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REMARKS ON THE LIECHTI-STRENNER'S EXAMPLES HAVING SMALL DILATATIONS

Ji-Young Ham and Joongul Lee

Abstract. We show that the Liechti-Strenner's example for the closed nonorientable surface in [13] minimizes the dilatation within the class of pseudo-Anosov homeomorphisms with an orientable invariant foliation and all but the first coefficient of the characteristic polynomial of the action induced on the first cohomology nonpositive. We also show that the Liechti-Strenner's example of orientation-reversing homeomorphism for the closed orientable surface in [13] minimizes the dilatation within the class of pseudo-Anosov homeomorphisms with an orientable invariant foliation and all but the first coefficient of the characteristic polynomial $p(x)$ of the action induced on the first cohomology nonpositive or all but the first coefficient of $p(x)(x \pm 1)^2$, $p(x)(x^2 \pm 1)$, or $p(x)(x^2 \pm x + 1)$ nonpositive.

1. Introduction

Let Σ_g be a closed surface of finite genus. A homeomorphism h of Σ_g is called *pseudo-Anosov* if there is a pair of transversely measured foliations $\mathcal{F}^u$ and $\mathcal{F}^s$ in Σ and a real number $\lambda > 1$ such that $h(\mathcal{F}^u) = \lambda\mathcal{F}^u$ and $h(\mathcal{F}^s) = 1/\lambda\mathcal{F}^s$ [5, 15]. The number λ is called the *dilatation* of h and the logarithm of λ is called the *topological entropy.* The set of dilatations of pseudo-Anosov homeomorphisms of the group of isotopy classes Σ_g is discrete [3, 9]. In particular, there exists the minimal dilatation.

The dilatation of a pseudo-Anosov homeomorphism of Σ_g measures its dynamical complexity. Furthermore, the collection of topological entropies has a geometric interpretation as the collection of Teichmüller distances between Riemann surfaces of the same topological type as Σ_g [2] if Σ_g is orientable (as a subcollection of Teichmüller distances between those of the same topological type as the orientable double cover of Σ_g if Σ_g is nonorientable). In particular, the logarithm of the minimal dilatation of a genus g surface gives the length of the systole for the genus g moduli space if Σ_g is orientable.

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For an orientable surface S_g, several results have been known on the bounds of the minimal dilatation, δ_g, for all pseudo-Anosov homeomorphisms of S_g. Penner gave upper and lower bounds for the dilatations of S_g, and proved that as g tends to infinity, the minimal dilatation tends to one (the logarithm of the minimal dilatation tends to zero on the order of $1/g$) [14]. The upper bound was improved by Bauer for closed surfaces of genus $g \geq 3$ [4].

However, the exact value of the minimal dilatation δ_g of S_g has been found only when the genus g is two [6].

More is known for the minimal dilatation of orientation-preserving pseudo-Anosov homeomorphisms on S_g with orientable invariant foliations. Denote the minimal dilatation of orientation-preserving pseudo-Anosov homeomorphisms on S_g with orientable invariant foliations by $\delta^+(S_g)$. The following Table 1 shows the known values.

TABLE 1. The known values of $\delta^+(S_g)$

g	$\delta^+(S_g) \approx$	Minimal polynomial of $\delta^+(S_g)$
1	2.61803	$x^2 - 3x + 1$
2	1.72208	$x^4 - x^3 - x^2 - x + 1$
3	1.40127	$x^6 - x^4 - x^3 - x^2 + 1$
4	1.28064	$x^8 - x^5 - x^4 - x^3 + 1$
5	1.17628	$x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1 = \frac{x^{12}-x^7-x^6-x^5+1}{x^2-x+1}$
7	1.11548	$x^{14} + x^{13} - x^9 - x^8 - x^7 - x^6 - x^5 + x + 1$
8	1.12876	$x^{16} - x^9 - x^8 - x^7 + 1$

The pseudo-Anosov homeomorphisms realizing $\delta^+(S_g)$ in Table 1 were constructed by Zhirov [16] for $g = 2$, Lanneau and Thiffeault [11] for $g = 3$ and 4, Leiniger [12] for $G = 5$, Kin and Takasawa [10] and Aaber and Dunfield [1] for $g = 7$, and Hironaka [8] for $g = 8$. Hironaka [8] then showed that all of the examples above except the $g = 7$ example arise from the fibration of the link complement of 6^2_2. From genus 6 to genus 8, each example identified the lower bound calculated by Lanneau and Thiffeault [11] as the minimal dilatation.

