (a_{st}q)$. In the case of $m = n$, we denote $M_{m\times n}(\mathbb{H})$ by $M_n(\mathbb{H})$. For $A, B, C \in M_n(\mathbb{H})$, $ABC = (AB)C = A(BC)$ holds. A is called *invertible* if $AB = BA = I$ for some $B \in M_n(\mathbb{H})$, where I denotes the identity matrix. The following is a well known fact.

Proposition 2.1 (Proposition 4.1 in [8]). *For $A, B \in M_n(\mathbb{H})$, if $AB = I$, then $BA = I$.*

As we express a quaternion $a = a_1 + a_2i + a_3j + a_4k = (a_1 + a_2i) + (a_3 + a_4i)j$, a quaternion matrix also can be expressed as $A = A_1 + A_2j \in M_n(\mathbb{H})$, where $A_1, A_2 \in M_n(\mathbb{C})$. Then, the matrix

$$\chi_A = \begin{pmatrix} A_1 & A_2 \\ -\bar{A}_2 & \bar{A}_1 \end{pmatrix} \in M_{2n}(\mathbb{C})$$

is called the *complex adjoint matrix* of A, and this is uniquely determined by A. We define the *determinant* of A to be the determinant of χ_A. The following proposition gives a necessary and sufficient condition for a matrix in $M_n(\mathbb{H})$ to be invertible.

Proposition 2.2 (Proposition 8.1 in [8]). *$A \in M_n(\mathbb{H})$ is invertible if and only if $\det(\chi_A) \neq 0$.*

By computing the determinant of a 3×3 quaternion matrix, we get the following result.

Theorem 2.3. *Let*

$$M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & l \end{pmatrix}$$

be a quaternion matrix in $M_3(\mathbb{H})$. Then M is invertible if and only if the following condition is satisfied:

$$\begin{aligned}\mathcal{F}_0(M) := {} & |a|^2|e|^2|l|^2 + |a|^2|f|^2|h|^2 + |b|^2|d|^2|l|^2 + |b|^2|f|^2|g|^2 + |c|^2|d|^2|h|^2 \\ & + |c|^2|e|^2|g|^2 + 2Re(\bar{b}c\bar{f}d\bar{g}h + \bar{e}f\bar{c}a\bar{g}h + \bar{c}b\bar{e}d\bar{g}l + \bar{f}e\bar{b}a\bar{g}l + \bar{f}d\bar{a}b\bar{h}l \\ & + \bar{c}a\bar{d}e\bar{h}l) - 2|a|^2Re(e\bar{h}l\bar{f}) - 2|b|^2Re(d\bar{g}l\bar{f}) - 2|c|^2Re(d\bar{g}h\bar{e}) \\ & - 2|d|^2Re(b\bar{h}l\bar{c}) - 2|e|^2Re(a\bar{g}l\bar{c}) - 2|f|^2Re(a\bar{g}h\bar{b}) - 2|g|^2Re(b\bar{e}f\bar{c}) \\ & - 2|h|^2Re(a\bar{d}f\bar{c}) - 2|l|^2Re(a\bar{d}e\bar{b}) \neq 0.\end{aligned}$$

In what follows, we use $Sl(3,\mathbb{H})$ to denote the collection of all 3×3 quaternion matrices M with $\mathcal{F}_0(M)=1$.

According to [1,2], one can embed $\mathbb{H}$ into $M_{2\times 2}(\mathbb{C})$ by

$$a \mapsto L(a) = \begin{pmatrix} a_1 + a_2 i & a_3 + a_4 i \\ -a_3 + a_4 i & a_1 - a_2 i \end{pmatrix}.$$

By replacing every entry a_{st} of A by the 2×2 complex matrix $L(a)$, we can embed $Sl(n,\mathbb{H})$ into $Sl(2n,\mathbb{C})$: $A \mapsto L(A)$. For the embed function $L(\cdot)$, the relation $L(AB) = L(A)L(B)$ holds.

3. Proof

In this section, we prove Theorem 1.1 and Theorem 2.3.

