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A NEW CLASS OF RIEMANNIAN METRICS ON TANGENT BUNDLE OF A RIEMANNIAN MANIFOLD

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Abstract. The class of isotropic almost complex structures, $J_{\delta,\sigma}$, define a class of Riemannian metrics, $g_{\delta,\sigma}$, on the tangent bundle of a Riemannian manifold which are a generalization of the Sasaki metric. This paper characterizes the metrics $g_{\delta,0}$ using the geometry of tangent bundle. As a by-product, some integrability results will be reported for $J_{\delta,\sigma}$.

1. Introduction

Assume (M, g) is a Riemannian manifold and ∇ represents the Levi-Civita connection of g and $\pi : \mathrm{T}M \to M$ is the tangent bundle of M. We denote by X^h and X^v the horizontal and vertical lifts of a vector field X on M, respectively. There are many papers [1, 2, 6, 7, 9, 10, 12–16] which are on differential geometric structures on tangent and cotangent bundles like the Riemannian metrics, harmonic sections, almost complex structures, connections and so on.

As a fundamental ingredient in studying the Riemannian manifolds, the almost complex structures have various applications in physics, signal processing and information geometry. Kähler manifolds as a special class of complex manifolds play an important role in signal processing. Choi and Mullhaupt [8] proved a correspondence between the information geometry of a signal filter and a Kähler manifold; the information geometry of a minimum-phase linear system with a finite complex cepstrum norm is a Kähler manifold. In [17], the authors investigated the necessary conditions for a divergence function on a manifold M such that the manifold $M \times M$ admits a Kähler structure. We know that starting with a metrical almost complex manifold, one can get to a symplectic manifold and vice versa; a symplectic manifold is equivalent to a metrical almost complex manifold. Lisi [11] investigated the applications of pseudo-holomorphic curves to problems in Hamiltonian dynamics using the structures of symplectic manifolds.

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The classical almost complex structure $J_{1,0} : \mathrm{TT}M \to \mathrm{TT}M$ is defined by

$$J_{1,0}(X^h) = X^v, \quad J_{1,0}(X^v) = -X^h$$

for vector field X on M. In [3], Aguilar generalized this structure to a class of almost complex structures and called them isotropic almost complex structures $J_{\delta,\sigma}$ with definition

$$J_{\delta,\sigma}(X^h) = \alpha X^v + \sigma X^h, \qquad J_{\delta,\sigma}(X^v) = -\sigma X^v - \delta X^h$$

for functions $\alpha, \delta, \sigma : \mathrm{T}M \to \mathbb{R}$ which satisfy $\alpha\delta - \sigma^2 = 1$. He showed that there exists an integrable isotropic almost complex structure on an open subset $\mathcal{A} \subset \mathrm{T}M$ if and only if the sectional curvature of $(\pi(\mathcal{A}), g)$ is constant.

Besides, he introduced special class of Riemannian metrics $g_{\delta,\sigma}$ constructed by the Liouville 1-form Θ on $\mathrm{T}M$ together with the isotropic almost complex structure $J_{\delta,\sigma}$ with definition

$$g_{\delta,\sigma}(A, B) = \mathrm{d}\Theta(J_{\delta,\sigma}A, B), \qquad A, B \in \mathrm{TT}M.$$

They are generalizations of the Sasaki metric and in some cases, intersect the class of g-natural metrics. It is easy to see that $(\mathrm{T}M, g_{\delta,\sigma}, J_{\delta,\sigma})$ is a Hermitian manifold and so in some cases are Kähler manifolds.

We will achieve some results on the integrability of $J_{\delta,\sigma}$ when the base manifold is the Euclidean space and the hyperbolic one using the complex function $z : \mathrm{T}M \to \mathbb{C}$ defined by $z(u) = \frac{\sigma+i}{\delta}(u)$. These results characterize the integrable isotropic almost complex structures in a comprehensible concepts compared with the Aguilar's ways. The following propositions state the results.

Proposition 1.1. *Let $J_{\delta,\sigma}$ be an isotropic almost complex structure on $\mathrm{T}\mathbb{R}^n = \mathbb{R}^{2n}$. Then $J_{\delta,\sigma}$ is integrable if and only if $z : (\mathbb{R}^{2n}, J_{\delta,\sigma}) \to \mathbb{C}$ is a holomorphic mapping.*

Let $(\mathbb{H}^n, g)$ be the hyperbolic space and let $e_1, \dots, e_n$ be an orthonormal frame field on $\mathbb{H}^n$. Suppose $v_i : \mathrm{T}\mathbb{H}^n \to \mathbb{R}$ are functions defined by $v_i(u_p) = g(e_i(p), u_p)$ for $i = 1, \dots, n$ and $u_p \in \mathrm{T}_p\mathbb{H}^n$. Using this notations we have:

Proposition 1.2. *Let $J_{\delta,\sigma}$ be an isotropic almost complex structure on $\mathrm{T}\mathbb{H}^n$. Then one can claim that $J_{\delta,\sigma}$ is integrable if and only if $\mathrm{d}(-z^2 + {v_1}^2 + \cdots + {v_n}^2)$ is a $(1,0)$-form on $(\mathrm{T}\mathbb{H}^n, J_{\delta,\sigma})$.*

Note that $(1,0)$-forms are zero on vectors $V = A + \sqrt{-1}J_{\delta,\sigma}A \in \mathrm{T}^{(0,1)}(\mathrm{T}M)$ for $A \in \mathrm{TT}M$.

Unlike the classical researches on the geometry of tangent bundle, we would like to characterize the metrics $g_{\delta,\sigma}$ under some geometric conditions. In the following theorems, we will show that the metric $g_{\delta,0}$ takes a special form by considering some conditions on the tangent bundle and base manifold.