Recently, Liechti and Strenner [13] determined the minimal dilatation of pseudo-Anosov homeomorphisms with orientable invariant foliations on the closed nonorientable surfaces of genus 4, 5, 6, 7, 8, 10, 12, 14, 16, 18 and 20 and the minimal dilatation of orientation-reversing pseudo-Anosov homeomorphisms with orientable invariant foliations on the closed orientable surfaces of genus 1, 3, 5, 7, 9, and 11. Denote by N_g the closed nonorientable surface of genus g and by $\delta^+(N_g)$ the minimal dilatation among pseudo-Anosov homeomorphisms of N_g with an orientable invariant foliation. Denote the minimal dilatation among orientation-reversing pseudo-Anosov homeomorphisms on S_g with orientable invariant foliations by $\delta^+_{rev}(S_g)$. The values determined by Liechti and Strenner [13] are in Table 2 and Table 3.

TABLE 2. The known values of $\delta^+(N_g)$

g	$\delta^+(N_g) \approx$	Minimal polynomial of $\delta^+(N_g)$
4	1.83929	$x^3 - x^2 - x - 1$
5	1.51288	$x^4 - x^3 - x^2 + x - 1$
6	1.42911	$x^5 - x^3 - x^2 - 1$
7	1.42198	$x^6 - x^5 - x^4 - x^3 + x - 1$
8	1.28845	$x^7 - x^4 - x^3 - 1$
10	1.21728	$x^{11} - x^6 - x^5 - 1$
12	1.17429	$x^{13} - x^7 - x^6 - 1$
14	1.14551	$x^{15} - x^8 - x^7 - 1$
16	1.12488	$x^{16} - x^9 - x^8 - x^7 + 1$
18	1.10938	$x^{17} - x^9 - x^8 - 1$
20	1.09730	$x^{19} - x^{10} - x^9 - 1$

TABLE 3. The known values of $\delta^+_{rev}(S_g)$

g	$\delta^+_{rev}(S_g) \approx$	Minimal polynomial of $\delta^+_{rev}(S_g)$
1	1.61803	$x^2 - x - 1$
3	1.25207	$\frac{x^8 - x^5 - x^3 - 1}{x^2+1}$
5	1.15973	$\frac{x^{12} - x^7 - x^5 - 1}{x^2+1}$
7	1.11707	$\frac{x^{16} - x^9 - x^7 - 1}{x^2+1}$
9	1.09244	$\frac{x^{20} - x^{11} - x^9 - 1}{x^2+1}$
11	1.07638	$x^{24} - x^{13} - x^{11} - 1$

The main purpose of the paper is to show that the Liechti-Strenner's example for the closed nonorientable surface in [13] minimizes the dilatation within the class of pseudo-Anosov homeomorphisms with an orientable invariant foliation and all but the first coefficient of the characteristic polynomial of the action induced on the first cohomology nonpositive. We also show that the Liechti-Strenner's example of orientation-reversing homeomorphism for the closed orientable surface in [13] minimizes the dilatation within the class of pseudo-Anosov homeomorphisms with an orientable invariant foliation and all but the first coefficient of the characteristic polynomial $p(x)$ of the action induced on the first cohomology nonpositive or all but the first coefficient of $p(x)(x \pm 1)^2$, $p(x)(x^2 \pm 1)$, or $p(x)(x^2 \pm x + 1)$ nonpositive.

Theorem 1.1. *Denote by N_{2k} the closed nonorientable surface of genus $2k$. For all $k \geq 2$, the largest positive root of*

$$x^{2k-1} - x^k - x^{k-1} - 1$$

minimizes the dilatation within the class of pseudo-Anosov homeomorphisms with an orientable invariant foliation and all but the first coefficient of the characteristic polynomial of the action induced on the first cohomology nonpositive.

