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ON A CLASS OF GENERALIZED RECURRENT (k, μ)-CONTACT METRIC MANIFOLDS

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Abstract. The goal of this paper is the introduction of hyper generalized ϕ-recurrent (k, μ)-contact metric manifolds and of quasi generalized ϕ-recurrent (k, μ)-contact metric manifolds, and the investigation of their properties. Their existence is guaranteed by examples.

1. Introduction

The concept of a (k, μ)-contact metric manifold was introduced by Blair et al. [4], and there are several reasons for studying it. One of its key features is that it contains both Sasakian and non-Sasakian manifolds. Sasakian manifolds were first studied by Sasaki [20]. A full classification of (k, μ)-spaces was given by Boeckx [5]. Recently, the properties of (k, μ)-spaces under certain conditions has been studied by many geometers; see [1, 2, 23] and references therein.

Cartan [6] introduced the concept of locally symmetric space, which has been weakened and studied by many geometers throughout the years to a great extent. The notion of locally ϕ-symmetric Sasakian manifolds was introduced by Takashi [24]. The generalization of ϕ-symmetric Sasakian manifolds was made by De et al. [9] and called it ϕ-recurrent Sasakian manifolds. Jun et al. [16] studied ϕ-recurrent (k, μ)-contact metric manifolds. De et al. [14] studied ϕ-Ricci symmetric (k, μ)-contact metric manifolds. Dubey [11] introduced the notion of generalized recurrent manifold. A non-flat Riemannian manifold is said to be a generalized recurrent manifold if its curvature tensor R satisfies

$$\nabla R = A \otimes R + B \otimes G, \tag{1}$$

where A and B are non-vanishing 1-forms defined by $A(X) = g(X, \gamma_1)$ and $B(X) = g(X, \gamma_2)$ and the tensor G is defined by

$$G(X, Y)Z = g(Y, Z)X - g(X, Z)Y \tag{2}$$

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for any vector fields X, Y, Z. Here, ∇ denotes the covariant differentiation with respect to the metric g. If the 1-form B vanishes, then (1) reduces to recurrent manifold [27].

A non-flat Riemannian manifold is said to be generalized Ricci-recurrent manifold [10] if the Ricci tensor S satisfies

$$\nabla S = A \otimes S + B \otimes g, \tag{3}$$

where A and B are 1-forms defined in (1). If 1-form B vanishes, then it reduces to the notion of Ricci-recurrent manifold [19].

Shaikh et al. [21] extended this concept to generalized ϕ-recurrent Sasakian manifold. Hui [15] studied generalized ϕ-recurrent generalized (k, μ)-contact metric manifold and obtained interesting results. A non-flat Riemannain manifold is said to be generalized ϕ-recurrent manifold if the curvature tensor R satisfies the condition

$$\phi^2((\nabla_W R)(X,Y)Z) = A(W)R(X,Y)Z + B(W)G(X,Y)Z \tag{4}$$

for all vector fields X, Y and Z. Here, tensor G is defined as in (2).

A Riemannain manifold is said to be hyper generalized recurrent manifold if its curvature tensor R satisfies the condition

$$\nabla R = A \otimes R + B \otimes (g \wedge S), \tag{5}$$

where A and B are 1-forms defined in (1).

Recently, Venkatesha et al. [25] extended the notion of hyper generalized recurrent manifolds (resp. quasi generalized recurrent manifolds) to hyper generalized ϕ-recurrent Sasakian manifolds (resp. quasi generalized ϕ-recurrent Sasakian manifolds) and obtained interesting results. Continuing this, we studied hyper generalized ϕ-recurrent (k, μ)-contact metric manifolds and prove its existence by giving a proper example. Similarly, quasi generalized ϕ-recurrent (k, μ)-contact metric manifolds was investigated. This paper has the following organization. After preliminaries, in Section 3, we study hyper generalized ϕ-recurrent (k, μ)-contact metric manifolds. And in Section 4, we construct an example to prove the existence of hyper generalized ϕ-recurrent (k, μ)-contact metric manifolds. Next, in Section 4, we study quasi generalized ϕ-recurrent (k, μ)-contact metric manifolds. Its existence is proved in Section 5 by constructing an example.

2. Preliminaries

In this section, we listed some of the basic formulae and definitions on (k, μ)-contact metric manifolds which will be used throughout the paper. It is well known that, the concept of (k, μ)-contact metric manifold contains both Sasakian and non-Sasakian manifolds. Recently, geometry of contact metric manifolds under various conditions has been studied by [10, 12, 13, 18, 19, 26]. A detailed study on (k, μ)-contact metric manifolds are available in [3–5, 8] and references therein.