Theorem 1.3. *Let $J_{\delta,0}$ be an isotropic almost complex structure on the tangent bundle of the Euclidean space $(\mathbb{R}^n, g)$. Then $(\mathrm{T}\mathbb{R}^n, g_{\delta,0})$ is an Einstein manifold* *if and only if*

$$g_{\delta,\sigma}(X^h, Y^h) = \alpha_0 g(X, Y),$$
$$g_{\delta,\sigma}(X^h, Y^v) = 0,$$
$$g_{\delta,\sigma}(X^v, Y^v) = \frac{1}{\alpha_0} g(X, Y),$$

where $\alpha_0 \in \mathbb{R}$ *is a constant number.*

Theorem 1.4. *Let* $J_{\delta,0}$ *be an isotropic almost complex structure on the tangent bundle of* (M, g) *and let* $(\mathrm{T}M, g_{\delta,0})$ *be of constant sectional curvature. Then*

$$g_{\delta,\sigma}(X^h, Y^h) = \alpha_0 g(X, Y),$$
$$g_{\delta,\sigma}(X^h, Y^v) = 0,$$
$$g_{\delta,\sigma}(X^v, Y^v) = \frac{1}{\alpha_0} g(X, Y),$$

where $\alpha_0 \in \mathbb{R}$ *is a constant number.*

These theorems will be proved after four technical lemmas.

This paper is organized as follows: Section 2 is devoted to an introduction to the isotropic almost complex structures and the Riemannian metrics $g_{\delta,\sigma}$ together with the proofs of Propositions 1.1 and 1.2. In Section 3, Theorems 1.3 and 1.4 will be proved. Section 4 is an appendix which contains some needed formulas.

2. Isotropic almost complex structures and related metrics

This section is devoted to study integrability conditions of $J_{\delta,\sigma}$ and to introduce the induced metrics $g_{\delta,\sigma}$. Note that manifold (M, g) will be supposed to be of arbitrary sectional curvature unless we indicate that it is constant.

2.1. Integrable almost complex structures

Aguilar [3] introduced two classes of integrable structures $J_{\delta,\sigma}$ on the tangent bundle of a space form (M, g) defined by the following two classes of functions

$$\delta^{-1} = \sqrt{2kE + b}, \qquad \sigma = 0,$$
$$\delta^{-2} = \frac{1}{2}\{2kE + b + \sqrt{(2kE + b)^2 + 4a^2k^2}\}, \qquad \sigma = ak\delta^2,\ a \neq 0,$$

where $E(u) = \frac{1}{2}g(u, u)$ is the energy density and k is the sectional curvature of (M, g) which is constant. It is worth mentioning that these classes are not the all of integrable structures. Aguilar proved that the necessary and sufficient conditions for integrability of $J_{\delta,\sigma}$ is the following equation

$$\mathrm{d}\sigma + k\delta\Theta - \sqrt{-1}(1 - \sqrt{-1}\sigma)\delta^{-1}\mathrm{d}\delta \equiv 0 \mod \{\zeta_1, \dots, \zeta_n\}, \tag{1}$$

where ζ_i, $i = 1, \dots, n$, supposed to be a basis for $(1, 0)$-forms of $(\mathrm{T}M, J_{\delta,\sigma})$ and k is the constant sectional curvature.

Now, let $(x^1, \dots, x^n, y^1, \dots, y^n)$ be the standard coordinate system on $\mathrm{T}\mathbb{R}^n$. Due to the definition of z, it plays an important role in integrablity of $J_{\delta,\sigma}$. In [4], the authors proved that $J_{\delta,\sigma} : \mathrm{TT}\mathbb{R}^n \to \mathrm{TT}\mathbb{R}^n$ is integrable if and only if

$$\frac{\partial z}{\partial x^l} + z\frac{\partial z}{\partial y^l} = 0 \qquad \forall\, l,\ 1 \le l \le n. \tag{2}$$

The proof of Proposition 1.1 shows this matter a little more.

Proof of Proposition 1.1

Let $V_k = \frac{\partial}{\partial x^k} + \sqrt{-1}J_{\delta,\sigma}\frac{\partial}{\partial x^k} = (1+\sqrt{-1}\sigma)\frac{\partial}{\partial x^k} + \sqrt{-1}\alpha\frac{\partial}{\partial y^k}$ for $k = 1, \dots, n$ be a basis for the space $\mathrm{T}^{(0,1)}(\mathrm{T}\mathbb{R}^n)$ with respect to the structure $J_{\delta,\sigma}$. Then z is holomorphic if and only if $V_k(z) = 0$. One can compute $V_k(z)$ as follows

$$V_k(z) = (1+\sqrt{-1}\sigma)\frac{\partial z}{\partial x^k} + \sqrt{-1}\alpha\frac{\partial z}{\partial y^k},$$

which can be written in the better form

$$(1+\sqrt{-1}\sigma)\Big(\frac{\partial z}{\partial x^k} + \frac{\sqrt{-1}\alpha}{1+\sqrt{-1}\sigma}\frac{\partial z}{\partial y^k}\Big) = (1+\sqrt{-1}\sigma)\Big(\frac{\partial z}{\partial x^k} + z\frac{\partial z}{\partial y^k}\Big).$$

So, z is holomorphic if and only if $\frac{\partial z}{\partial x^k} + z\frac{\partial z}{\partial y^k} = 0$. But $\frac{\partial z}{\partial x^k} + z\frac{\partial z}{\partial y^k} = 0$ if and only if $J_{\delta,\sigma}$ is integrable and the proof is complete.

Now, we give the proof of Proposition 1.2. The techniques of the proof are from [1]"Solutions of equations characterizing a complex structure".

