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L_K-BIHARMONIC HYPERSURFACES IN SPACE FORMS WITH THREE DISTINCT PRINCIPAL CURVATURES

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Abstract. In this paper we consider L_k-conjecture introduced in [5,6] for hypersurface M^n in space form $R^{n+1}(c)$ with three principal curvatures. When $c = 0, -1$, we show that every L_1-biharmonic hypersurface with three principal curvatures and H_1 is constant, has $H_2 = 0$ and at least one of the multiplicities of principal curvatures is one, where H_1 and H_2 are first and second mean curvature of M and we show that there is not L_2-biharmonic hypersurface with three disjoint principal curvatures and, H_1 and H_2 is constant. For $c = 1$, by considering having three principal curvatures, we classify L_1-biharmonic hypersurfaces with multiplicities greater than one, H_1 is constant and $H_2 = 0$, proper L_1-biharmonic hypersurfaces which H_1 is constant, and L_2-biharmonic hypersurfaces which H_1 and H_2 is constant.

1. Introduction and statement of result

B. Y. Chen in [12] made the conjecture: Any biharmonic submanifold of a Euclidean space is minimal. Several authors have proved it under some conditions, see for example, [1, 14–17, 20]. Also this conjecture has been generalized in [9]: Any biharmonic submanifold of a Riemannian manifold of non-positive sectional curvature is minimal. This generalized conjecture has been proved for constant sectional curvature ambient spaces in numerous cases as in [2, 7, 9, 19, 24, 25]. The Generalized Chen conjecture has been shown to be false by constructing foliations of proper biharmonic hyperplanes in a 5-dimensional conformally flat space of non-constant negative sectional curvature in [26]. In case of positive sectional curvature ambient spaces, there are several families of biharmonic submanifolds which are not minimal. For example in [8], the authors classified proper biharmonic hypersurfaces in the unit Euclidean sphere with at most two distinct principal curvatures.

Let $\varphi : M^n \to \mathbb{R}^{n+1}$ be an isometric immersion from a connected oriented Riemannian manifold into the Euclidean space $\mathbb{R}^{n+1}$ with N as the unit normal

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direction. We have, [3],

$$L_k\varphi = (k+1)\binom{n}{k+1}H_{k+1}N,$$

where $k = 0,\dots,n-1$ and H_{k+1} is $(k+1)$-th mean curvature of M. When $k = 0$, the above equation reduces to $\Delta\varphi = nH_1N = n\vec{H}$ which is the Beltrami equation. In [5], we proposed the L_k-conjecture: Every Euclidean hypersurface $\varphi : M^n \to \mathbb{R}^{n+1}$ satisfying the condition $L_k^2\varphi = 0$ for some k, $0 \leq k \leq n-1$, has zero $(k+1)$-th mean curvature, namely it is k-minimal. We have proved that the L_k-conjecture is true for Euclidean hypersurfaces with at most two principal curvatures, [5]. Hereafter in [6], we have generalized the notions of tension and bitension fields to introduce L_k-harmonic and L_k-biharmonic maps.

Let M be a connected, oriented isometrically immersed Riemannian hypersurface in a simply connected space form $R^{n+1}(c), c = 0, \pm1$. Then M is called an L_k-biharmonic hypersurface if the following equations are satisfied:

$$\binom{n}{k+1}H_{k+1}\nabla H_{k+1} + 2(S\circ P_k)(\nabla H_{k+1}) = 0, \tag{1}$$

$$L_kH_{k+1} - \binom{n}{k+1}H_{k+1}\left(nH_1H_{k+1} - (n-k-1)H_{k+2} - c(k+1)H_k\right) = 0. \tag{2}$$

In addition M is called a proper L_k-biharmonic hypersurface if M is an L_k-biharmonic hypersurface and $H_{k+1} \neq 0$.