Let M be a smooth connected manifold of dimension $(2n+1)$. Then, M is called an almost contact metric manifold if it is equipped with an almost contact structure (ϕ,ξ,η,g) which satisfies the following relations:

$$\phi^2X = -X+\eta(X)\xi,\ \eta(\xi)=1,\ g(X,\xi)=\eta(X), \tag{6}$$

$$\phi\xi=0,\ \eta\xi=0,\ g(X,\phi Y)=-g(\phi X,Y),\ g(X,\phi X)=0, \tag{7}$$

$$g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y), \tag{8}$$

where η is a 1-form, ξ is a vector field, ϕ is a tensor field of type (1,1) and g is a Riemannian metric on M. An almost contact metric manifold satisfying $g(X,\phi Y)=d\eta(X,Y)$, is called a contact metric manifold. We consider on $M(\phi,\xi,\eta,g)$, a symmetric (1,1) tensor field h defined by $h=\frac{1}{2}L_\xi\phi$, where L denotes Lie differentiation, and satisfies $h\xi=0$, $h\phi=-\phi h$, $trh=tr\phi h=0$.

The (k,μ)-nullity distribution on the manifold $M(\phi,\xi,\eta,g)$ is a distribution [4]

$$\begin{aligned} N(k,\mu): p\to N_p(k,\mu)=\{Z\in T_p(M): R(X,Y)Z &= k(g(Y,Z)X-g(X,Z)Y)\\ &+\mu(g(Y,Z)hX-g(X,Z)hY)\} \end{aligned} \tag{9}$$

for any $X,Y\in T_pM$ and $k,\mu\in\mathbb{R}^2$. A contact metric manifold with ξ belongings to (k,μ)-nullity distribution is called a (k,μ)-contact metric manifold. A (k,μ)-contact metric manifold becomes Sasakian manifold for $k=1$, $\mu=0$; and the notion of (k,μ)-nullity distribution reduces to k-nullity distribution for $\mu=0$.

In a (k,μ)-contact metric manifold the following properties are true [4]:

$$h^2=(k-1)\phi^2,\ k\le 1, \tag{10}$$

$$\nabla_X\xi=-\phi X-\phi hX,\ (\nabla_X\phi)(Y)=g(X+hX,Y)\xi-\eta(Y)(X+hX), \tag{11}$$

$$R(X,Y)\xi=k[\eta(Y)X-\eta(X)Y]+\mu[\eta(Y)hX-\eta(X)hY], \tag{12}$$

$$R(\xi,X)Y=k[g(X,Y)\xi-\eta(Y)X]+\mu[g(hX,Y)\xi-\eta(Y)hX], \tag{13}$$

$$\begin{aligned} S(X,Y) &= [2(n-1)-n\mu]g(X,Y)+[2(n-1)+\mu]g(hX,Y)\\ &+[2(1-n)+n(2k+\mu)]\eta(X)\eta(Y), \end{aligned} \tag{14}$$

$$S(X,\xi)=2nk\eta(X), \tag{15}$$

$$r=2n(2n-2+k-n\mu), \tag{16}$$

$$S(\phi X,\phi Y)=S(X,Y)-2nk\eta(X)\eta(Y)-2(2n-2+\mu)g(hX,Y), \tag{17}$$

where S is the Ricci tensor of type $(0,2)$ and r is the scalar curvature of the manifold M. So

$$(\nabla_X\eta(Y))=g(X+hX,\phi Y), \tag{18}$$

$$(\nabla_X hY) = \big[(1-k)g(X,\phi Y) + g(X,h\phi Y)\big]\xi + \eta(Y)\big[h(\phi X + \phi hX)\big] - \mu\eta(X)\phi hY \tag{19}$$

for all $X, Y \in \chi(M)$.

Definition. A $(2n+1)$-dimensional (k,μ)-contact metric manifold is said to be η-Einstein if its Ricci tensor S is of the form

$$S(X,Y) = \alpha g(X,Y) + \beta\eta(X)\eta(Y),$$

for any vector fields X and Y, where α and β are constants. If $\beta = 0$, then the manifold M is an Einstein manifold.

3. Hyper generalized ϕ-recurrent (k,μ)-contact metric manifold

In the paper [22], the author studied hyper generalized recurrent manifolds. Recently, the author [25] studied hyper generalized ϕ-recurrent Sasakian manifold and obtained important results. By observing this, we extended it to (k,μ)-contact metric manifold. In this section, we study hyper generalized ϕ-recurrent (k,μ)-contact metric manifold.