Proof of Proposition 1.2

Let $(\mathbb{H}^n, g)$ be the hyperbolic space, $e_1, \dots, e_n$ be an orthonormal frame field on $\mathbb{H}^n$ and $\omega^1, \dots, \omega^n$ be its dual 1-forms. It is easy to deduce that the structure equations with respect to this frame can be written as follows:

$$\begin{aligned} \mathrm{d}\omega_i &= -\omega_{ij} \wedge \omega_j, \\ \mathrm{d}\omega_{ij} &= -\omega_{ik} \wedge \omega_{kj} - \omega_i \wedge \omega_j. \end{aligned}$$

If we define functions $v_i : \mathrm{T}\mathbb{H}^n \to \mathbb{R}$ by $v_i(u) = g(e_i, u)$ for $i = 1, \dots, n$ and $\eta_i = \mathrm{d}v_i + v_j\omega_{ij}$, then the $2n$ 1-forms η_i, ω_i is a basis of 1-forms on $\mathrm{T}\mathbb{H}^n$ such that ω_i are zero on vertical vector fields and η_i are zero on horizontal vector fields for $i = 1, \dots, n$. It is easy to see that $\zeta_k = \eta_k - z\omega_k$ for $k = 1, \dots, n$ is a basis for $(1,0)$-forms. One can compute $\mathrm{d}\zeta_k$ in the following useful form

$$\begin{aligned} \mathrm{d}\zeta_k &= \mathrm{d}\eta_k - \mathrm{d}z \wedge \omega_k - z\omega_k \\ &= \zeta_j \wedge (\omega_{kj} - \frac{v_j}{z}\omega_k) + \Big(v_j\omega_j - \mathrm{d}z + \frac{v_j}{z}(\mathrm{d}v_j + v_i\omega_{ji}) - z\omega_j\Big) \wedge \omega_k. \end{aligned}$$

Since $v_iv_j\omega_{ij} = 0$ then we get

$$\mathrm{d}\zeta_k = \zeta_j \wedge (\omega_{kj} - \frac{v_j}{z}\omega_k) + \frac{1}{2z}\mathrm{d}(-z^2 + v_1^2 + \cdots + v_n^2) \wedge \omega_k.$$

[1] http://mathoverflow.net/questions/230574/solutions-of-equations-characterizing-a-complex-structure

On the other hand $\omega_k = (\zeta_k - \overline{\zeta_k})/(z - \overline{z})$, and so

$$\mathrm{d}\zeta_k \equiv \frac{\mathrm{d}(-z^2 + {v_1}^2 + \cdots + {v_n}^2)}{2z(\overline{z} - z)} \wedge \overline{\zeta_k} \quad \text{modulo } \zeta_1, \ldots, \zeta_n .$$

This yields $J_{\delta,\sigma}$ is integrable if and only if

$$\mathrm{d}(-z^2 + {v_1}^2 + \cdots + {v_n}^2) \equiv 0 \quad \text{modulo } \zeta_1, \ldots, \zeta_n,$$

which means that $J_{\delta,\sigma}$ is integrable if and only if $\mathrm{d}(-z^2 + {v_1}^2 + \cdots + {v_n}^2)$ is a $(1,0)$-form.

2.2. Induced Riemannian metrics

Suppose Θ is the Liouvill 1-form defined by $\Theta_v(A) = g_{\pi(v)}(\pi_*(A), v)$ for all $A \in \mathrm{T}_v\mathrm{T}M$ and $v \in \mathrm{T}M$. Then the $(0,2)$-tensor

$$g_{\delta,\sigma}(A, B) = \mathrm{d}\Theta(J_{\delta,\sigma}A, B), \qquad A, B \in \mathrm{TT}M,$$

is a symmetric tensor and defines a Riemannian metric on $\mathrm{T}M$ if $\alpha > 0$. For vector fields X, Y on M, this metric can be expressed by

$$\begin{aligned} g_{\delta,\sigma}(X^h, Y^h) &= \alpha g(X, Y), \\ g_{\delta,\sigma}(X^h, Y^v) &= -\sigma g(X, Y), \\ g_{\delta,\sigma}(X^v, Y^v) &= \delta g(X, Y). \end{aligned}$$

Remark 2.1. When we work with Θ, it is convenient to work with a locally orthonormal frame field on (M, g) like $X_1, \ldots, X_n$. Because, if we suppose that $\pi : \mathrm{T}M \to M$, $K : \mathrm{TT}M \to \mathrm{T}M$ are the natural projection and the connection map, respectively and if we suppose θ^i is the dual 1-forms of X_i, then

$$\mathrm{d}\Theta = \sum_{i=1}^{n} (\theta^i \circ K) \wedge (\pi^* \theta^i),$$

where $\{\theta^i \circ K, \pi^*\theta^i\}$ is the dual basis of $\{X_i^v, X_i^h\}$.

Here after, we will put $\sigma = 0$ and represent $g_{\delta,0}$ by $\bar{g}$ (note that in this case we have $\alpha = \frac{1}{\delta}$). In [4], the authors calculated the Levi-Civita connection of $g_{\delta,\sigma}$. By putting $\sigma = 0$ we get the Levi-Civita connection of $\bar{g}$.