Proof of Theorem 1.1. We follow the same strategy in [2]. Let us consider the group homomorphism embedding $Sl(3,\mathbb{H})$ into $Sl(6,\mathbb{C})$:

$$M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & l \end{pmatrix} \mapsto L(M) = \begin{pmatrix} a_1+a_2i & a_3+a_4i & b_1+b_2i & b_3+b_4i & c_1+c_2i & c_3+c_4i \\ -a_3+a_4i & a_1-a_2i & -b_3+b_4i & b_1-b_2i & -c_3+c_4i & c_1-c_2i \\ d_1+d_2i & d_3+d_4i & e_1+e_2i & e_3+e_4i & f_1+f_2i & f_3+f_4i \\ -d_3+d_4i & d_1-d_2i & -e_3+e_4i & e_1-e_2i & -f_3+f_4i & f_1-f_2i \\ g_1+g_2i & g_3+g_4i & h_1+h_2i & h_3+h_4i & l_1+l_2i & l_3+l_4i \\ -g_3+g_4i & g_1-g_2i & -h_3+h_4i & h_1-h_2i & -l_3+l_4i & l_1-l_2i \end{pmatrix},$$

where $a = (a_1+a_2i)+(a_3+a_4i)j, \ldots, l = (l_1+l_2i)+(l_3+l_4i)j$. By calculating the characteristic polynomial of $L(M)$, we will obtain the conjugate invariant functions. Let σ_ι be the sum of all $(n-\iota)$-principal minors of M for $\iota = 0,1,2,3,4,5$. Then

$$x^6 + \alpha_5 x^5 + \alpha_4 x^4 + \alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0 = 0$$

is the characteristic polynomial of $L(M)$, where $\alpha_\iota = (-1)^{n-\iota}\sigma_\iota$ for $\iota = 0,1,2,3,4,5$. Moreover, α_ι's are conjugate invariant functions in the variables $a_1, a_2, \cdots, l_3, l_4$. We claim that $\mathcal{F}_\iota(M) = \alpha_\iota$ for $\iota = 0,1,2,3,4,5$.

For $\iota = 5$, we have

$$\begin{aligned} \alpha_5 &= -(a_1 + a_2 i + a_1 - a_2 i + e_1 + e_2 i + e_1 - e_2 i + l_1 + l_2 i + l_1 - l_2 i) \\ &= -2Re(a+e+l) \\ &= F_5(M). \end{aligned}$$

For $\iota = 4$, it is easy to see that each 2-principal minor of $L(M)$ must be a 2-principal minor of the three following matrices:

$$M_1 = \begin{pmatrix} a_1+a_2i & a_3+a_4i & b_1+b_2i & b_3+b_4i \\ -a_3+a_4i & a_1-a_2i & -b_3+b_4i & b_1-b_2i \\ d_1+d_2i & d_3+d_4i & e_1+e_2i & e_3+e_4i \\ -d_3+d_4i & d_1-d_2i & -e_3+e_4i & e_1-e_2i \end{pmatrix},$$

$$M_2 = \begin{pmatrix} a_1+a_2i & a_3+a_4i & c_1+c_2i & c_3+c_4i \\ -a_3+a_4i & a_1-a_2i & -c_3+c_4i & c_1-c_2i \\ g_1+g_2i & g_3+g_4i & l_1+l_2i & l_3+l_4i \\ -g_3+g_4i & g_1-g_2i & -l_3+l_4i & l_1-l_2i \end{pmatrix},$$

$$M_3 = \begin{pmatrix} e_1 + e_2 i & e_3 + e_4 i & f_1 + f_2 i & f_3 + f_4 i \\ -e_3 + e_4 i & e_1 - e_2 i & -f_3 + f_4 i & f_1 - f_2 i \\ h_1 + h_2 i & h_3 + h_4 i & l_1 + l_2 i & l_3 + l_4 i \\ -h_3 + h_4 i & h_1 - h_2 i & -l_3 + l_4 i & l_1 - l_2 i \end{pmatrix}.$$

The sum of all 2-principal minors of M_1 is

$$\begin{aligned} & |a|^2 + \begin{vmatrix} a_1 + a_2 i & b_1 + b_2 i \\ d_1 + d_2 i & e_1 + e_2 i \end{vmatrix} + \begin{vmatrix} a_1 + a_2 i & b_3 + b_4 i \\ -d_3 + d_4 i & e_1 - e_2 i \end{vmatrix} \\ & + |e|^2 + \begin{vmatrix} a_1 - a_2 i & -b_3 + b_4 i \\ d_3 + d_4 i & e_1 + e_2 i \end{vmatrix} + \begin{vmatrix} a_1 - a_2 i & b_1 - b_2 i \\ d_1 - d_2 i & e_1 - e_2 i \end{vmatrix} \\ = & |a|^2 + |e|^2 + 4a_1 e_1 - 2Re(bd). \end{aligned}$$