***L_k*-conjecture 1.1** ([6]). *Let $\varphi : M^n \to R^{n+1}(c)$, $c = 0, \pm1$, be a connected oriented hypersurface immersed into a simply connected space form $R^{n+1}(c)$. If M is an L_k-biharmonic hypersurface, then H_{k+1} is zero.*

The L_k-conjecture has been proved in some cases. For $c = 0, -1$, the L_k-conjecture is proved as hypersurface M has two principal curvatures, or M is weakly convex, or M is complete with some constraint on it and on L_k, and it is shown that there is not any L_k-biharmonic hypersurface M^n in $\mathbb{H}^{n+1}$ with two principal curvatures of multiplicities greater than one, [6].

In this paper we consider L_k-conjecture for hypersurface M^n in space form $R^{n+1}(c)$ with three principal curvatures. When $c = 0, -1$, in Theorem 1.2, we show that every L_1-biharmonic hypersurface with three principal curvatures and H_1 is constant, has $H_2 = 0$ and at least one of the multiplicities of principal curvatures is one, and we show that there is not L_2-biharmonic hypersurface with three disjoint principal curvatures and, H_1 and H_2 is constant. Recently, in [22] for the case $c = 0$, the authors prove that the L_1-conjecture is true for L_1-biharmonic hypersurfaces with three distinct principal curvatures and constant mean curvature of a Euclidean space, meanwhile in our paper we give more result in this case and also we consider L_2-conjecture and we give some classification for cases $c = 0, 1, -1$ which are completely different.

For the case $c = 1$, the L_k-conjecture is false by considering hypersurface $\mathbb{S}^n(\frac{\sqrt{2}}{2})$ in the n-dimensional unit Euclidean sphere $\mathbb{S}^n$, so $\mathbb{S}^n(\frac{\sqrt{2}}{2})$ is a proper

L_k-biharmonic hypersurface. This result has been extended to hypersurfaces having two distinct principal curvatures and it's shown that they are open pieces of the standard products of spheres, [4].

For $c = 1$, in Theorem 1.2, by considering hypersurfaces having three principal curvatures in the unit Euclidean sphere, we classify L_1-biharmonic hypersurfaces with multiplicities greater than one, H_1 is constant and $H_2 = 0$, proper L_1-biharmonic hypersurfaces which H_1 is constant, and L_2-biharmonic hypersurfaces which H_1 and H_2 is constants.

Theorem 1.2. *Let M^n be a connected, oriented isometrically immersed hypersurface in space form $R^{n+1}(c)$. Suppose that M has three distinct principal curvatures and $H_1, \ldots, H_k$ are constant. Let $c = 0, -1$. If $k = 1$ and M is L_1-biharmonic, then $H_2 = 0$ and at least one of the multiplicities of principal curvatures is one. If $k = 2$, then M is not L_2-biharmonic. Let $c = 1$. If $k = 1$ and M is L_1-biharmonic, then H_2 is constant, and if $H_2 = 0$ and multiplicities of principal curvatures are greater than one, and or M is proper L_1-biharmonic, then M is an isoparametric hypersurface. If $k = 2$ and M is L_2-biharmonic, then M is an isoparametric hypersurface.*

Assume that $k_1 > k_2 > k_3$ denote the principal curvatures of an isoparametric hypersurface in the unit Euclidean Sphere $\mathbb{S}^{n+1}$. Then multiplicities of principal curvatures is equal, say m, m is either $1, 2, 4$ or 8, and $k_2 = \frac{k_1-\sqrt{3}}{1+\sqrt{3}k_1}, k_3 = \frac{k_1+\sqrt{3}}{1-\sqrt{3}k_1}$, and there is a homogeneous polynomial F of degree 3 over $\mathbb{R}^{n+2}$ where for any $a \in (-1, 1)$, $f^{-1}(a) = F|_{\mathbb{S}^{n+1}}^{-1}(a)$ is an isoparametric hypersurface (see Theorem 2.1).