Theorem 2.2. *Let (M, g) be a Riemannian manifold and $(\mathrm{T}M, \bar{g})$ be its tangent bundle equipped with the Riemannian metric $\bar{g}$ induced by the isotropic almost complex structure $J_{\delta,0}$. Then the Levi-Civita connection of $\bar{g}$ at a point $(p, u) \in \mathrm{T}M$ is given by,*

$$\begin{aligned} (3) \qquad \bar{\nabla}_{X^h} Y^h = {} & (\nabla_X Y)^h + \frac{1}{2\alpha} X^h(\alpha) Y^h + \frac{1}{2\alpha} Y^h(\alpha) X^h - \frac{1}{2}(R(X,Y)u)^v \\ & - \frac{1}{2} g(X,Y) \bar{\nabla}\alpha, \end{aligned}$$

$$(4) \qquad \bar{\nabla}_{X^h} Y^v = \frac{1}{2\alpha^2}(R(u,Y)X)^h + \frac{1}{2\alpha} Y^v(\alpha) X^h + (\nabla_X Y)^v$$

$$-\frac{1}{2\alpha}X^h(\alpha)Y^v,$$

$$(5)\qquad \bar{\nabla}_{X^v}Y^h = \frac{1}{2\alpha^2}(R(u,X)Y)^h + \frac{1}{2\alpha}X^v(\alpha)Y^h - \frac{1}{2\alpha}Y^h(\alpha)X^v,$$

$$(6)\qquad \bar{\nabla}_{X^v}Y^v = -\frac{1}{2\alpha}X^v(\alpha)Y^v - \frac{1}{2\alpha}Y^v(\alpha)X^v + \frac{1}{2\alpha^2}g(X,Y)\bar{\nabla}\alpha,$$

where X and Y are vector fields on M, $\bar{\nabla}\alpha$ is the gradient vector field of α with respect to $\bar{g}$ and R is the Riemannian curvature of g.

3. Proof of Theorems 1.3 and 1.4

To prove these theorems, first we will prove four lemmas.

Lemma 3.1. *Suppose (M,g) is a Riemannian manifold and let $(\mathrm{T}M,\bar{g})$ be of constant sectional curvature $\bar{K}$. Then (M,g) is flat.*

Proof. If $(\mathrm{T}M,\bar{g})$ is of constant sectional curvature $\bar{K}$, then we have

$$\bar{R}(X^v,Y^v)Z^h = \bar{K}\{\bar{g}(Y^v,Z^h)X^v - \bar{g}(X^v,Z^h)Y^v\}.$$

Since the vertical and horizontal sub-bundles are perpendicular to each other, one can write

$$\bar{R}(X^v,Y^v)Z^h = 0.$$

By setting $u=0$ in (14) we get

$$0 = \frac{1}{\alpha^2}(R(X,Y)Z)^h.$$

Since, $\pi_* : \mathcal{H}\mathrm{T}M \longrightarrow \mathrm{T}M$ is an isomorphism, one can get the flatness of (M,g). □

In the following statements, by supposing that (M,g) is the Euclidean space and $(\mathrm{T}\mathbb{R}^n,\bar{g})$ is an Einstein manifold, we shall investigate what happen for α.

Lemma 3.2. *Suppose $\mathcal{A}$ is an open subset of $\mathrm{T}\mathbb{R}^n$. If $(\pi(\mathcal{A}),g)$ is the Euclidean space and $(\mathcal{A},\bar{g})$ is an Einstein manifold, then there exists an open subset $\mathcal{B}\subset\mathcal{A}$ such that at least one of the vector fields $v\bar{\nabla}\alpha$ or $h\bar{\nabla}\alpha$ vanishes on this open set.*

Proof. Let X and Y be two arbitrary vector fields on $\pi(\mathcal{A})$. By using the equation (16) one can write

$$\bar{g}(\bar{Q}(X^v),Y^h) = \frac{1}{2\alpha}\bar{g}(\bar{\nabla}_{X^v}h\bar{\nabla}\alpha,Y^h) - \frac{1}{2\alpha}\bar{g}(\bar{\nabla}_{X^v}\bar{\nabla}\alpha,Y^h) + \frac{3-2n}{4\alpha^2}X^v(\alpha)Y^h(\alpha).$$

Since, the vertical and horizontal vectors are perpendicular, we have

$$\bar{g}(h\bar{\nabla}\alpha,Y^h) = \bar{g}(\bar{\nabla}\alpha,Y^h) = Y^h(\alpha).$$

By using the compatibility of $\bar{g}$ with $\bar{\nabla}$ we get

$$\bar{g}(\bar{Q}(X^v), Y^h) = \frac{1}{2\alpha}X^v(Y^h(\alpha)) - \frac{1}{2\alpha}\bar{g}(h\bar{\nabla}\alpha, \bar{\nabla}_{X^v}Y^h) - \frac{1}{2\alpha}X^v(Y^h(\alpha)) + \frac{1}{2\alpha}\bar{g}(\bar{\nabla}\alpha, \bar{\nabla}_{X^v}Y^h) + \frac{3-2n}{4\alpha^2}X^v(\alpha)Y^h(\alpha).$$

But, the formula (5) says that $\bar{g}(\bar{\nabla}\alpha, \bar{\nabla}_{X^v}Y^h) = \bar{g}(h\bar{\nabla}\alpha, \bar{\nabla}_{X^v}Y^h)$. So,

$$\bar{g}(\bar{Q}(X^v), Y^h) = \frac{3-2n}{4\alpha^2}X^v(\alpha)Y^h(\alpha). \tag{7}$$