In the same way, the sum of all 2-principal minors of M_2 is

$$|a|^2 + |l|^2 + 4a_1 l_1 - 2Re(cg)$$

and the sum of all 2-principal minors of M_3 is

$$|e|^2 + |l|^2 + 4e_1 l_1 - 2Re(fh).$$

Note that $|a|^2$, $|e|^2$ and $|l|^2$ are multiply counted. Hence we have

$$\begin{aligned} \alpha_4 &= |a|^2 + |e|^2 + |l|^2 + 4Re(a)Re(e) + 4Re(a)Re(l) + 4Re(e)Re(l) \\ &\quad - 2Re(bd) - 2Re(cg) - 2Re(fh) \\ &= F_4(M). \end{aligned}$$

For $\iota \leq 3$, it is difficult to compare α_ι and $F_\iota(M)$ directly. According to [6], one can also embed $\mathbb{H}$ into $Sl(4, \mathbb{R})$ by

$$a = a_1 + a_2 i + a_3 k + a_4 j \mapsto \mathcal{LR}(a) = \begin{pmatrix} \mathrm{a}_1 & -\mathrm{a}_2 & \mathrm{a}_4 & -\mathrm{a}_3 \\ \mathrm{a}_2 & \mathrm{a}_1 & -\mathrm{a}_3 & -\mathrm{a}_4 \\ -\mathrm{a}_4 & \mathrm{a}_3 & \mathrm{a}_1 & -\mathrm{a}_2 \\ \mathrm{a}_3 & \mathrm{a}_4 & \mathrm{a}_2 & \mathrm{a}_1 \end{pmatrix}.$$

Moreover, $\mathcal{LR}(ab) = \mathcal{LR}(a)\mathcal{LR}(b)$ and $\mathcal{LR}(ab)_{(1,1)} = Re(ab)$ for $a, b \in \mathbb{H}$. We then use a computer to verify the other invariants (see Appendix for the details). □

Proof of Theorem 2.3. Similar to the above proof, we verify that $det\big(L(M)\big) = \mathcal{F}_0(M)$ by a computer. We thus finish the proof by Proposition 2.2. □

4. The review of $Sl(2, \mathbb{H})$

In this section, we review Foreman's result and show that our results (Theorem 1.1 and Theorem 2.3) imply Foreman's result. The set $Sl(2, \mathbb{H})$ is the collection of all 2×2 quaternion matrices

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $\alpha(A) := |a|^2|d|^2 + |b|^2|c|^2 - 2Re[a\bar{c}d\bar{b}] = 1$. One can verify that $\alpha(A)$ is equal to the determinant of the complex adjoint matrix of A. In [2], Foreman proves the following theorem (we use the corrected version in [7]).

Theorem 4.1 (Foreman)**.** *Suppose that*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is an element in $Sl(2, \mathbb{H})$. *Then the following functions on* $Sl(2, \mathbb{H})$ *are conjugate invariant*:

$$\begin{aligned} \beta(A) &:= Re[(ad - bc)\bar{a} + (da - cb)\bar{d}], \\ \gamma(A) &:= |a + d|^2 + 2Re[ad - bc], \\ \delta(A) &:= Re[a + d]. \end{aligned}$$

Proof. Let

$$\Gamma(A) := \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ for } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sl(2, \mathbb{H}).$$

