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PSEUDO-HERMITIAN MAGNETIC CURVES IN NORMAL ALMOST CONTACT METRIC 3-MANIFOLDS

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Abstract. In this article, we show that a pseudo-Hermitian magnetic curve in a normal almost contact metric 3-manifold equipped with the canonical affine connection $\hat{\nabla}^t$ is a slant helix with pseudo-Hermitian curvature $\hat{\kappa} = |q| \sin\theta$ and pseudo-Hermitian torsion $\hat{\tau} = q\cos\theta$. Moreover, we prove that every pseudo-Hermitian magnetic curve in normal almost contact metric 3-manifolds except quasi-Sasakian 3-manifolds is a slant helix as a Riemannian geometric sense. On the other hand we will show that a pseudo-Hermitian magnetic curve γ in a quasi-Sasakian 3-manifold M is a slant curve with curvature $\kappa = |(t-\alpha)\cos\theta + q|\sin\theta$ and torsion $\tau = \alpha + \{(t-\alpha)\cos\theta + q\}\cos\theta$. These curves are not helices, in general. Note that if the ambient space M is an α-Sasakian 3-manifold, then γ is a slant helix.

1. Introduction

Magnetic curves represent, in physics, the trajectories of the charged particles moving on a Riemannian manifold under the action of magnetic fields. A *magnetic field* F on a Riemannian manifold (M, g) is a *closed* 2-form Φ and the *Lorentz force* associated to F is an endomorphism field L defined by

$$g(LX, Y) = F(X, Y), \quad X, Y \in \Gamma(TM).$$

The magnetic trajectories of F are curves γ satisfying the *Lorentz equation*:

$$\nabla_{\gamma'}\gamma' = L\gamma'.$$

One can see that every magnetic trajectory has constant speed. Unit speed magnetic curves are called *normal magnetic curves.*

Now let $M = (M, \varphi, \xi, \eta, g)$ be an almost contact metric manifold with *closed* fundamental 2-form Φ. Then $F_{\xi,q} = -q\Phi$ gives a magnetic field on M, called

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the *contact magnetic field* on M. Here q is a constant called the *strength* of $F_{\xi,q}$. The associated Lorentz force is $L_{\xi,q} = q\varphi$. The Lorentz equation of a magnetic curve with respect to $F_{\xi,q}$ is

$$\nabla_{\gamma'}\gamma' = q\varphi\gamma'. \tag{1}$$

A magnetic curve in a quasi-Sasakian 3-manifold M with respect to the magnetic field $F_{\xi,q}$ is a slant curve. Moreover, the magnetic curve γ in a quasi-Sasakian 3-manifold M with respect to the magnetic field $F_{\xi,q}$ has constant curvature $\kappa = |q|\sin\theta$ and torsion $\tau = \alpha + q\cos\theta$ (see [10]). Since α is a function on a quasi-Sasakian 3-manifold M, a magnetic curve γ is not a helix in general. In case M is an α-Sasakian 3-manifold, γ is a slant helix with curvature $\kappa = |q|\sin\theta$ and torsion $\tau = \alpha_0 + q\cos\theta$. In particular, if M is a Sasakian 3-manifold, then $\alpha_0 = 1$.

Druţă-Romaniuc, Inoguchi, Munteanu and Nistor studied a magnetic curves in cosymplectic manifolds in [5]. They proved that if a non-geodesic Legendre curve is a magnetic curve in cosymplectic manifold, then it is a circle.

The trajectory equation (1) is valid for curves in arbitrary almost contact metric manifolds even if Φ is not closed. We consider curves satisfying (1) in normal almost contact metric 3-manifolds from a pseudo-Hermitian geometric point of view. More precisely we study magnetic curves by virtue of the *canonical affine connection* $\hat{\nabla}^t$ (see (7)).

The first contribution on pseudo-Hermitian geometric studies of magnetic curves were obtained by Ikawa when the ambient space is Sasakian [7]. He gave an interpretation of contact magnetic curves in Sasakian manifolds in terms of the canonical affine connection $\hat{\nabla}^1$. Next contact magnetic curves in quasi-Sasakian 3-manifolds are investigated by the canonical affine connection $\hat{\nabla}^t$ in [5]. In this article we introduce a notion of pseudo-Hermitian magnetic curve in the following manner.