Definition. A $(2n+1)$-dimensional (k,μ)-contact metric manifold is said to be a hyper generalized ϕ-recurrent if its curvature tensor R satisfies

$$\phi^2((\nabla_W R)(X,Y)Z) = A(W)R(X,Y)Z + B(W)H(X,Y)Z \tag{20}$$

for all vector fields X, Y and Z. Here, A and B are two non-vanishing 1-forms such that $A(X) = g(X,\rho_1)$, $B(X) = g(X,\rho_2)$ and the tensor H is defined by

$$H(X,Y)Z = S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY \tag{21}$$

for all vector fields X, Y and Z. Here, Q is the Ricci operator, ρ_1 and ρ_2 are vector fields associated with 1-forms A and B respectively. If the 1-form B vanishes, then (20) reduces to the notion of ϕ-recurrent manifolds.

Theorem 3.1. *In a hyper generalized ϕ-recurrent (k,μ)-contact metric manifold, the 1-forms A and B satisfy the relation*

$$kA(W) + [n(2k-\mu+2)-2]B(W) = 0.$$

Proof. Let us consider hyper generalized ϕ-recurrent (k,μ)-contact metric manifold. In view of (20) and (6) we obtain

$$-(\nabla_W R)(X,Y)Z + \eta((\nabla_W R)(X,Y)Z)\xi = A(W)R(X,Y)Z + B(W)H(X,Y)Z. \tag{22}$$

Taking an inner product with U in (22), we get

$$-g((\nabla_W R)(X,Y)Z) + \eta((\nabla_W R)(X,Y)Z)\eta(U) = A(W)g(R(X,Y)Z,U) + B(W)g(H(X,Y)Z,U). \tag{23}$$

Contracting over X and U in (22) gives

$$-(\nabla_W S)(Y,Z) + \eta((\nabla_W R)(\xi,Y)Z)$$

$$= [A(W) + (2n-1)B(W)]S(Y,Z) + rB(W)g(Y,Z). \tag{24}$$

Taking $Z = \xi$ in (24) and using the fact that $\eta((\nabla_W R)(\xi,Y)\xi) = 0$ we obtain

$$-(\nabla_W S)(Y,\xi) = [2nk(A(W) + (2n-1)B(W)) + rB(W)]\eta(Y). \tag{25}$$

Putting $Y = \xi$ in above equation gives

$$2nk[A(W) + (2n-1)B(W)] + rB(W) = 0. \tag{26}$$

Using (16) in (26), we obtain

$$kA(W) + [n(2k-\mu+2) - 2]B(W) = 0 \tag{27}$$

for any vector field W. This completes the proof. □

Taking $r = 0$ in (26), we are in a position to state the following corollary.

Corollary 3.2. *In a hyper generalized ϕ-recurrent (k,μ)-contact metric manifold, if the scalar curvature of the manifold vanishes then, either*

1. 1-*forms A and B are co-directional, or*
2. *it is $\left(0, \frac{2(n-1)}{n}\right)$-contact metric manifold.*

Let $\{e_i\}_{i=1}^{2n+1}$ be an orthonormal basis of the manifold. Putting $Y = Z = e_i$ in (24) and taking summation over $i, 1 \le i \le 2n+1$, and using (6), (11) and (15) we obtain

$$-dr(W) = r[A(W) + 4nB(W)]. \tag{28}$$

This led us to the following theorem.

Theorem 3.3. *In a hyper generalized ϕ-recurrent (k,μ)-contact metric manifold, if the scalar curvature of the manifold is a non-zero constant, then $A(W) + 4nB(W) = 0$ for any vector field W.*

Theorem 3.4. *In a hyper generalized ϕ-recurrent (k,μ)-contact metric manifold, the associated vector fields ρ_1 and ρ_2 corresponding to 1-forms A and B satisfy the relation*

$$r\eta(\rho_1) + 2(2n-1)(r-2nk)\eta(\rho_2) = 0.$$

Proof. Changing X, Y, Z cyclically in (23) and using Bianchi's identity we get

$$\begin{aligned} &A(W)g(R(X,Y)Z,U) + A(X)g(R(Y,W)Z,U) \\ &+ A(Y)g(R(W,X)Z,U) + B(W)g(H(X,Y)Z,U) \\ &+ B(X)g(H(Y,W)Z,U) + B(Y)g(H(W,X)Z,U) = 0. \end{aligned} \tag{29}$$

Contracting over Y and Z and using (9), we obtain

$$\begin{aligned} &A(W)S(X,U) - A(X)S(W,U) - kg(X,U)A(W) + kg(W,U)A(X) \\ &- \mu g(hW,U)A(X) + B(W)[rg(X,U) + (2n-1)S(X,U)] \\ &+ B(X)[-rg(W,U) - (2n-1)S(W,U)] + B(QX)g(W,U) \\ &- B(QW)g(X,U) + B(X)S(W,U) - B(W)S(X,U) = 0. \end{aligned} \tag{30}$$

Again contracting (30) over X and U yields

$$\begin{aligned}&(r+2nk)A(W)-A(QW)+\mu A(hW)\\&+(4nr-2r)B(W)-(4n-2)B(QW)=0.\end{aligned}\tag{31}$$