Then $\Gamma\big(Sl(2, \mathbb{H})\big)$ is a subgroup of $Sl(3, \mathbb{H})$ and $\mathcal{F}_i$'s are conjugacy invariants on $\Gamma\big(Sl(2, \mathbb{H})\big)$ for $i = 0, 1, 2, 3, 4, 5$. By applying Theorem 1.1 and Theorem 2.3, we have

$$\begin{aligned}
\mathcal{F}_0\big(\Gamma(A)\big) &= |a|^2|d|^2 + |b|^2|c|^2 - 2Re(a\bar{c}d\bar{b}) = \alpha(A) = 1, \\
\mathcal{F}_1\big(\Gamma(A)\big) &= -2Re(a)|d|^2 - 2Re(d)|a|^2 + 2Re(a\bar{c}\bar{b} + d\bar{b}\bar{c}) \\
&\quad - 2\big(|a|^2|d|^2 + |b|^2|c|^2 - 2Re(a\bar{c}d\bar{b})\big) \\
&= -2Re(a)|d|^2 - 2Re(d)|a|^2 + 2Re(a\bar{c}\bar{b} + d\bar{b}\bar{c}) - 2\alpha(A) \\
&= -2\beta(A) - 2, \\
\mathcal{F}_2\big(\Gamma(A)\big) &= 4Re(d)|a|^2 - 4Re(a\bar{c}\bar{b}) + 4Re(a)|d|^2 - 4Re(d\bar{b}\bar{c}) + 4Re(a)Re(d) \\
&\quad - 2Re(bc) + |a|^2|d|^2 + |b|^2|c|^2 - 2Re(a\bar{c}d\bar{b}) + |a|^2 + |d|^2, \\
&= 4\beta(A) + \gamma(A) + \alpha(A) = 4\beta(A) + \gamma(A) + 1, \\
\mathcal{F}_3\big(\Gamma(A)\big) &= -2Re(a)|d|^2 - 2Re(a) - 2Re(d)|a|^2 - 2Re(d) - 2|a|^2 - 2|d|^2 \\
&\quad + 4Re(bc) - 8Re(a)Re(d) + 2Re(a\bar{c}\bar{b} + d\bar{b}\bar{c}) \\
&= -2\beta(A) - 2\gamma(A) - 2\delta(A), \\
\mathcal{F}_4\big(\Gamma(A)\big) &= |a|^2 + |d|^2 + 1 + 4Re(a)Re(d) + 4Re(a) + 4Re(d) - 2Re(bc) \\
&= \gamma(A) + 4\delta(A) + 1, \\
\mathcal{F}_5\big(\Gamma(A)\big) &= -2Re(a + d) - 2 = -2\delta(A) - 2.
\end{aligned}$$

Note that $Sl(2, \mathbb{H})$ and $\Gamma\big(Sl(2, \mathbb{H})\big)$ are isomorphic. Observing the above identities, since $\mathcal{F}_1(\Gamma(A)) = -2\beta - 2$ is a conjugate invariant, β is a conjugate invariant. Furthermore, since $\mathcal{F}_2(\Gamma(A)) = 4\beta + \gamma + 1$ is a conjugate invariant, γ

is a conjugate invariant. Similarly, from $\mathcal{F}_5(\Gamma(A)) = -2\delta - 2$, δ is a conjugate invariant. Hence we conclude that β, γ, δ are conjugate invariants in $Sl(2, \mathbb{H})$. □

5. A conjugacy invariant in high dimensions

Let $Sl(n, \mathbb{H})$ be the collection of all $n \times n$ quaternionic matrices whose complex adjoint matrix is of determinant 1. We have the following result.

Theorem 5.1. *Let $M = (m_{ij})_{n\times n}$ be an arbitrary matrix in $Sl(n, \mathbb{H})$. Then the following function on $Sl(n, \mathbb{H})$ is an invariant*:

$$\sum_{i=1}^{n} |m_{ii}|^2 + 4\sum_{i<j} Re(m_{ii})Re(m_{jj}) - 2\sum_{i<j} Re(m_{ij}m_{ji}).$$

Proof. One can obtain this invariant by calculating the characteristic polynomial of $L(M)$. This conjugation invariant is equal to the sum of all 2-principal minors of $L(M)$. For an arbitrary 2-principal minor of $L(M)$, it must be a 2-principal minor of the matrix

$$M_{ij} := \begin{pmatrix} L(m_{ii}) & L(m_{ij}) \\ L(m_{ji}) & L(m_{jj}) \end{pmatrix}$$

for some $1 \leq i < j \leq n$. The sum of all 2-principal minors of M_{ij} is

$$|m_{ii}|^2 + |m_{jj}|^2 + 4Re(m_{ii})Re(m_{jj}) - 2Re(m_{ij}m_{ji}).$$

On the other hand, $|m_{ii}|^2$ is multiply counted for $i = 1, 2, \ldots, n$. Hence the sum of all 2-principal minors of M is

$$\sum_{i=1}^{n} |m_{ii}|^2 + 4\sum_{i<j} Re(m_{ii})Re(m_{jj}) - 2\sum_{i<j} Re(m_{ij}m_{ji}).$$ □