A regular curve γ is said to be a *pseudo-Hermitian magnetic curve* in normal almost contact metric 3-manifolds if it satisfies the Lorentz equation with respect to the canonical affine connection $\hat{\nabla}^t$:

$$\hat{\nabla}^t_{\gamma'}\gamma' = q\varphi\gamma'.$$

In this article, we find the necessary and sufficient condition for pseudo-hermitian magnetic curve in normal almost contact metric 3-manifolds. From this and Frenet-Serret equations, we study the relationship between pseudo-Hermitian magnetic curve and magnetic curve. In Section 3, we show that a pseudo-Hermitian magnetic curve in a normal almost contact metric 3-manifold equipped with the canonical affine connection $\hat{\nabla}^t$ is a slant helix with pseudo-Hermitian curvature $\hat{\kappa} = |q|\sin\theta$ and pseudo-Hermitian torsion $\hat{\tau} = q\cos\theta$. In Section 4, we prove that every pseudo-Hermitian magnetic curve in normal almost contact metric 3-manifolds except quasi-Sasakian 3-manifolds is a slant helix as a Riemannian geometric sense. On the other hand we will show that a pseudo-Hermitian magnetic curve γ in a quasi-Sasakian 3-manifold M is a slant curve with curvature $\kappa = |(t-\alpha)\cos\theta \ + q|\sin\theta$ and torsion $\tau = \alpha + \{(t-\alpha)\cos\theta + q\}\cos\theta$. These curves are not helices, in general. Note that if the ambient space M is an α-Sasakian 3-manifold, then γ is a slant helix.

2. Preliminaries

2.1. Normal almost contact manifolds

Let M be a manifold of odd dimension $m = 2n+1$. Then M is said to be an *almost contact manifold* if its structure group $\mathrm{GL}_m\mathbb{R}$ of the linear frame bundle is reducible to $\mathrm{U}(n) \times \{1\}$. This is equivalent to existence of a tensor field φ of type $(1,1)$, a vector field ξ and a 1-form η satisfying

$$\varphi^2 = -X + \eta \otimes \xi, \quad \eta(\xi) = 1.$$

From these conditions one can deduce that

$$\varphi\xi = 0, \quad \eta \circ \varphi = 0.$$

Moreover, since $\mathrm{U}(n) \times \{1\} \subset \mathrm{SO}(2n+1)$, M admits a Riemannian metric g satisfying

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y)$$

for all $X,\ Y \in \mathfrak{X}(M)$. Here $\mathfrak{X}(M) = \Gamma(TM)$ denotes the Lie algebra of all smooth vector fields on M. Such a metric is called an *associated metric* of the almost contact manifold $M = (M, \varphi, \xi, \eta)$. With respect to the associated metric g, η is metrically dual to ξ, that is

$$g(X,\xi) = \eta(X)$$

for all $X \in \mathfrak{X}(M)$. A structure (φ, ξ, η, g) on M is called an *almost contact metric structure*, and a manifold M equipped with an almost contact metric structure is said to be an *almost contact metric manifold.*

The *fundamental* 2*-form* of $(M, \varphi, \xi, \eta, g)$ is defined by

$$\Phi(X,Y) = g(X, \varphi Y), \quad X, Y \in \mathfrak{X}(M).$$

An almost contact metric manifold M is said to be a *contact metric manifold* if $\Phi = \mathrm{d}\eta$. On a contact metric manifold, η is a *contact form, i.e.*, $(\mathrm{d}\eta)^n \wedge \eta \neq 0$.

On the direct product manifold $M \times \mathbb{R}$ of an almost contact metric manifold and the real line $\mathbb{R}$, any tangent vector field can be represented as the form $(X, f\mathrm{d}/\mathrm{d}t)$, where $X \in \mathfrak{X}(M)$ and f is a function on $M \times \mathbb{R}$ and t is the Cartesian coordinate on the real line $\mathbb{R}$.

Define an almost complex structure J on $M \times \mathbb{R}$ by

$$J(X, \lambda\mathrm{d}/\mathrm{d}t) = (\varphi X - \lambda\xi, \eta(X)\mathrm{d}/\mathrm{d}t).$$

If J is integrable, then M is said to be *normal.*

Equivalently, M is normal if and only if

$$[\varphi,\varphi](X,Y) + 2\mathrm{d}\eta(X,Y)\xi = 0,$$

where $[\varphi, \varphi]$ is the *Nijenhuis torsion* of φ defined by

$$[\varphi, \varphi](X, Y) = [\varphi X, \varphi Y] + \varphi^2[X, Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y]$$

for any $X, Y \in \mathfrak{X}(M)$.

For more details on almost contact metric manifolds, we refer to Blair's monograph [1].