Replacing W by ξ in (31) results in

$$r\eta(\rho_1)+2(2n-1)(r-2nk)\eta(\rho_2)=0.\tag{32}$$

This completes the proof. □

Making use of relation $g((\nabla_W R)(X,Y)Z,U)=-g((\nabla_W R)(X,Y)U,Z)$ we obtain the following relation

$$\begin{aligned}g((\nabla_W R)(\xi,Y)Z,\xi)=\mu[&\{(1-k)g(W,\phi Y)+g(W,h\phi Y)\\&-g(hY,\phi(W+hW))\}\eta(Z)-\mu\eta(W)g(\phi hY,Z)].\end{aligned}\tag{33}$$

Considering (33) and (23) we can state the following theorem.

Theorem 3.5. *A hyper generalized ϕ-recurrent (k,μ)-contact metric manifold is generalized Ricci recurrent if and only if the following relation holds*:

$$\begin{aligned}g((\nabla_W R)(\xi,Y)Z,\xi)=\mu[&\{(1-k)g(W,\phi Y)+g(W,h\phi Y)\\&-g(hY,\phi(W+hW))\}\eta(Z)-\mu\eta(W)g(\phi hY,Z)]=0.\end{aligned}$$

Theorem 3.6. *A hyper generalized ϕ-recurrent (k,μ)-contact metric manifold is an η-Einstein manifold.*

Proof. Since we have

$$(\nabla_W S)(Y,\xi)=\nabla_W S(Y,\xi)-S(\nabla_W Y,\xi)-S(Y,\nabla_W\xi).\tag{34}$$

Using (11) and (15) in (34) we get

$$(\nabla_W S)(Y,\xi)=-2nkg(\phi W+\phi hW,Y)+S(Y,\phi W+\phi hW).\tag{35}$$

From (27) and (35) we obtain

$$\begin{aligned}&2nkg(\phi W+\phi hW,Y)-S(Y,\phi W+\phi hW)\\&=\Big[2nk\{A(W)+(2n-1)B(W)\}+rB(W)\Big]\eta(Y).\end{aligned}\tag{36}$$

Taking $Y=\phi Y$ in (36) gives

$$\begin{aligned}S(Y,W)+S(Y,hW)=2nkg(Y,W)&+[2nk+2(2n-2+\mu)]g(Y,hW)\\&+2(2n-2+\mu)(k-1)g(Y,-W+\eta(W)\xi).\end{aligned}\tag{37}$$

Using

$$\begin{aligned}S(Y,hW)=(2n-2-n\mu)g(Y,hW)&-(2n-2+\mu)(k-1)g(Y,W)\\&+(2n-2+\mu)(k-1)\eta(W)\eta(Y),\end{aligned}$$

and (14) in (37) led us to the following relation

$$S(Y,W)=\alpha g(Y,W)+\beta\eta(Y)\eta(W),\tag{38}$$

where

$$\alpha=\tfrac{[2(nk+n-1)+\mu(n+2)][2(n-1)-n\mu]-[2(n-1)+\mu][\mu(1-k)+2(n-1)+2k]}{2nk+\mu(n+1)},$$
$$\beta=\tfrac{[2(nk+n-1)+\mu(n+2)][2(1-n)+n(2k+\mu)]-(k-1)[2(n-1)+\mu]^2}{2nk+\mu(n+1)}.$$

This completes the proof. □

Theorem 3.7. *In a hyper generalized ϕ-recurrent (k,μ)-contact metric manifold, the 1-forms A and B satisfy the relation*

$$2nkA(\phi W)+[r+2nk(2n-1)]B(\phi W)=0.$$

Proof. In view of (9), (11) and (12) we get

$$\begin{aligned}(\nabla_W R)(X,Y)\xi = {}& k[g(W+hW,\phi Y)X-g(W+hW,\phi X)Y]\\ &+\mu[g(W+hW,\phi Y)hX-g(W+hW,\phi X)hY\\ &+\{(1-k)g(W,\phi X)+g(W,h\phi X)\}\eta(Y)\xi\\ &-\{(1-k)g(W,\phi Y)+g(W,h\phi Y)\}\eta(X)\xi\\ &+\mu\eta(W)\{\eta(X)\phi hY-\eta(Y)\phi hX\}]\\ &+R(X,Y)\phi W+R(X,Y)\phi hW.\end{aligned}\tag{39}$$