6. Appendix

Below is a **Mathematica** program. The readers can use it to verify the invariants.

$$LM = \begin{pmatrix} \mathrm{a1}+i\mathrm{a2} & \mathrm{a3}+i\mathrm{a4} & \mathrm{b1}+i\mathrm{b2} & \mathrm{b3}+i\mathrm{b4} & \mathrm{c1}+i\mathrm{c2} & \mathrm{c3}+i\mathrm{c4} \\ -\mathrm{a3}+i\mathrm{a4} & \mathrm{a1}-i\mathrm{a2} & -\mathrm{b3}+i\mathrm{b4} & \mathrm{b1}-i\mathrm{b2} & -\mathrm{c3}+i\mathrm{c4} & \mathrm{c1}-i\mathrm{c2} \\ \mathrm{d1}+i\mathrm{d2} & \mathrm{d3}+i\mathrm{d4} & \mathrm{e1}+i\mathrm{e2} & \mathrm{e3}+i\mathrm{e4} & \mathrm{f1}+i\mathrm{f2} & \mathrm{f3}+i\mathrm{f4} \\ -\mathrm{d3}+i\mathrm{d4} & \mathrm{d1}-i\mathrm{d2} & -\mathrm{e3}+i\mathrm{e4} & \mathrm{e1}-i\mathrm{e2} & -\mathrm{f3}+i\mathrm{f4} & \mathrm{f1}-i\mathrm{f2} \\ \mathrm{g1}+i\mathrm{g2} & \mathrm{g3}+i\mathrm{g4} & \mathrm{h1}+i\mathrm{h2} & \mathrm{h3}+i\mathrm{h4} & \mathrm{l1}+i\mathrm{l2} & \mathrm{l3}+i\mathrm{l4} \\ -\mathrm{g3}+i\mathrm{g4} & \mathrm{g1}-i\mathrm{g2} & -\mathrm{h3}+i\mathrm{h4} & \mathrm{h1}-i\mathrm{h2} & -\mathrm{l3}+i\mathrm{l4} & \mathrm{l1}-i\mathrm{l2} \end{pmatrix};$$

$I4 = IdentityMatrix[4];$

$A1 = \{\{a1, -a2, a4, -a3\}, \{a2, a1, -a3, -a4\}, \{-a4, a3, a1, -a2\}, \{a3, a4, a2, a1\}\};$

$A2 = Dot[A1, -I4] + 2DiagonalMatrix[\{a1, a1, a1, a1\}];$

$B1 = \{\{b1, -b2, b4, -b3\}, \{b2, b1, -b3, -b4\}, \{-b4, b3, b1, -b2\}, \{b3, b4, b2, b1\}\};$