For an arbitrary almost contact metric 3-manifold M, we have ([12]):

$$(\nabla_X \varphi)Y = g(\varphi \nabla_X \xi, Y)\xi - \eta(Y)\varphi \nabla_X \xi, \tag{2}$$

where ∇ is the Levi-Civita connection on M.

Olszak [12] showed that a 3-dimensional almost contact metric manifold M is normal if and only if $\nabla \xi \circ \varphi = \varphi \circ \nabla \xi$ or, equivalently,

$$\nabla_X \xi = -\alpha \varphi X + \beta(X - \eta(X)\xi), \quad X \in \Gamma(TM), \tag{3}$$

where α and β are the functions defined by

$$\alpha = \frac{1}{2}\text{Trace}\,(\varphi \nabla \xi), \quad \beta = \frac{1}{2}\text{Trace}\,(\nabla \xi) = \text{div}\,\xi. \tag{4}$$

We call the pair (α, β) the *type* of a normal almost contact metric 3-manifold M.

Using (2) and (3) we note that the covariant derivative $\nabla \varphi$ of a 3-dimensional normal almost contact metric manifold is given by

$$(\nabla_X \varphi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\varphi X, Y)\xi - \eta(Y)\varphi X). \tag{5}$$

Moreover M satisfies

$$2\alpha\beta + \xi(\alpha) = 0.$$

Thus if α is a nonzero constant, then $\beta = 0$. In particular, a normal almost contact metric 3-manifold is said to be

- *cosymplectic* (or *coKähler*) *manifold* if $\alpha = \beta = 0$,
- *quasi-Sasakian manifold* if $\beta = 0$ and $\xi(\alpha) = 0$.
- *α-Sasakian manifold* if α is a nonzero constant and $\beta = 0$,
- *β-Kenmotsu manifold* if $\alpha = 0$ and β is a nonzero constant.

1-Sasakian manifolds and 1-Kenmotsu manifolds are simply called *Sasakian manifolds* and *Kenmotsu manifolds*, respectively.

The exterior derivative $\mathrm{d}\Phi$ of Φ is given by

$$\mathrm{d}\Phi = \beta\, \eta \wedge \Phi.$$

Thus Φ is a magnetic field if and only if $\beta = \text{div}\,\xi = 0$, that is, M is quasi-Sasakian.

2.2. Frenet-Serret equations

Let $\gamma : I \to M^3$ be a curve parameterized by arc-length in an almost contact metric 3-manifold M^3. We may define a Frenet frame fields (T, N, B) along γ. Then they satisfy the following

$$(6) \qquad \begin{cases} \nabla_T T = \kappa N, \\ \nabla_T N = -\kappa T + \tau B, \\ \nabla_T B = -\tau N, \end{cases}$$

where $\kappa = |\nabla_T T|$ is the *geodesic curvature* of γ and τ its *geodesic torsion.*

A *helix* is a curve with constant geodesic curvature and geodesic torsion. In particular, curves with constant nonzero geodesic curvature and zero geodesic torsion are called (*Riemannian*) *circles.* Note that geodesics are regarded as helices with zero geodesics curvature and torsion.

2.3. Canonical affine connections

Let $M = (M, \varphi, \xi, \eta, g)$ be an almost contact metric manifold. Define a tensor field $A = A^t$ of type $(1, 2)$ by

$$(7) \qquad A^t_X Y = -\frac{1}{2}\varphi(\nabla_X \varphi)Y - \frac{1}{2}\eta(Y)\nabla_X \xi - t\eta(X)\varphi Y + (\nabla_X \eta)(Y)\xi$$

for all vector fields X and Y. Here t is a real constant. We define an affine connection $\hat{\nabla}^t$ on M by [9]:

$$\hat{\nabla}^t_X Y = \nabla_X Y + A^t_X Y.$$

We call the connection $\hat{\nabla}^t$ the *canonical affine connection* of M. Note that the connection ∇^0 is the (φ, ξ, η)-connection introduced by Sasaki and Hatakeyama in [13]. Moreover $\hat{\nabla}^1$ was introduced by Cho [2]. When M is a strongly pseudo-convex CR-manifold,

$$\hat{\nabla}^t_X Y = \nabla_X Y - t\eta(X)\varphi Y + \eta(Y)\varphi(I + h)X - g(\varphi(I + h)X, Y)\xi.$$

This formula shows that when M is a strongly pseudo-convex CR-manifold, $\hat{\nabla}^t|_{t=-1}$ is the *Tanaka-Webster connection.* The canonical affine connection $\hat{\nabla}^t$ on an almost contact metric manifold satisfies the following conditions:

$$\hat{\nabla}\varphi = 0, \quad \hat{\nabla}\xi = 0, \quad \hat{\nabla}\eta = 0, \quad \hat{\nabla}g = 0.$$