Using (39) in (22) results in the following relation

$$\begin{aligned}&k[g(W+hW,\phi Y)\eta(X)-g(W+hW,\phi Y)\eta(Y)]\xi\\ &+\mu[(1-k)g(W,\phi X)\eta(Y)+g(W,h\phi X)\eta(Y)\\ &-(1-k)g(W,\phi Y)\eta(X)-g(W,h\phi Y)\eta(X)]\xi\\ &+k[g(Y,\phi W)\eta(X)-g(X,\phi W)\eta(Y)\\ &+g(Y,\phi hW)\eta(X)-g(X,\phi hW)\eta(Y)]\xi\\ &-k[g(W+hW,\phi Y)X-g(W+hW,\phi X)Y]\\ &-\mu[g(W+hW,\phi Y)hX-g(W+hW,\phi X)hY\\ &+\{(1-k)g(W,\phi X)+g(W,h\phi X)\}\eta(Y)\xi\\ &-\{(1-k)g(W,\phi Y)+g(W,h\phi Y)\}\eta(X)\xi\\ &+\mu\eta(W)\{\eta(X)\phi hY-\eta(Y)\phi hX\}]\\ &+R(X,Y)\phi W+R(X,Y)\phi hW\\ =\ &A(W)\{k[\eta(Y)X-\eta(X)Y]+\mu[\eta(Y)hX-\eta(X)hY]\}\\ &+B(W)\{2nk[\eta(Y)X-\eta(X)Y]+\eta(Y)QX-\eta(X)QY\}.\end{aligned}\tag{40}$$

Putting $Y=\xi$ in (40) we get

$$\begin{aligned}&A(W)[k(X-\eta(X)\xi)+\mu hX]+B(W)[2nkX-4nk\eta(X)\xi+QX]\\ &+\mu^2\eta(W)\phi hX=0.\end{aligned}\tag{41}$$

Taking $W=\phi W$ and contracting over X in (41) gives

$$2nkA(\phi W)+[r+2nk(2n-1)]B(\phi W)=0.\tag{42}$$

This completes the proof. □

4. Example of hyper generalized ϕ-recurrent (k,μ)-contact metric manifold

In this section, we construct an example of hyper generalized ϕ-recurrent (k,μ)-contact metric manifold. We consider a 3-dimensional manifold $M^3 = \{(x,y,z)\in\mathbb{R}^3 : x\neq 0\}$ where (x,y,z) are the standard coordinates in $\mathbb{R}^3$. Let $\{E_1,E_2,E_3\}$ be linearly independent vector fields in M^3 which satisfy

$$[E_1,E_2]=2xE_1,\ [E_2,E_3]=0,\ [E_1,E_3]=0.$$

Let g be Riemannian metric defined by

$$g(E_1,E_1)=g(E_2,E_2)=g(E_3,E_3)=1,$$
$$g(E_1,E_2)=g(E_2,E_3)=g(E_1,E_3)=0.$$

Let η be the 1-form defined by

$$\eta(X)=g(X,E_3)$$

for any vector field X. Let ϕ be (1,1)-tensor field defined by

$$\phi E_1=E_2,\ \phi E_2=-E_1,\ \phi E_3=0.$$

Then we have

$$\eta(E_3)=1,\ \phi^2(X)=-X+\phi(X)E_3$$

and

$$g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y).$$

Moreover

$$hE_3=0,\ hE_1=-E_1,\ hE_2=E_2.$$

Thus for $E_3=\xi$, (ϕ,ξ,η,g) defines a contact metric structure on M^3. Let ∇ be the Riemannian connection of g. Using Koszul formula we obtain

$$\nabla_{E_1}E_1=-2xE_2,\ \nabla_{E_1}E_2=2xE_1,\ \nabla_{E_1}E_3=0,$$
$$\nabla_{E_2}E_1=0,\ \nabla_{E_2}E_2=0,\ \nabla_{E_2}E_3=0,$$
$$\nabla_{E_3}E_1=0,\ \nabla_{E_3}E_2=0,\ \nabla_{E_3}E_3=0.$$

Thus the metric $M^3(\phi,\xi,\eta,g)$ under consideration is a (k,μ)-contact metric manifold. Now, we will show that it is a 3-dimensional hyper generalized ϕ-recurrent (k,μ)-contact metric manifold. The non-vanishing components of curvature tensor and Ricci tensor are

$$R(E_1,E_2)E_1=4x^2E_2,\ R(E_1,E_2)E_2=-4x^2E_1,$$
$$S(E_1,E_1)=S(E_2,E_2)=-4x^2.$$

Since $\{E_1,E_2,E_3\}$ forms the orthonormal basis of the 3-dimensional (k,μ)-contact metric manifold any vector fields can be expressed as

$$X=a_1E_1+b_1E_2+c_1E_3,$$
$$Y=a_2E_1+b_2E_2+c_2E_3,$$

$$Z = a_3E_1 + b_3E_2 + c_3E_3.$$

Then,

$$R(X,Y)Z = u_1E_1 + u_2E_2, \tag{43}$$

where $u_1 = 4x^2b_3(a_2b_1 - a_1b_2)$ and $u_2 = -4x^2a_3(a_2b_1 - a_1b_2)$,
and