Setting $\bar{Q}(X^v) = \rho X^v$ in (7) gives us $X^v(\alpha)Y^h(\alpha) = 0$. Now, suppose that $((x^1, \ldots, x^n, y^1, \ldots, y^n), \mathcal{A})$ be the standard coordinate system on $\mathcal{A}$. If the zero sets of the functions $\frac{\partial\alpha}{\partial y^i}$ for $i = 1, \ldots, n$ are dense in $\mathcal{A}$, then $v\bar{\nabla}\alpha = 0$ on $\mathcal{A}$. If there exists $i_0 \in \{1, \ldots, n\}$ such that the zero set of $\frac{\partial\alpha}{\partial y^{i_0}}$ is not dense, then there exists an open set $\mathcal{B} \subset \mathcal{A}$ such that for all $(p, u) \in \mathcal{B}$ we have $\frac{\partial\alpha}{\partial y^{i_0}}(p, u) \neq 0$. On the other hand

$$\frac{\partial\alpha}{\partial y^{i_0}}\frac{\partial\alpha}{\partial x^j} = 0 \qquad \forall j = 1, \ldots, n,$$

on $\mathcal{B}$, that is, $\frac{\partial\alpha}{\partial x^j} = 0, \forall j = 1, \ldots, n$. This implies that $h\bar{\nabla}\alpha = 0$ on $\mathcal{B}$. Note that when we talk about the Euclidean space with coordinate system $(x^1, \ldots, x^n)$, the horizontal space will be spanned by vector fields $\frac{\partial}{\partial x^i}$ for $i = 1, \ldots, n$. $\square$

Remark 3.3. If (M, g) is an Einstein Riemannian manifold of dimension greater than 3, then from the contracted second Bianchi identity we can conclude that the $(1, 1)$-Ricci tensor is a constant multiple of identity.

Now, suppose $\mathcal{B}$ is an open subset of $\mathrm{T}\mathbb{R}^n$. If $(\mathcal{B}, \bar{g})$ is an Einstein manifold and $h\bar{\nabla}\alpha = 0$ on $\mathcal{B}$, it will be shown that α is a constant function on $\mathcal{B}$.

Lemma 3.4. *Let $\mathcal{B}$ be an open subset of $\mathrm{T}\mathbb{R}^n$ and $\bar{g}$ be a Riemannian metric defined by α such that $h\bar{\nabla}\alpha = 0$ on $\mathcal{B}$ with $n \geq 2$. If $(\mathcal{B}, \bar{g})$ is an Einstein manifold, then α must be a constant function on $\mathcal{B}$.*

Proof. Let $(x^1, \ldots, x^n)$ be the standard coordinate system on $\pi(\mathcal{B}) \subset \mathbb{R}^n$ and suppose that $(x^1, \ldots, x^n, y^1, \ldots, y^n)$ is the standard coordinate system on $\mathcal{B}$. Moreover, suppose $\bar{Q}$ is the Ricci operator on $\mathcal{B}$ given by the equations (15) and (16). If $\bar{Q}(X^h) = \rho X^h$, then by setting $X = \frac{\partial}{\partial x^j}$ in the equation (15) one can get

$$\rho\frac{\partial}{\partial x^j} = \left\{\frac{1}{4\alpha^2}||v\bar{\nabla}\alpha||^2 - \frac{1}{2\alpha}\Delta_{\bar{g}}\alpha\right\}\frac{\partial}{\partial x^j} - \frac{1}{2\alpha}v\bar{\nabla}_{\frac{\partial}{\partial x^j}}\left(\alpha\sum_{i=1}^{n}\frac{\partial\alpha}{\partial y^i}\frac{\partial}{\partial y^i}\right) + \frac{1}{2\alpha}\bar{\nabla}_{\frac{\partial}{\partial x^j}}\left(\alpha\sum_{i=1}^{n}\frac{\partial\alpha}{\partial y^i}\frac{\partial}{\partial y^i}\right).$$

Using the equation (4) and the fact that $h\bar{\nabla}\alpha = 0$ give us

$$\rho\frac{\partial}{\partial x^j} = \left\{\frac{1}{4\alpha^2}||v\bar{\nabla}\alpha||^2 - \frac{1}{2\alpha}\Delta_{\bar{g}}\alpha\right\}\frac{\partial}{\partial x^j} + \frac{1}{4\alpha}\sum_{i=1}^{n}\left(\frac{\partial\alpha}{\partial y^i}\right)^2\frac{\partial}{\partial x^j},$$

which is equivalent to

$$\rho = \frac{1}{2\alpha}\sum_{i=1}^{n}\left(\frac{\partial\alpha}{\partial y^i}\right)^2 - \frac{1}{2\alpha}\Delta_{\bar{g}}\alpha. \tag{8}$$

On the other hand, working on the equation (16) leads to the following equation

$$\begin{aligned}\rho\frac{\partial}{\partial y^j} = &\left\{-\frac{1}{\alpha}\sum_{i=1}^{n}\left(\frac{\partial\alpha}{\partial y^i}\right)^2 + \frac{1}{2\alpha}\Delta_{\bar{g}}\alpha\right\}\frac{\partial}{\partial y^j}\\ &+\sum_{i=1}^{n}\left(\frac{1-n}{2\alpha}\frac{\partial\alpha}{\partial y^i}\frac{\partial\alpha}{\partial y^j} - \frac{\partial^2\alpha}{\partial y^i\partial y^j}\right)\frac{\partial}{\partial y^i} + \frac{1}{2\alpha}\sum_{i=1}^{n}\left(\frac{\partial\alpha}{\partial y^i}\right)^2\frac{\partial}{\partial y^j},\end{aligned}$$

which gives us

$$\rho = -\frac{1}{2\alpha}\sum_{i=1}^{n}\left(\frac{\partial\alpha}{\partial y^i}\right)^2 + \frac{1}{2\alpha}\Delta_{\bar{g}}\alpha + \frac{1-n}{2\alpha}\left(\frac{\partial\alpha}{\partial y^j}\right)^2 - \frac{\partial^2\alpha}{\partial y^j\partial y^j},\ j = 1,\dots,n, \tag{9}$$

and

$$0 = \frac{1-n}{2\alpha}\frac{\partial\alpha}{\partial y^j}\frac{\partial\alpha}{\partial y^i} - \frac{\partial^2\alpha}{\partial y^i\partial y^j},\quad i,j = 1,\dots,n \text{ and } i \neq j. \tag{10}$$