Remark 2.1 (Generalized Tanaka-Webster connection). Let M be a contact metric manifold. Tanno introduced the following affine connection on M ([14]):

$$^{\mathsf{T}}\nabla_X Y := \nabla_X Y + \eta(X)\varphi Y + \eta(Y)\varphi(I + h)X - g(\varphi(I + h)X, Y)\xi.$$

This affine connection is called the *generalized Tanaka-Webster connection.* In case, the associated almost CR-structure S is integrable, generalized Tanaka-Webster connection coincides with our canonical connection $\hat{\nabla}^t|_{t=-1}$. The generalized Tanaka-Webster connection does not coincide with $\hat{\nabla}^t|_{t=-1}$ if S

is non-integrable. In fact, ξ, η and g are parallel with respect to ${}^{\mathsf{T}}\nabla$ but for φ, ${}^{\mathsf{T}}\nabla$ satisfies

$$({}^{\mathsf{T}}\nabla_X\varphi)Y = \mathcal{Q}(Y,X)$$

holds. Here $\mathcal{Q}$ is the *Tanno tensor field*. Hence we notice that on a contact metric manifold M, ${}^{\mathsf{T}}\nabla = \hat{\nabla}^t|_{t=-1}$ if and only if its associated CR-structure is integrable.

3. Pseudo-Hermitian magnetic curves in normal almost contact metric 3-manifolds

In this section we assume that M is a normal almost contact metric 3-manifold (or more generally, trans-Sasakian manifold of general dimension) of type (α,β). Then (7) is reduced to

$$A^t_X Y = \alpha\{g(X,\varphi Y)\xi + \eta(Y)\varphi X\} + \beta\{g(X,Y)\xi - \eta(Y)X\} - t\eta(X)\varphi Y. \tag{8}$$

The torsion tensor field $\mathfrak{T}^t$ of $\hat{\nabla}^t$ is given by

$$\begin{aligned}\hat{\mathfrak{T}}^t(X,Y) = {}& \alpha\{2g(X,\varphi Y)\xi - \eta(X)\varphi Y + \eta(Y)\varphi X\}\\ &+ \eta(X)(\beta Y - t\varphi Y) - \eta(Y)(\beta X - t\varphi X).\end{aligned} \tag{9}$$

For a Sasakian manifold ($\alpha = 1$ and $\beta = 0$) we get

$$\begin{aligned}A^t_X Y &= g(X,\varphi Y)\xi + \eta(Y)\varphi X - t\eta(X)\varphi Y,\\ \hat{\mathfrak{T}}^t(X,Y) &= 2g(X,\varphi Y)\xi - (1+t)\eta(X)\varphi Y + (1+t)\eta(Y)\varphi X.\end{aligned} \tag{10}$$

On a Sasakian 3-manifolds, canonical affine connection $\hat{\nabla}^t$ coincides with the linear connection introduced by Okumura. In particular, $\hat{\nabla}^1$ is called the *Okumura connection* [11].

Now, let us consider a contact magnetic curve in normal almost contact metric 3-manifolds from a pseudo-Hermitian geometrical point of view.

Definition. A regular curve γ is said to be a *pseudo-Hermitian magnetic curve* in an almost contact metric manifold M if it satisfies the Lorentz equation with respect to the canonical affine connection:

$$\hat{\nabla}^t_{\gamma'}\gamma' = q\varphi\gamma'. \tag{11}$$

It should be remarked that the notion of pseudo-Hermitian magnetic curve is introduced by Güvenç and Özgür [6], independently to ours. However their notion is different from ours. A regular curve in a *Sasakian manifold M* is said to be a pseudo-Hermitian magnetic curve in the sense of Güvenç and Özgür if it satisfies

$$\hat{\nabla}^t_{\gamma'}\gamma'|_{t=-1} = (q + 2\eta(\gamma'))\varphi\gamma'.$$

Obviously their notion coincides with ours when and only when γ is a *Legendre curve*, *i.e.*, $\eta(\gamma') = 0$.