$$F(X,Y)Z = v_1E_1 + v_2E_2 + v_3E_3, \tag{44}$$

where

$$\begin{aligned} v_1 &= 4x^2[a_1(a_1a_2 + b_1b_2)(a_1a_3 + b_1b_3 + c_1c_3) \\ &\quad + b_3(a_2b_1 - a_1b_2) - a_2(a_1a_2 + b_1b_2)(a_2a_3 + b_2b_3 + c_2c_3)], \\ v_2 &= 4x^2[b_1(a_1a_3 + b_1b_3 + c_1c_3)(a_1a_2 + b_1b_2) \\ &\quad - a_3(a_2b_1 - a_1b_2) - b_2(a_1a_2 + b_1b_2)(a_2a_3 + b_2b_3 + c_2c_3)] \end{aligned}$$

and

$$\begin{aligned} v_3 &= 4x^2[c_1(a_1a_3 + b_1b_3 + c_1c_3)(a_1a_2 + b_1b_2) - c_1(a_2a_3 + b_2b_3) \\ &\quad + c_2(a_1a_3 + b_1b_3) - c_2(a_1a_2 + b_1b_2)(a_2a_3 + b_2b_3 + c_2c_3)]. \end{aligned}$$

By virtue of (43), we have the following

$$\begin{aligned} &(\nabla_{E_1}R)(X,Y)Z = 8x^3(a_1b_2 - a_2b_1)(b_3E_1 - a_3E_2), \\ &(\nabla_{E_2}R)(X,Y)Z = 0, \\ &(\nabla_{E_3}R)(X,Y)Z = 0. \end{aligned} \tag{45}$$

Form (43) one can easily obtain the following

$$\phi^2(\nabla_{E_i}R)(X,Y)Z = p_iE_1 + q_iE_2,\ i = 1,2,3, \tag{46}$$

where $p_1 = -8x^3b_3(a_1b_2 - a_2b_1),\ q_1 = 8x^3a_3(a_1b_2 - a_2b_1),\ p_2 = 0,\ q_2 = 0,\ p_3 = 0,\ q_3 = 0$.

Let the 1-forms be defined as

$$\begin{aligned} &A(E_1) = \frac{p_1v_2 - v_1q_1}{u_1v_2 - v_1u_2},\ B(E_1) = \frac{u_1q_1 - p_1u_2}{u_1v_2 - v_1u_2}, \\ &A(E_2) = 0,\ B(E_2) = 0, \\ &A(E_3) = 0,\ B(E_3) = 0, \end{aligned} \tag{47}$$

satisfying, $p_1v_2 - v_1q_1 \neq 0$, $u_1v_2 - v_1u_2 \neq 0$, $u_1q_1 - p_1u_2 \neq 0$ and $v_3 = 0$.
In view of (43), (44) and (46) it is easy to show the following relation:

$$\phi^2(\nabla_{E_i}R)(X,Y)Z) = A(E_i)R(X,Y)Z + B(E_i)F(X,Y)Z,\ i = 1,2,3. \tag{48}$$

Hence, the metric M^3 under consideration is a 3-dimensional hyper generalized ϕ-recurrent (k, μ)-contact metric manifold which is neither ϕ-symmetric nor ϕ-recurrent.

We can state the following.

Theorem 4.1. *There exists a 3-dimensional hyper generalized ϕ-recurrent (k, μ)-contact metric manifold which is neither ϕ-symmetric nor ϕ-recurrent.*

5. Quasi generalized ϕ-recurrent (k, μ)-contact metric manifold

Recently, the author [25] studied quasi generalized ϕ-recurrent Sasakian manifolds. A brief study on quasi generalized recurrent manifolds was done by Shaikh [23] and obtained interesting results. In this section, we will study quasi generalized ϕ-recurrent (k, μ)-contact metric manifolds.

Definition. A $(2n + 1)$-dimensional (k, μ)-contact metric manifold is said to be a quasi generalized ϕ-recurrent if its curvature tensor R satisfies

$$\phi^2((\nabla_W R)(X, Y)Z) = D(W)R(X, Y)Z + E(W)F(X, Y)Z \tag{49}$$

for all vector fields X, Y and Z. Here, D and E are two non-vanishing 1-forms such that $D(X) = g(X, \mu_1)$, $E(X) = g(X, \mu_2)$ and the tensor F is define by

$$\begin{aligned} F(X, Y)Z = g(Y, Z)X - g(X, Z)Y + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y \\ + g(Y, Z)\eta(Y)\xi - g(X, Z)\eta(Y)\xi \end{aligned} \tag{50}$$

for all vector fields X, Y and Z. Here, μ_1 and μ_2 are vector fields associated with 1-forms D and E respectively.