Theorem 1.2. *Denote by S_{2k-1} the closed orientable surface of genus $2k-1$. For all $k \geq 2$, the largest positive root of*

$$x^{4k} - x^{2k+1} - x^{2k-1} - 1$$

minimizes the dilatation within the class of pseudo-Anosov homeomorphisms with an orientable invariant foliation and all but the first coefficient of the characteristic polynomial $p(x)$ of the action induced on the first cohomology nonpositive or all but the first coefficient of $p(x)(x \pm 1)^2$, $p(x)(x^2 \pm 1)$, or $p(x)(x^2 \pm x + 1)$ nonpositive.

2. Perron-Frobenius matrix

Definition. Let $M = \big(m_{ij}\big)$ and $N = \big(n_{ij}\big)$ be two nonnegative $d \times d$ matrices. We say $M > N$ if $m_{ij} \geq n_{ij}$ for all i, $j \in \{1, 2, \ldots, d\}$ and strict inequality holds for at least one entry.

Lemma 2.1 (Perron-Frobenius). *Let T and L be two Perron-Frobenius matrices such that $T > L$. Then the spectral radius λ_T of T is strictly bigger than the spectral radius λ_L of L.*

Proof. Let $t > 0$ be a left eigenvector of T corresponding to λ_T and let $l > 0$ be a right eigenvector of L corresponding to λ_L. Then

$$Tl > Ll = \lambda_L l$$

and

$$\lambda_T tl = tTl > \lambda_L tl.$$

Therefore $\lambda_T > \lambda_L$. □

Proposition 2.2. *The transpose of the Frobenius companion matrix of*

$$x^{2k-1} - a_{2k-2}x^{2k-2} - \cdots - a_1 x - 1$$

is

$$\begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots 1 & \\ 1 & a_1 & a_2 & \ldots & a_{2k-2} \end{bmatrix},$$

and the transpose of the Frobenius companion matrix of

$$x^{4k} - a_{4k-1}x^{4k-1} - \cdots - a_1 x - 1$$

is

$$\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots 1 & \\ 1 & a_1 & a_2 & \dots & a_{4k-1} \end{bmatrix}.$$

Proposition 2.3 ([13, Proposition 4.1])**.** *Let $\psi : N_g \to N_g$ be a pseudo-Anosov map with a transversely orientable invariant foliation on the closed nonorientable surface N_g of genus g. Then its dilatation λ is a root of a* (*not necessarily irreducible*) *polynomial $p(x) \in \mathbb{Z}[x]$ with the following properties*:

(1) $\deg(p) = g - 1$.
(2) *$p(x)$ is monic and its constant coefficient is ± 1.*
(3) *The absolute values of the roots of $p(x)$ other than λ lie in the open interval (λ^{-1}, λ). In particular, $p(x)$ is not reciprocal or anti-reciprocal.*
(4) *$p(x)$ is reciprocal* mod 2.

Proposition 2.4 ([13, Proposition 4.3])**.** *Let $\psi : S_g \to S_g$ be an orientation-reversing pseudo-Anosov map with transversely orientable invariant foliations. Then its dilatation λ is a root of a* (*not necessarily irreducible*) *polynomial $p(x) \in \mathbb{Z}[x]$ with the following properties*:

(1) $\deg(p) = 2g$.
(2) *$p(x)$ is monic and its constant coefficient is $(-1)^g$.*
(3) $p(x) = (-1)^g x^{2g} p(-x^{-1})$.
(4) *The absolute values of the roots of $p(x)$ other than λ and $-\lambda^{-1}$ lie in the open interval (λ^{-1}, λ).*

Lemma 2.5. *Let A be the adjacency matrix for a graph G with m vertices. Then A is primitive if and only if G is strongly connected and the* gcd *of the lengths of the loops in $G(A)$ is one.*