3.1. Frenet-Serret equations

Let $\gamma = \gamma(s) : I \to M^3$ be a curve parameterized by arc-length in normal almost contact metric 3-manifold M^3. We may define the Frenet frame field $\hat{F} = (\hat{T}, \hat{N}, \hat{B})$ along γ with respect to the canonical affine connection $\hat{\nabla}^t$, since $\hat{\nabla}^t$ is a metric connection. Then $\hat{F}$ satisfies the following Frenet-Serret equations with respect to $\hat{\nabla}^t$:

$$\begin{cases} \hat{\nabla}^t_{\gamma'}\hat{T} = & \hat{\kappa}\hat{N}, \\ \hat{\nabla}^t_{\gamma'}\hat{N} = -\hat{\kappa}\hat{T} & +\hat{\tau}\hat{B}, \\ \hat{\nabla}^t_{\gamma'}\hat{B} = & -\hat{\tau}\hat{N}, \end{cases} \tag{12}$$

where $\hat{\kappa} = |\hat{\nabla}^t_{\gamma'}\hat{T}|$ is the *pseudo-Hermitian curvature* of γ and $\hat{\tau}$ its *pseudo-Hermitian torsion* for the canonical affine connection $\hat{\nabla}^t$. A non-geodesic curve γ is said to be a *pseudo-Hermitian circle* if $\hat{\kappa}$ is nonzero constant and $\hat{\tau} = 0$. A *pseudo-Hermitian helix* is a non-geodesic curve with nonzero constant pseudo-Hermitian curvature $\hat{\kappa}$ and pseudo-Hermitian torsion $\hat{\tau}$.

3.2. Pseudo-Hermitian magnetic curves

From the equation (11) and Frenet-Serret equations (12) for the canonical affine connection $\hat{\nabla}^t$ we have the pseudo-Hermitian curvature

$$\hat{\kappa} = |q|\sqrt{1-\eta(\gamma')^2},$$

and the normal vector field $\hat{N} = \frac{\varepsilon\varphi\gamma'}{\sqrt{1-\eta(\gamma')^2}}$, where $\varepsilon = \frac{q}{|q|}$.

Differentiating the pseudo-Hermitian normal vector field $\hat{N}$ then

$$\begin{aligned} \hat{\nabla}^t_{\gamma'}\hat{N} &= \hat{\nabla}^t_{\gamma'}\frac{\varepsilon\varphi\gamma'}{\sqrt{1-\eta(\gamma')^2}} \\ &= \varepsilon\left[-\frac{q}{\sqrt{1-\eta(\gamma')^2}}\gamma' + \frac{\eta(\gamma')\eta(\gamma')'}{(\sqrt{1-\eta(\gamma')^2})^3}\varphi\gamma' + \frac{q\eta(\gamma')}{\sqrt{1-\eta(\gamma')^2}}\xi\right]. \end{aligned} \tag{13}$$

The pseudo-Hermitian binormal vector field $\hat{B}$ is calculated by

$$\hat{B} = \gamma' \times \hat{N} = \frac{\varepsilon}{\sqrt{1-\eta(\gamma')^2}}\{\xi - \eta(\gamma')\gamma'\}.$$

From the equation (13) and the Frenet-Serret equations (12) for the canonical affine connection $\hat{\nabla}^t$ we get pseudo-Hermitian torsion

$$\hat{\tau} = q\eta(\gamma')$$

and $\eta(\gamma')$ is a constant.

As a generalization of Legendre curve (in contact metric manifold), the notion of slant curves was introduced in [3]. A unit speed curve γ in an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ is said to be *slant* if its tangent vector field makes constant *contact angle* θ with ξ, *i.e.*, $\cos\theta := \eta(\gamma')$ is constant along γ [3]. By definition, slant curves with constant angle 0 are trajectories of ξ. Slant curves with constant angle $\pi/2$ are called *almost Legendre curves* or *almost contact curves*. We studied almost contact curves in normal almost contact metric 3-manifolds, see also [8].

Hence we have:

Theorem 3.1. *Let γ be a pseudo-Hermitian magnetic curve in an almost contact metric 3-manifold with the canonical affine connection $\hat{\nabla}^t$. Then γ is a slant helix with*

$$\hat{\kappa} = |q| \sin\theta, \quad \hat{\tau} = q \cos\theta. \tag{14}$$

Moreover, its ratio of $\hat{\kappa}$ and $\hat{\tau}$ is constant.

Conversely, let γ be a non-geodesic slant helix with constant curvature $\hat{\kappa}$ and torsion $\hat{\tau}$. Since γ is a slant curve, it is defined by $g(\gamma', \xi) = \cos\theta$ for a constant angle θ.