Note that $\bar{\nabla}\alpha = \alpha\sum_{i=1}^{n}\frac{\partial\alpha}{\partial y^i}\frac{\partial}{\partial y^i}$ and $\Delta_{\bar{g}}\alpha$ can be calculated from (13) as $\Delta_{\bar{g}}\alpha = \sum_{i=1}(\alpha\frac{\partial^2\alpha}{\partial(y^i)^2} + (\frac{\partial\alpha}{\partial y^i})^2)$. The equations (8), (9) and (10) are equivalent to the following system of PDE's

$$\begin{aligned}&\sum_{i=1}^{n}\frac{\partial^2\alpha}{\partial(y^i)^2} = -2\rho,\\ &\frac{1-n}{2\alpha}\left(\frac{\partial\alpha}{\partial y^j}\right)^2 - \frac{\partial^2\alpha}{\partial(y^j)^2} = 2\rho,\quad j = 1,\dots,n,\\ &\frac{1-n}{2\alpha}\frac{\partial\alpha}{\partial y^j}\frac{\partial\alpha}{\partial y^i} - \frac{\partial^2\alpha}{\partial y^j\partial y^i} = 0,\quad i,j = 1,\dots,n \ \textit{and}\ i \neq j.\end{aligned}$$

The third equation can be written as $\frac{\partial^2}{\partial y^i\partial y^j}(\alpha^{(n+1)/2}) = 0$. So, it follows that

$$\alpha^{(n+1)/2} = \sum_i f_i(y^i)$$

for some mappings f_i, $i = 1,\dots,n$ (note that $h\bar{\nabla}\alpha = 0$). Now, the second condition says that

$$\frac{\partial^2}{\partial(y^i)^2}(\alpha^{(n+1)/2}) = -(n+1)\rho\alpha^{(n-1)/2}.$$

Since the left hand side depends only on y_i, this is possible when α is a function of only y^i. If we consider all indexes i, α must be constant. Note that according to Remark 3.3, ρ is a constant function. □

It is natural to think of what happen for α when $v\bar{\nabla}\alpha$ vanishes on an open set $\mathcal{B} \subset \mathrm{T}\mathbb{R}^n$ whenever $\mathcal{B}$ equipped with an Einstein metric $\bar{g}$. Next lemma shows that α must be a constant function in this case, too.

Lemma 3.5. *Let $\mathcal{B}$ be an open subset of $\mathrm{T}\mathbb{R}^n$ and $\bar{g}$ be a Riemannian metric defined by α such that $v\bar{\nabla}\alpha = 0$ on $\mathcal{B}$. If $(\mathcal{B}, \bar{g})$ is an Einstein manifold, then α must be a constant function.*

Proof. First, note that $\bar{\nabla}\alpha = \frac{1}{\alpha}\sum_{i=1}^n \frac{\partial\alpha}{\partial x^i}\frac{\partial}{\partial x^i}$ and $\Delta_{\bar{g}}\alpha = \sum_{i=1}^n(\frac{1}{\alpha}\frac{\partial^2\alpha}{\partial(x^i)^2} - \frac{1}{\alpha^2}(\frac{\partial\alpha}{\partial x^i})^2)$. Like the proof of the last lemma, using the equations (15) and (16) and after some routine calculations we get the following system of PDE's

$$\rho = \frac{1}{2\alpha^2}\sum_{j=1}^n \frac{\partial^2\alpha}{\partial(x^j)^2} - \frac{1}{\alpha^3}\sum_{j=1}^n \left(\frac{\partial\alpha}{\partial x^j}\right)^2,$$

$$2\rho = \frac{1}{\alpha^2}\frac{\partial^2\alpha}{\partial(x^i)^2} - \frac{n+3}{2\alpha^3}\left(\frac{\partial\alpha}{\partial x^i}\right)^2, \qquad i = 1, \ldots, n,$$

$$\frac{\partial^2\alpha}{\partial x^i\partial x^j} = \frac{n+3}{2\alpha}\frac{\partial\alpha}{\partial x^i}\frac{\partial\alpha}{\partial x^j}, \qquad i \neq j, \qquad i, j = 1, \ldots, n.$$

The third equation is equivalent to the following

$$\frac{\partial^2}{\partial x^i\partial x^j}\alpha^{-(n+1/2)} = 0,$$

which says that

$$\alpha^{-(n+1/2)} = \sum_{i=1}^n g_i(x^i), \tag{11}$$

for some functions g_i, $i = 1, \ldots, n$ (note that $v\bar{\nabla}\alpha = 0$). Also, the second equation gives us

$$-(n+1)\rho\alpha^{\frac{1-n}{2}} = \frac{\partial^2}{\partial(x^i)^2}\alpha^{-(n+1/2)}. \tag{12}$$

The equations (11) and (12) show that α must be a constant function. □

Proof of Theorem 1.3

Suppose α is not constant. So, there exists $v \in \mathrm{T}\mathbb{R}^n$ such that $\bar{\nabla}\alpha \neq 0$ at v. This implies that there exists an open set $\mathcal{A}$ of $\mathrm{T}\mathbb{R}^n$ such that $\bar{\nabla}\alpha \neq 0$ on $\mathcal{A}$. But, the last three lemmas showed that $\bar{\nabla}\alpha$ vanishes on an open subset of $\mathcal{A}$ and this is a contradiction. So, α is a constant mapping.

Now, let $\bar{\nabla}\alpha = 0$ on $\mathrm{T}\mathbb{R}^n$. Since the Euclidean space is a flat space then using the equations (15) and (16) calculated for $\bar{Q}(X^h)$ and $\bar{Q}(X^v)$ shows that $\bar{Q}$ must be vanished and this implies that $(\mathrm{T}\mathbb{R}^n, \bar{g})$ is an Einstein manifold.