$B2 = Dot[B1, -I4] + 2DiagonalMatrix[\{b1, b1, b1, b1\}];$

$C1 = \{\{c1, -c2, c4, -c3\}, \{c2, c1, -c3, -c4\}, \{-c4, c3, c1, -c2\}, \{c3, c4, c2, c1\}\};$
$C2 = Dot[C1, -I4] + 2DiagonalMatrix[c1, c1, c1, c1];$
$D1 = \{\{d1, -d2, d4, -d3\}, \{d2, d1, -d3, -d4\}, \{-d4, d3, d1, -d2\}, \{d3, d4, d2, d1\}\};$
$D2 = Dot[D1, -I4] + 2DiagonalMatrix[\{d1, d1, d1, d1\}];$
$E1 = \{\{e1, -e2, e4, -e3\}, \{e2, e1, -e3, -e4\}, \{-e4, e3, e1, -e2\}, \{e3, e4, e2, e1\}\};$
$E2 = Dot[E1, -I4] + 2DiagonalMatrix[\{e1, e1, e1, e1\}];$
$F1 = \{\{f1, -f2, f4, -f3\}, \{f2, f1, -f3, -f4\}, \{-f4, f3, f1, -f2\}, \{f3, f4, f2, f1\}\};$
$F2 = Dot[F1, -I4] + 2DiagonalMatrix[\{f1, f1, f1, f1\}];$
$G1 = \{\{g1, -g2, g4, -g3\}, \{g2, g1, -g3, -g4\}, \{-g4, g3, g1, -g2\}, \{g3, g4, g2, g1\}\};$
$G2 = Dot[G1, -I4] + 2DiagonalMatrix[\{g1, g1, g1, g1\}];$
$H1 = \{\{h1, -h2, h4, -h3\}, \{h2, h1, -h3, -h4\}, \{-h4, h3, h1, -h2\}, \{h3, h4, h2, h1\}\};$
$H2 = Dot[H1, -I4] + 2DiagonalMatrix[\{h1, h1, h1, h1\}];$
$L1 = \{\{l1, -l2, l4, -l3\}, \{l2, l1, -l3, -l4\}, \{-l4, l3, l1, -l2\}, \{l3, l4, l2, l1\}\};$
$L2 = Dot[L1, -I4] + 2DiagonalMatrix[\{l1, l1, l1, l1\}];$
$\alpha_0 = Det[LM];$
$\alpha_1 = -Tr[Minors[LM, 5]];$
$\alpha_2 = Tr[Minors[LM, 4]];$
$\alpha_3 = -Tr[Minors[LM, 3]];$
$\alpha_4 = Tr[Minors[LM, 2]];$
$\alpha_5 = -Tr[Minors[LM, 1]];$

We use $A1, A2, \ldots, L1, L2$ to denote $\mathcal{LR}(a), \mathcal{LR}(\bar{a}), \ldots, \mathcal{LR}(l), \mathcal{LR}(\bar{l})$. For a matrix M, $M[[m,n]]$ is the entry of M in the m-th row and n-th column. From the definition of $\mathcal{LR}$, $A1[[1,1]]$ is the real part of a. For matrices $M_1, M_2, \ldots, M_n$, $Dot[M_1, M_2, \ldots, M_n]$ is the standard matrix product $M_1M_2 \cdots M_n$. From the property of $\mathcal{LR}$, $Dot[A_1, B_1]$ is equal to $\mathcal{LR}(ab)$ and $Dot[A_1, B_1][[1,1]]$ is $Re(ab)$. $Tr[Minors[LM, \iota]]$ is the sum of all $(6-\iota)$-principal minors of LM for $\iota = 0, 1, 2, \ldots, 5$. If one run the following commands:

$$Simplify[-2 * A1[[1,1]] - 2 * E1[[1,1]] - 2 * L1[[1,1]] - \alpha_5].$$

One will obtain 0. Hence $\alpha_5 = \mathcal{F}_5(M)$. Similarly, one can verify the other invariants.

References

[1] H. Aslaksen, *Quaternionic determinants*, Math. Intelligencer **18** (1996), no. 3, 57–65. https://doi.org/10.1007/BF03024312

[2] B. Foreman, *Conjugacy invariants of* Sl(2, $\mathbb{H}$), Linear Algebra Appl. **381** (2004), 25–35. https://doi.org/10.1016/j.laa.2003.11.002

[3] J. Kim, *Quaternionic hyperbolic Fuchsian groups*, Linear Algebra Appl. **438** (2013), no. 9, 3610–3617. https://doi.org/10.1016/j.laa.2013.02.001

[4] S. Kim and J. Kim, *Complex and quaternionic hyperbolic Kleinian groups with real trace fields*, J. Lond. Math. Soc. (2) **93** (2016), no. 1, 101–122. https://doi.org/10.1112/jlms/jdv063
[5] ———, *A characterization of quaternionic Kleinian groups in dimension 2 with complex trace fields*, Algebr. Geom. Topol. **18** (2018), no. 2, 957–974. https://doi.org/10.2140/agt.2018.18.957
[6] J. P. Morais, S. Georgiev, and W. Sprößig, *Real Quaternionic Calculus Handbook*, BirkhSpräuser, Basel, 2014.
[7] J. R. Parker and I. Short, *Conjugacy classification of quaternionic Möbius transformations*, Comput. Meth. Funct. Th. **9** (2009), no. 1, 13–25.
[8] F. Zhang, *Quaternions and matrices of quaternions*, Linear Algebra Appl. **251** (1997), 21–57. https://doi.org/10.1016/0024-3795(95)00543-9

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