Differentiating $g(\gamma', \xi) = \cos\theta$, since γ is a non-geodesic curve, we get $\eta(\hat{N}) = 0$. So $\hat{N}$ is orthogonal to both γ' and ξ. Thus N has the form $\hat{N} = \lambda\xi \times \gamma'$. This implies

$$1 = |\hat{N}| = |\lambda| \sin\theta.$$

Hence λ is a constant. Note that since we assumed that γ is non-geodesic, $\sin\theta \neq 0$. Thus $\hat{N}$ has the form

$$\hat{N} = \frac{\varepsilon}{\sin\theta} \varphi\gamma'.$$

This formula implies that

$$\hat{\nabla}^t_{\gamma'} \gamma' = q\varphi\gamma'$$

with

$$q = \frac{\varepsilon}{\sin\theta} \hat{\kappa}.$$

Hence γ is a magnetic curve with respect to the pseudo-Hermitian magnetic field of strength q.

The pseudo-Hermitian binormal $\hat{B}$ is given by

$$\hat{B} = \gamma' \times \hat{N} = \frac{\varepsilon}{\sin\theta}(\xi - \cos\theta\, \gamma').$$

The pseudo-Hermitian torsion of γ is computed as

$$\hat{\tau} = \frac{\varepsilon \cos\theta}{\sin\theta} \hat{\kappa}.$$

Theorem 3.2. *Let γ be a non-geodesic slant helix in a normal almost contact metric 3-manifold with the canonical affine connection $\hat{\nabla}^t$. Then γ is a pseudo-Hermitian magnetic curve with strength $q = \varepsilon\hat{\kappa}/\sin\theta$.*

4. Magnetic curves in normal almost contact metric 3-manifolds

As we have seen before pseudo-Hermitian magnetic curves are slant curves. In this section, we study pseudo-Hermitian magnetic curves from Riemannian geometric point of view by virtue of (3).

Proposition 4.1. *Let M be a normal almost contact metric 3-manifold. Then γ is a pseudo-Hermitian magnetic curve if and only if it satisfies*

$$\nabla_{\gamma'}\gamma' = \{(t-\alpha)\cos\theta + q\}\varphi\gamma' + \beta\{\cos\theta\,\gamma' - \xi\}. \tag{15}$$

4.1. Quasi-Sasakian 3-manifold

From the equation (15), we get:

Proposition 4.2. *Let M be a quasi-Sasakian 3-manifold. Then γ is a pseudo-Hermitian magnetic curve if and only if it satisfies*

$$\nabla_{\gamma'}\gamma' = \{(t-\alpha)\cos\theta + q\}\varphi\gamma'. \tag{16}$$

Remark 4.3. In quasi-Sasakian 3-manifolds, a pseudo-Hermitian magnetic curve means a contact magnetic curve with non-constant strength $(t-\alpha)\cos\theta + q$.

From the above equation (16) and Frenet-Serret equations (6) we have the geodesic curvature

$$\kappa = |(t-\alpha)\cos\theta\ + q|\sin\theta,$$

and the normal vector field $N = \frac{\varepsilon}{\sin\theta}\varphi\gamma'$, where $\varepsilon = \frac{(t-\alpha)\cos\theta+q}{|(t-\alpha)\cos\theta+q|}$.

Thus, the binormal vector field B is computed as

$$B = \gamma' \times N = \frac{\varepsilon}{\sin\theta}\{\xi - \cos\theta\gamma'\}. \tag{17}$$

Differentiating the above equation (17) then we have

$$\begin{aligned}\nabla_{\gamma'}B &= \nabla_{\gamma'}\frac{\varepsilon}{\sin\theta}\{\xi - \eta(\gamma')\gamma'\}\\ &= \frac{\varepsilon}{\sin\theta}\{-\alpha\varphi\gamma' - g(\kappa N,\xi) - g(\gamma', -\alpha\varphi\gamma')\gamma' - \eta(\gamma')\kappa N\}\\ &= -\frac{\varepsilon}{\sin\theta}\{\alpha + ((t-\alpha)\eta(\gamma') + q)\cos\theta\}\varphi\gamma'.\end{aligned}$$

Since the normal vector field $N = \frac{\varepsilon}{\sin\theta}\varphi\gamma'$, using the Frenet-Serret equations (6) we have:

Theorem 4.4. *Let γ be a pseudo-Hermitian magnetic curve in quasi-Sasakian 3-manifolds M. Then γ is a slant curve with*

$$\kappa = |(t-\alpha)\cos\theta\ + q|\sin\theta, \quad \tau = \alpha + \{(t-\alpha)\cos\theta + q\}\cos\theta.$$

Moreover, the ratio of κ and $\tau - \alpha$ is constant.