So, we get that $(\mathrm{T}\mathbb{R}^n, \bar{g})$ is an Einstein manifold if and only if α is a constant function.

Proof of Theorem 1.4

We know that every space form is an Einstein manifold. So, $(\mathrm{T}M, \bar{g})$ is an Einstein manifold. Moreover, due to the lemma 3.1 the base manifold is locally flat. Now, without loss of generality, one can assume that M is the Euclidean space. So, from the theorem 1.3 one can get that α is a constant function. This proves the theorem.

Corollary 3.6. *If* $(\mathrm{T}M, \bar{g})$ *is of constant sectional curvature, then* $J_{\delta,0}$ *is integrable.*

Proof. We know that if the base manifold is flat with locally conformal flat coordinate system $(x^1, \ldots, x^n)$, then $J_{\delta,\sigma}$ is integrable if and only if

$$\frac{\partial z}{\partial x^l} + z\frac{\partial z}{\partial y^l} = 0,$$

for all $l = 1, \ldots, n$ where $(x^1, \ldots, x^n, y^1, \ldots, y^n)$ is the related coordinate system on $\mathrm{T}M$. So, according to the theorem 1.4, z satisfies the above equation and therefore, $J_{\delta,0}$ is integrable. □

Appendix

Here, one can find the needed formulas of curvatures and Laplacian. Their calculations can be found in [5].

Definition. Let (M, g) be a Riemannian manifold and ∇ be the Levi-Civita connection of g. Moreover, let $C^\infty M$ be the set of all smooth functions on M. The differential operator $\Delta_g : C^\infty M \longrightarrow C^\infty M$ given by

$$\Delta_g(f) = \sum_{i=1}^{n} \{\nabla_{E_i}\nabla_{E_i}(f) - \nabla_{\nabla_{E_i}E_i}(f)\},$$

is called rough Laplacian on functions, where $\{E_1, \ldots, E_n\}$ is a locally orthonormal frame on M and $f \in C^\infty M$.

Let $\{E_1, \ldots, E_n\}$ be a locally orthonormal frame on (M, g) around $p \in M$ such that $\nabla_{E_i}E_j = 0$ at p. Then, it is obvious that

$$\left\{\frac{E_1^h}{\sqrt{\alpha}}, \ldots, \frac{E_1^h}{\sqrt{\alpha}}, \sqrt{\alpha}E_1^v, \ldots, \sqrt{\alpha}E_n^v\right\}$$

is a locally orthonormal frame on $(\mathrm{T}M, \bar{g})$. The Laplacian of α at p is calculated as follow,

$$\Delta_{\bar{g}}\alpha(p) = \sum_{i=1}^{n} \left\{\frac{1}{\alpha}E_i^h(E_i^h(\alpha)) + \alpha E_i^v(E_i^v(\alpha))\right.$$

$$-\frac{1}{\alpha^2}E_i^h(\alpha)E_i^h(\alpha)+E_i^v(\alpha)E_i^v(\alpha)\Big\}(p). \tag{13}$$

Denote by ∇ and R the Levi-Civita connection and the Riemannian curvature tensor of (M,g), respectively. Moreover, let $\{E_1,\dots,E_n\}$ be an orthonormal locally frame on M. Then one can get the following formulas for curvature tensor $\bar{R}$ of $\bar{g}$ in special case and Ricci tensor $\bar{Q}$ as follows.

$$\begin{aligned}\bar{R}(X^v,Y^v)Z^h &= \frac{1}{\alpha^2}(R(X,Y)Z)^h-\frac{1}{\alpha^3}X^v(\alpha)(R(u,Y)Z)^h\\
&+\frac{1}{\alpha^3}Y^v(\alpha)(R(u,X)Z)^h+\frac{1}{4\alpha^4}(R(u,X)R(u,Y)Z)^h\\
&-\frac{1}{4\alpha^4}(R(u,Y)R(u,X)Z)^h+\{-\frac{1}{4\alpha^3}(R(u,Y)Z)^h(\alpha)\\
&+\frac{1}{2\alpha}Y^v(Z^h(\alpha))-\frac{3}{4\alpha^2}Y^v(\alpha)Z^h(\alpha)\}X^v\\
&+\{\frac{1}{4\alpha^3}(R(u,X)Z)^h(\alpha)-\frac{1}{2\alpha}X^v(Z^h(\alpha))\\
&+\frac{3}{4\alpha^2}X^v(\alpha)Z^h(\alpha)\}Y^v,\end{aligned} \tag{14}$$

and

$$\begin{aligned}\bar{Q}(X^h) &= \frac{1}{\alpha}Q^h(X)+\{\frac{1}{4\alpha^2}||v\bar{\nabla}\alpha||^2-\frac{1}{2\alpha}\Delta_{\bar{g}}\alpha\}X^h\\
&+\frac{1}{2\alpha}\{h\bar{\nabla}_{X^h}h\bar{\nabla}\alpha-v\bar{\nabla}_{X^h}v\bar{\nabla}\alpha\}\\
&+\frac{1}{2\alpha}\bar{\nabla}_{X^h}\bar{\nabla}\alpha-\frac{2n+1}{4\alpha^2}X^h(\alpha)\bar{\nabla}\alpha-\frac{1}{4\alpha^2}X^h(\alpha)h\bar{\nabla}\alpha\\
&+\frac{1}{\alpha^2}X^h(\alpha)v\bar{\nabla}\alpha\\
&+\Sigma_{i=1}^n\{\frac{3}{4\alpha^3}(R(u,R(X,E_i)u)E_i)^h+\frac{1}{4\alpha^2}(R(X,E_i)u)^v(\alpha)E_i^h\\
&-\frac{1}{4\alpha^3}(R(u,E_i)R(u,E_i)X)^h+\frac{1}{2\alpha}((\nabla_{E_i}R)(X,E_i)u)^v\\
&-\frac{3}{2\alpha^2}E_i^h(\alpha)(R(X,E_i)u)^v+\frac{1}{4\alpha^2}(R(u,E_i)X)^h(\alpha)E_i^v\},\end{aligned} \tag{15}$$