Proof. See [7, Chapter 6, Section 1, Remarks 6, 8]. □

Lemma 2.6. (1) *Let $\psi : N_{2k} \to N_{2k}$ be a pseudo-Anosov map with a transversely orientable invariant foliation on the closed nonorientable surface N_{2k} of genus $2k$. Let $x^{2k-1} - a_{2k-2}x^{2k-2} - \cdots - a_1 x - 1$ be the characteristic polynomial of the action of ψ induced on first cohomology of N_{2k}. Then $a_i \equiv a_{2k-1-i}$* mod 2 *and at least for one a_i with* $\gcd(2k-1, 2k-1-i) = 1$, *$a_i \neq 0$ for $1 \le i \le k-1$.*

(2) *Let $\psi : S_{2k-1} \to S_{2k-1}$ be an orientation-reversing pseudo-Anosov map with transversely orientable invariant foliations. If*

$$p(x) = x^{4k-2} - a_{4k-3}x^{4k-3} - \cdots - a_1 x - 1$$

be the characteristic polynomial of the action of ψ induced on first cohomology of S_{2k-1}, then $a_i \equiv a_{4k-2-i}$ mod 2 *and at least for one a_i with* $\gcd(4k-2, 4k-2-i) = 1$, *$a_i \neq 0$ for $1 \le i \le 2k-1$.*

(3) *Let* $\psi : S_{2k-1} \to S_{2k-1}$ *be an orientation-reversing pseudo-Anosov map with transversely orientable invariant foliations. Let* $p(x)$ *be the characteristic polynomial of the action of* ψ *induced on first cohomology of* S_{2k-1}*. If*

$$x^{4k} - a_{4k-1}x^{4k-1} - \cdots - a_1 x - 1$$

is one of $p(x)(x \pm 1)^2$, $p(x)(x^2 \pm 1)$, *or* $p(x)(x^2 \pm x + 1)$, *then* $a_i \equiv a_{4k-i} \bmod 2$ *and at least for one* a_i *with* $\gcd(4k, 4k-i) = 1$, $a_i \neq 0$ *for* $1 \leq i \leq 2k-1$.

Proof. (1) can be obtained from Proposition 2.3(1) and (4) and Lemma 2.5.

(2) can be obtained from Proposition 2.4(1) and (3) and Lemma 2.5.

Note that $p(x)(x\pm 1)^2$, $p(x)(x^2 \pm 1)$, or $p(x)(x^2 \pm x+1)$ is either $p(x)(x^2+1)$, or $p(x)(x^2+x+1)$ mod 2. Hence,

(3) can be obtained from (2) and Lemma 2.5. □

By a spectral radius of a polynomial, we mean the spectral radius of the Frobenius companion matrix of it.

Lemma 2.7. *Let* $n \geq 2$.

$$x^{2n} - x^{n+1} - x^{n-1} - 1$$

gives the minimum spectral radius among the polynomials $x^{2n} - a_{2n-1}x^{2n-1} - \cdots - a_1 x - 1$ *with* $a_i \equiv a_{2n-i} = 1$ *for only one* i $(1 \leq i \leq n-1)$ *and* $a_j = 0$ *for all* j *with* $j \neq i$ *and* $j \neq 2n-i$.

Proof. Let $g(x)$ be $g(x) = x^{2n} - x^{2n-i} - x^i - 1$. Then $g(1) = -2 < 0$ and $g(2) = 2^{2n} - 2^{2n-i} - 2^i - 1 = (2^{2n-i} - 1)(2^i - 1) - 2 > 0$. $g'(x) = (2n)x^{2n-1} - (2n-i)x^{2n-i-1} - ix^{i-1}$. $g'(1) = 2n - (2n-i) - i = 0$ and $g'(x) = x^{2n-1}\left((2n) - (2n-i-1)x^{-i} - ix^{-2n+i}\right) \geq 0$ for $x \geq 1$. Hence the largest root of $g(x)$ lies between 1 and 2 and it is the only root bigger than 1. We can regard g as a two variable polynomial $G(x,i)$.

$$\frac{\partial G}{\partial i} = x^i \ln x \left(x^{2n-2i} - 1\right) \geq 0$$

if $x \geq 1$. Hence,

$$x^{2n} - x^{2n+1} - x^{2n-1} - 1$$

gives the minimum spectral considering Lemma 2.1. □

3. Proof of Theorem 1.1 and Theorem 1.2

Given a pseudo-Anosov homeomorphism ψ, let ψ^* be the induced action of ψ on the first cohomology and $p_\psi(x)$ be the characteristic polynomial of ψ^*.