Remark 4.5. If M is an α-Sasakian 3-manifold, then γ is a slant helix.

For a Sasakian 3-manifold with respect to the Tanaka-Webster connection, that is $t = -1$ and $\alpha = 1$, we have:

Corollary 4.6 (cf. [6]). *Let γ be a pseudo-Hermitian magnetic curve in Sasakian 3-manifolds M. Then γ is a slant helix with*

$$\kappa = |q - 2\cos\theta|\sin\theta, \quad \tau = 1 + \{q - 2\cos\theta\}\cos\theta.$$

Moreover, the ratio of κ and $\tau - 1$ is constant.

Example 4.7 (cf. [3, 4]). The Heisenberg group $\mathbb{H}_3$ is a Cartesian 3-space $\mathbb{R}^3(x, y, z)$ furnished with the group structure

$$(x', y', z') \cdot (x, y, z) = (x' + x,\ y' + y,\ z' + z + (x'y - y'x)/2).$$

Define the left-invariant metric g by

$$g = \frac{dx^2 + dy^2}{4} + \eta \otimes \eta, \quad \eta = \frac{1}{2}\{dz + \frac{1}{2}(ydx - xdy)\}.$$

We take a left-invariant orthonormal frame field (e_1, e_2, e_3):

$$e_1 = 2\frac{\partial}{\partial x} - y\frac{\partial}{\partial z},\ e_2 = 2\frac{\partial}{\partial y} + x\frac{\partial}{\partial z},\ e_3 = 2\frac{\partial}{\partial z}.$$

Then the commutative relations are derived as follows:

$$[e_1, e_2] = 2e_3,\ [e_2, e_3] = [e_3, e_1] = 0. \tag{18}$$

Then the 1-form η is a contact form and the vector field $\xi = e_3$ is the characteristic vector field on $\mathbb{H}_3$.

We define a (1,1)-tensor field φ by

$$\varphi e_1 = e_2,\ \varphi e_2 = -e_1,\ \varphi\xi = 0.$$

Then we find

$$d\eta(X, Y) = g(X, \varphi Y), \tag{19}$$

and hence, (η, ξ, φ, g) is a contact Riemannian structure. Moreover, we see that it becomes a Sasakian structure.

Now, for a slant curve in the Heigenberg group $\mathbb{H}_3$, we put the tangent vector field

$$T(s) = \sin\alpha_0\cos\beta(s)e_1 + \sin\alpha_0\sin\beta(s)e_2 + \cos\alpha_0 e_3,$$

then we get

$$\nabla_{\gamma'}\gamma' = \sin\alpha_0(\beta'(s) - 2\cos\alpha_0)(-\sin\beta(s)e_1 + \cos\beta(s)e_2). \tag{20}$$

Since $\varphi\gamma' = \sin\alpha_0(-\sin\beta(s)e_1 + \cos\beta(s)e_2)$, from (16) and (20) we have $\beta' = q$ and $\beta(s) = qs + b,\ b \in \mathbb{R}$.

Hence every pseudo-Hermitian magnetic curve in $\mathbb{H}_3$ is represented as

$$\begin{cases} x(s) = \frac{1}{q}\sin\alpha_0\sin(qs + b) + x_0, \\ y(s) = -\frac{1}{q}\sin\alpha_0\cos(qs + b) + y_0, \\ z(s) = \{\cos\alpha_0 + \sin^2\alpha_0/(2q)\}s - \frac{\sin\alpha_0}{2q}\{x_0\cos(qs+b) + y_0\sin(qs + b)\} + z_0, \end{cases}$$

where q, b, x_0, y_0, z_0 are constants.

From the equation (16) if γ is an almost Legendre curve, then it satisfies $\nabla_{\gamma'}\gamma' = q\varphi\gamma'$, hence we have:

Corollary 4.8. *If γ is an almost Legendre curve in a quasi-Sasakian 3-manifold M, then γ is a pseudo-Hermitian magnetic curve if and only if γ is a contact magnetic curve. Moreover, it has*

$$\kappa = |q|, \quad \tau = \alpha.$$

Lemma 4.9 ([10])**.** *Let γ be a contact magnetic curve in quasi-Sasakian 3-manifolds M. Then γ is a slant curve with*

$$\kappa = |q| \sin\theta, \qquad \tau = \alpha + q\cos\theta,$$

and the ratio of κ and $\tau - \alpha$ is constant.