Theorem 5.1. *In a quasi generalized ϕ-recurrent (k, μ)-contact metric manifold, the associated 1-forms D and E are related by $kD(W) + 2E(W) = 0$.*

Proof. Consider a quasi generalized ϕ-recurrent (k, μ)-contact metric manifold. From (49) we get

$$\begin{aligned} & -((\nabla_W R)(X, Y)Z) + \eta((\nabla_W R)(X, Y)Z)\xi \\ & = D(W)R(X, Y)Z + E(W)F(X, Y)Z. \end{aligned} \tag{51}$$

Taking the same steps as in Theorem 3.1, we obtain the relation:

$$kD(W) + 2E(W) = 0. \tag{52}$$

This completes the proof. □

Contracting over X in (51) gives

$$\begin{aligned} & -(\nabla_W S)(Y, Z) + \eta((\nabla_W R)(\xi, Y)Z) \\ & = D(W)S(Y, Z) + [(2n + 1)g(Y, Z) + (2n - 1)\eta(Y)\eta(Z)]E(W). \end{aligned} \tag{53}$$

Putting $Y = Z = e_i$, (53) reduce to

$$-dr(W) = rD(W) + 2n(2n + 3)E(W). \tag{54}$$

We are in a position to state the following.

Theorem 5.2. *In a quasi generalized ϕ-recurrent (k, μ)-contact metric manifold, if the scalar curvature is a non-zero constant, then*

$$rD(W) + 2n(2n + 3)E(W) = 0.$$

From (53), we can state the following.

Theorem 5.3. *A quasi generalized ϕ-recurrent (k, μ)-contact metric manifold is a super generalized Ricci recurrent manifold if and only if*

$$g((\nabla_W R)(\xi, Y)Z, \xi) = \mu[\{(1-k)g(W, \phi Y) + g(W, h\phi Y) - g(hY, \phi(W + hW))\}\eta(Z) - \mu\eta(W)g(\phi hY, Z)] = 0.$$

Theorem 5.4. *In a quasi generalized ϕ-recurrent (k, μ)-contact metric manifold, the scalar curvature of the manifold satisfy the relation $r = k[n(5+2n^2)] + 2(2n-1)]$.*

Proof. Changing X, Y, Z cyclically in (51) and making use of Bianchi's identity we get

$$D(W)R(X,Y)Z + D(X)R(Y,W)Z + D(Y)R(W,X)Z + E(W)F(X,Y)Z + E(X)F(Y,W)Z + E(Y)F(W,X)Z = 0. \tag{55}$$

Contracting over X in (55) we get

$$\begin{aligned} &D(W)S(Y,Z) + D(R(Y,W)Z) - D(Y)S(W,Z) \\ &+ E(W)[(2n+1)g(Y,Z) + (2n-1)\eta(Y)\eta(Z)] + E(Y)g(W,Z) \\ &- g(Y,Z)E(W) + \eta(W)\eta(Z)E(X) - \eta(Y)\eta(Z)E(W) \\ &+ g(W,Z)\eta(Y)\eta(\mu_2) - g(Y,Z)\eta(W)\eta(\mu_2) \\ &- E(Y)[(2n+1)g(W,Z) + (2n+1)\eta(Z)\eta(W)] = 0. \end{aligned} \tag{56}$$

Putting $Y = Z = e_i, 1 \le i \le 2n+1$ in (56) we obtain

$$rD(W) - 2nkD(W) + \mu D(hW) - D(QW) + 2(2n^2 + n - 1)E(W) + 2(1-2n)\eta(W)\eta(\mu_2) = 0. \tag{57}$$

Replacing W with ξ in (57) gives

$$r = k[n(5 + 2n^2)] + 2(2n-1)]. \tag{58}$$

This completes the proof. □

Corollary 5.5. *In a quasi generalized ϕ-recurrent (k, μ)-contact metric manifold, if $k = 0$, then the scalar curvature is constant.*

Proceeding like in Theorem 3.6, one can easily show that the manifold is an η-Einstein manifold. Hence, we get the following statement.