3.1. Proof of Theorem 1.1

Let Ψ be the following set:

$$\Psi = \left\{ \begin{array}{l} \psi : \psi \text{ is a pseudo-Anosov homeomorphism of } N_{2k} \text{ with an orientable invariant} \\ \quad \text{foliation and } p_\psi(x) \text{ has nonpositive coefficients except the first one} \end{array} \right\}.$$

Let Λ be the following set:

$$\Lambda = \{\lambda : \lambda \text{ is the dilatation of a pseudo-Anosov homeomorphism } \psi,\ \psi \in \Psi\}.$$

Let $\mathcal{G}$ be the following set:

$$\mathcal{G} = \left\{ p_\psi(x) : \psi \in \Psi \right\}.$$

Note that

$$x^{2k-1} - x^k - x^{k-1} - 1$$

is in $\mathcal{G}$ by [13, Proposition 2.6]. Since any polynomial in $\mathcal{G}$ can be considered as the characteristic polynomial of nonnegative Frobenius companion matrix which is also primitive, the largest root of

$$x^{2k-1} - x^k - x^{k-1} - 1$$

is the minimal dilation among λ's in Λ by Lemma 2.6(1), Lemma 2.7, and Lemma 2.1.

3.2. Proof of Theorem 1.2

Let Ψ be the following set:

$$\Psi = \left\{ \psi : \begin{array}{l} \psi \text{ is a pseudo-Anosov homeomorphism of } S_{2k-1}, \text{ which is orientation} \\ \text{reversing with orientable invariant foliations and either } p_\psi(x) \text{ or one} \\ \text{of } p(x)(x \pm 1)^2,\ p(x)(x^2 \pm 1), \text{ or } p(x)(x^2 \pm x + 1) \text{ has nonpositive} \\ \text{coefficients except the first one} \end{array} \right\}.$$

Let Λ be the following set:

$$\Lambda = \{\lambda : \lambda \text{ is the dilatation of a pseudo-Anosov homeomorphism } \psi,\ \psi \in \Psi\}.$$

Let $\mathcal{G}$ be the following set:

$$\mathcal{G} = \left\{ g(x) : \begin{array}{l} \text{g(x) has nonpositive coefficients except the first one and } g(x) \text{ is} \\ \text{one of } p_\psi(x),\ p_\psi(x)(x \pm 1)^2,\ p_\psi(x)(x^2 \pm 1), \text{ or } p_\psi(x)(x^2 \pm x + 1), \\ \psi \in \Psi \end{array} \right\}.$$

Note that

$$x^{4k} - x^{2k+1} - x^{2k-1} - 1$$

is in $\mathcal{G}$ by [13, Proposition 3.3]. Observe that the largest positive root of

$$x^{4k} - x^{2k+1} - x^{2k-1} - 1$$

is less than the largest positive root of

$$x^{4k-2} - x^{2k} - x^{2k-2} - 1.$$

Since any polynomial in $\mathcal{G}$ can be considered as the characteristic polynomial of nonnegative Frobenius companion matrix which is also primitive, the largest positive root of

$$x^{4k} - x^{2k+1} - x^{2k-1} - 1$$

is the minimal dilation among λ's in Λ by Lemma 2.6(2) & (3), Lemma 2.7, and Lemma 2.1.

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Ji-Young Ham
Department of Science
Hongik University
Seoul 04066, Korea
Email address: jyham@hongik.ac.kr

Joongul Lee
Department of Mathematics Education
Hongik University
Seoul 04066, Korea
Email address: jglee@hongik.ac.kr