4.2. β-Kenmotsu 3-manifold

From the equation (15), we get:

Proposition 4.10. *Let M be a β-Kenmotsu 3-manifold. Then γ is a pseudo-Hermitian magnetic curve if and only if it satisfies*

$$\nabla_{\gamma'}\gamma' = \{t\cos\theta + q\}\varphi\gamma' + \beta\{\cos\theta\, \gamma' - \xi\}. \tag{21}$$

From the equation (21) and the Frenet-Serret equations (6) we have the constant curvature

$$\kappa = \sqrt{(\beta^2 + (t\cos\theta + q)^2)}\sin\theta,$$

and the normal vector field $N = \frac{1}{\kappa}[\beta(\cos\theta\gamma' - \xi) + (t\cos\theta + q)\varphi\gamma']$.

The binormal vector field B is computed as

$$B = \gamma' \times N = \frac{1}{\kappa}[\beta\varphi\gamma' + (t\eta(\gamma') + q)(\xi - \eta(\gamma')\gamma')].$$

Differentiating the binormal vector field B we have

$$\nabla_{\gamma'}B = -\frac{1}{\kappa}\cos\theta(t\cos\theta + q)[\beta(\eta(\cos\theta\gamma' - \xi) + (t\cos\theta + q)\varphi\gamma'].\tag{22}$$

From (22) and Frenet-Serret equations (6) we have the constant torsion

$$\tau = \cos\theta(t\cos\theta + q).$$

Hence we have:

Proposition 4.11. *Let γ be a pseudo-Hermitian magnetic curve in β-Kenmotsu 3-manifold. Then γ is a slant helix with*

$$\kappa = \sqrt{(\beta^2 + (t\cos\theta + q)^2)}\sin\theta, \quad \tau = \cos\theta(t\cos\theta + q).$$

Thus, from the equation (21), if γ is a Legendre curve, since $\eta(\gamma') := \cos\theta = 0$, it satisfies $\nabla_{\gamma'}\gamma' = q\varphi\gamma' - \beta\xi$. So, we have:

Corollary 4.12. *Let γ be pseudo-Hermitian Legendre magnetic curves in a β-Kenmotsu 3-manifold. Then it is an almost Legendre circle with*

$$\kappa = \sqrt{\beta^2 + q^2}, \quad \tau = 0.$$

Remark 4.13. Let γ be a contact magnetic curve in a β-Kenmotsu 3-manifold M. Then it is a slant helix with

$$\kappa = \mid q \mid \sin\theta, \quad \tau = q\cos\theta.$$

4.3. In cosymplectic 3-manifold

From the equation (15), we get:

Proposition 4.14. *Let γ be a pseudo-Hermitian magnetic curve in cosymplectic 3-manifold if and only if*

$$\nabla_{\gamma'}\gamma' = \{t\cos\theta + q\}\varphi\gamma'. \tag{23}$$

From the equation (23) and Frenet-Serret equations (6) we have the constant curvature

$$\kappa = \sqrt{(t\cos\theta + q)^2}\sin\theta,$$

and normal vector field $N = \frac{1}{\kappa}(t\cos\theta + q)\varphi\gamma'$.

The binormal vector field B is computed as

$$B = \gamma' \times N = \frac{1}{\kappa}(t\cos\theta + q)(\xi - \cos\theta\,\gamma').$$

Differentiating the binormal vector field B we have

$$\nabla_{\gamma'}B = -\frac{1}{\kappa}\cos\theta(t\cos\theta + q)^2\varphi\gamma'.$$

From (4.3) and Frenet-Serret equations (6) we have the constant torsion

$$\tau = \cos\theta(t\cos\theta + q).$$

Hence we have:

Proposition 4.15. *Let γ be a pseudo-Hermitian magnetic curve in cosymplectic 3-manifold. Then γ is a slant helix with*

$$\kappa = |t\cos\theta + q|\sin\theta, \quad \tau = (t\cos\theta + q)\cos\theta,$$

and its ratio of κ and τ is constant.

Thus, from the equation (23) we have:

Corollary 4.16. *If γ is an almost Legendre curve in a cosymplectic 3-manifold M, then γ is a pseudo-Hermitian magnetic curve if and only if γ is a contact magnetic curve. Moreover, it is a Legendre circle with*

$$\kappa = |q|, \quad \tau = 0.$$

Remark 4.17. If γ is a contact magnetic curve in a cosymplectic 3-manifold M, then it satisfies

$$\kappa = |q|\sin\theta, \quad \tau = q\cos\theta.$$

Moreover, for an almost Legendre curve γ, it is a circle with $\kappa = |q|$, $\tau = 0$. ([5])

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