Theorem 5.6. *A quasi generalized ϕ-recurrent (k, μ)-contact metric manifold is an η-Einstein manifold i.e.,*

$$S(Y,W) = \alpha g(Y,W) + \beta\eta(Y)\eta(W),$$

where

$$\alpha = \frac{[2(nk+n-1)+\mu(n+2)][2(n-1)-n\mu]-[2(n-1)+\mu][\mu(1-k)+2(n-1)+2k]}{2nk+\mu(n+1)},$$

$$\beta = \frac{[2(nk+n-1)+\mu(n+2)][2(1-n)+n(2k+\mu)]-(k-1)[2(n-1)+\mu]^2}{2nk+\mu(n+1)}.$$

6. Example of a quasi generalized ϕ-recurrent (k, μ)-contact metric manifold

In this section we give an example of a quasi generalized ϕ-recurrent (k, μ)-contact metric manifold. We consider a 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3 : x \neq 0, y \neq 0\}$, where $\{x, y, z\}$ are the standard coordinates in $\mathbb{R}^3$. Let $\{E_1, E_2, E_3\}$ be the global coordinate frame on M given by

$$E_1 = \frac{\partial}{\partial y},\ E_2 = 2xy\frac{\partial}{\partial z},\ E_3 = \frac{\partial}{\partial z}.$$

Hui [15] has shown that M is a 3-dimensional (k, μ)-contact metric manifold with $k = -\frac{1}{y}$ and $\mu = -\frac{1}{y}$. We will show that the manifold M is a 3-dimensional quasi generalized ϕ-recurrent (k, μ)-contact metric manifold. Any vector fields X, Y, Z on M can be expressed as

$$\begin{aligned} X &= a_1E_1 + b_1E_2 + c_1E_3, \\ Y &= a_2E_1 + b_2E_2 + c_2E_3, \\ Z &= a_3E_1 + b_3E_2 + c_3E_3, \end{aligned}$$

where $a_i, b_i, c_i \in \mathbb{R}^+$ (set of positive numbers). Then the Riemannian curvature R becomes

$$R(X, Y)Z = v_1E_1 + v_2E_2, \tag{59}$$

where $v_1 = -\frac{2b_3}{y^2}(a_1b_2 - a_2b_1)$ and $v_2 = \frac{2a_3}{y^2}(a_1b_2 - a_2b_1)$.
Also

$$\begin{aligned} F(X, Y)Z = {} & (b_3u_1 + 2c_3u_2)E_1 + (2c_3u_3 - a_3u_1)E_2 \\ & - 2(a_3u_2 - b_3u_3)E_3, \end{aligned} \tag{60}$$

where $u_1 = (a_1b_2 - b_1a_2)$, $u_2 = (a_1c_2 - a_2c_1)$, $u_3 = (b_1c_2 - b_2c_1)$.

From (59) we obtained

$$(\nabla_{E_1}R)(X, Y)Z = \frac{4}{y^3}(a_1b_2 - a_2b_1)(b_3E_1 - a_3E_2), \tag{61}$$

$$(\nabla_{E_2}R)(X, Y)Z = 0, \tag{62}$$

$$(\nabla_{E_3}R)(X, Y)Z = 0. \tag{63}$$

Making use of (61), (62) and (63) we get the following

$$\phi^2((\nabla_{E_i}R)(X, Y)Z) = p_iE_1 + q_iE_2,\ i = 1, 2, 3, \tag{64}$$

where

$$\begin{aligned} & p_1 = -\frac{4b_3}{y^3}(a_1b_2 - a_2b_1),\ q_1 = \frac{4a_3}{y^3}(a_1b_2 - a_2b_1), \\ & p_2 = 0,\ q_2 = 0,\ p_3 = 0,\ q_3 = 0. \end{aligned}$$

Let us define 1-forms A and B by

$$(65)\qquad \begin{aligned} A(E_1) &= \frac{a_3p_1(2c_3u_2 - b_3u_1) - q_1b_3(b_3u_1 + 2c_3u_2)}{v_1a_3(2c_3u_2 - b_3u_1) - b_3v_3u_2(a_3 + 2c_3)}, \\ B(E_1) &= \frac{b_3(q_1v_1 - p_1v_2)}{v_1a_3(2c_3u_2 - b_3u_1) - b_3v_3u_2(a_3 + 2c_3)}, \\ A(E_2) &= 0,\ B(E_2) = 0, \\ A(E_3) &= 0,\ B(E_3) = 0, \end{aligned}$$

where $a_3p_1(2c_3u_2 - b_3u_1) - q_1b_3(b_3u_1 + 2c_3u_2) \neq 0$, $b_3(q_1v_1 - p_1v_2) \neq 0$ and $v_1a_3(2c_3u_2 - b_3u_1) - b_3v_3u_2(a_3 + 2c_3) \neq 0$.

Using (61), (64) and (65) one can easily show that

$$(66)\quad \phi^2((\nabla_{E_i}R)(X,Y)Z) = A(E_i)R(X,Y)Z + B(E_i)F(X,Y)Z,\ i = 1,2,3.$$

Hence, the manifold under consideration is a 3-dimensional quasi generalized ϕ-recurrent (k,μ)-contact metric manifold. Thus we can state the following.

Theorem 6.1. *There exists a 3-dimensional quasi generalized ϕ-recurrent (k,μ)-contact metric manifold which is neither ϕ-symmetric nor ϕ-recurrent.*

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