and

$$\begin{aligned}\bar{Q}(X^v) &= \{-\frac{1}{4\alpha^2}||v\bar{\nabla}\alpha||^2-\frac{3}{4\alpha^2}||\bar{\nabla}\alpha||^2+\frac{1}{2\alpha}\Delta_{\bar{g}}\alpha\}X^v\\
&+\frac{1}{2\alpha}\{h\bar{\nabla}_{X^v}h\bar{\nabla}\alpha-v\bar{\nabla}_{X^v}v\bar{\nabla}\alpha\}\\
&-\frac{1}{2\alpha}\bar{\nabla}_{X^v}\bar{\nabla}\alpha+\frac{3-2n}{4\alpha^2}X^v(\alpha)\bar{\nabla}\alpha+\frac{3}{4\alpha^2}X^v(\alpha)v\bar{\nabla}\alpha\\
&+\Sigma_{i=1}^n\{-\frac{1}{2\alpha^3}((\nabla_{E_i}R)(u,X)E_i)^h+\frac{3}{2\alpha^4}E_i^h(\alpha)(R(u,X)E_i)^h\end{aligned} \tag{16}$$

$$-\frac{1}{4\alpha^4}(R(u,X)E_i)^h(\alpha)E_i^h + \frac{1}{4\alpha^3}(R(E_i,R(u,X)E_i)u)^v\}.$$

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References

[1] M. T. K. Abbassi, G. Calvaruso, and D. Perrone, *Harmonic sections of tangent bundles equipped with Riemannian g-natural metrics*, Q. J. Math. **62** (2011), no. 2, 259–288. `https://doi.org/10.1093/qmath/hap040`

[2] M. T. K. Abbassi and M. Sarih, *On some hereditary properties of Riemannian g-natural metrics on tangent bundles of Riemannian manifolds*, Differential Geom. Appl. **22** (2005), no. 1, 19–47. `https://doi.org/10.1016/j.difgeo.2004.07.003`

[3] R. M. Aguilar, *Isotropic almost complex structures on tangent bundles*, Manuscripta Math. **90** (1996), no. 4, 429–436. `https://doi.org/10.1007/BF02568316`

[4] A. Baghban and E. Abedi, *On the integrability of the isotropic almost complex structures and harmonic unit vector fields*, arXiv:1606.09198v1, 2016.

[5] ———, *Curvatures of metrics induced by the isotropic almost complex structures*, arXiv preprint arXiv:1607.07018, 2016.

[6] ———, *A new class of almost complex structures on tangent bundle of a Riemannian manifold*, Commun. Math. **26** (2018), no. 2, 137–145. `https://doi.org/10.2478/cm-2018-0010`

[7] I. Biswas, J. Loftin, and M. Stemmler, *Flat bundles on affine manifolds*, Arab. J. Math. (Springer) **2** (2013), no. 2, 159–175. `https://doi.org/10.1007/s40065-012-0064-8`

[8] J. Choi and A. P. Mullhaupt, *Kählerian information geometry for signal processing*, Entropy **17** (2015), no. 4, 1581–1605. `https://doi.org/10.3390/e17041581`

[9] R. M. Friswell and C. M. Wood, *Harmonic vector fields on pseudo-Riemannian manifolds*, J. Geom. Phys. **112** (2017), 45–58. `https://doi.org/10.1016/j.geomphys.2016.10.015`

[10] C. Kuzu and N. Cengiz, *Curvature tensor in tangette bundles of semi-riemannian manifold*, NTMSCI **6** (2018), no. 3, 168–173.

[11] S. T. Lisi, *Applications of symplectic geometry to Hamiltonian mechanics*, ProQuest LLC, Ann Arbor, MI, 2006.

[12] E. Peyghan, A. Baghban, and E. Sharahi, *Linear Poisson structures and Hom-Lie algebroids*, Rev. Un. Mat. Argentina **60** (2019), no. 2, 299–313. `https://doi.org/10.33044/revuma.v60n2a01`

[13] E. Peyghan, A. Heydari, and L. Nourmohammadi Far, *On the geometry of tangent bundles with a class of metrics*, Ann. Polon. Math. **103** (2012), no. 3, 229–246. `https://doi.org/10.4064/ap103-3-2`

[14] E. Peyghan, H. Nasrabadi, and A. Tayebi, *The homogeneous lift to the* $(1,1)$*-tensor bundle of a Riemannian metric*, Int. J. Geom. Methods Mod. Phys. **10** (2013), no. 4, 1350006, 18 pp. `https://doi.org/10.1142/S0219887813500060`

[15] E. Peyghan and L. Nourmohammadi Far, *Foliations and a class of metrics on tangent bundle*, Turkish J. Math. **37** (2013), no. 2, 348–359. `https://doi.org/10.3906/mat-1102-36`

[16] A. Salimov and A. Gezer, *On the geometry of the* $(1,1)$*-tensor bundle with Sasaki type metric*, Chin. Ann. Math. Ser. B **32** (2011), no. 3, 369–386. `https://doi.org/10.1007/s11401-011-0646-3`

[17] J. Zhang and F. Li, *Symplectic and Kähler structures on statistical manifolds induced from divergence functions*, in Geometric science of information, 595–603, Lecture Notes in Comput. Sci., 8085, Springer, Heidelberg, 2013. https://doi.org/10.1007/978-3-642-40020-9_66

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