(a) *Let $k = 1$ and M be L_1-biharmonic, $H_2 = 0$ and the multiplicities of principal curvatures be greater than one. Then we have the followings:*
- *If $m = 2$, then k_1, k_2, k_3 approximately are $k_1 \approx 3.286, k_2 \approx 0.232, k_3 \approx -1.069$ or $k_1 \approx 1.069, k_2 \approx -0.232, k_3 \approx -3.286$. So M is congruent to an open part of $f^{-1}(a)$ and a tube of radius θ around the standard embedding of a complex projective plane $\mathbb{C}\boldsymbol{P}^2$ into $\mathbb{S}^7$ where $a \approx 0.632$ and $\theta \approx \pi/10.634$.*
- *If $m = 4$, then k_1, k_2, k_3 approximately are $k_1 \approx 2.527, k_2 \approx 0.147, k_3 \approx -1.261$, or $k_1 \approx 1.261, k_2 \approx -0.147, k_3 \approx -2.527$. So M is congruent to an open part of $f^{-1}(a)$ and a tube of radius θ around the standard embedding of a quaternionic projective plane $\mathbb{H}\boldsymbol{P}^2$ into $\mathbb{S}^{13}$ where $a \approx 0.426$ and $\theta \approx \pi/8.337$.*
- *If $m = 8$, then k_1, k_2, k_3 approximately are $k_1 \approx 2.216, k_2 \approx 0.1, k_3 \approx -1.39$ or $k_1 \approx 1.39, k_2 \approx -0.1, k_3 \approx -2.216$. So M is congruent to an open part of $f^{-1}(a)$ and a tube of radius θ around the standard embedding of a Cayley projective plane $\mathbb{O}\boldsymbol{P}^2$ into $\mathbb{S}^{25}$ where $a \approx 0.294$ and $\theta \approx \pi/7.411$.*

(b) *Let $k = 1$ and M be proper L_1-biharmonic. Then we have the followings:*

- *If* $m = 1$*, then* k_1, k_2, k_3 *satisfy the following equation*
$$3H_1H_2 - H_3 - 2H_1 = 0,$$
so that $k_1 = \sqrt{3}$*,* $k_2 = 0$*,* $k_3 = -\sqrt{3}$*. Therefore* M *is congruent to an open part of* $f^{-1}(0)$ *and a tube of radius* $\pi/6$ *around the standard embedding of a real projective plane* $\mathbb{R}\boldsymbol{P}^2$ *into* $\mathbb{S}^4$*. Also* M *is a Cartan minimal hypersurface of dimension* 3*.*
- *If* $m = 2$*, then* k_1, k_2, k_3 *satisfy the following equation*
$$6H_1H_2 - 4H_3 - 2H_1 = 0,$$
so that either $k_1 = \sqrt{3}, k_2 = 0, k_3 = -\sqrt{3}$ *or approximately* $k_1 \approx 1.369, k_2 \approx -0.107, k_3 \approx -2.261$ *or* $k_1 \approx 2.261, k_2 \approx 0.107, k_3 \approx -1.369$*. If* $k_1 = \sqrt{3}, k_2 = 0, k_3 = -\sqrt{3}$*, then* M *is congruent to an open part of* $f^{-1}(0)$ *and a tube of radius* $\pi/6$ *around the standard embedding of a complex projective plane* $\mathbb{C}\boldsymbol{P}^2$ *into* $\mathbb{S}^7$*. Also* M *is a Cartan minimal hypersurface of dimension* 6*. If* $k_1 \approx 1.369, k_2 \approx -0.107, k_3 \approx -2.261$ *or* $k_1 \approx 2.261, k_2 \approx 0.107, k_3 \approx -1.369$*, then* M *is congruent to an open part of* $f^{-1}(a)$ *and a tube of radius* θ *around the standard embedding of a complex projective plane* $\mathbb{C}\boldsymbol{P}^2$ *into* $\mathbb{S}^7$ *where* $a \approx 0.316$ *and* $\theta \approx \pi/7.544$*.*
- *If* $m = 4$*, then* k_1, k_2, k_3 *satisfy the following equation*
$$12H_1H_2 - 10H_3 - 2H_1 = 0,$$
so that $k_1 = \sqrt{3}, k_2 = 0, k_3 = -\sqrt{3}$*. Therefore* M *is congruent to an open part of* $f^{-1}(0)$ *and a tube of radius* $\pi/6$ *around the standard embedding of a quaternionic projective plane* $\mathbb{H}\boldsymbol{P}^2$ *into* $\mathbb{S}^{13}$*. Also* M *is a Cartan minimal hypersurface of dimension* 12*.*
- *If* $m = 8$*, then* k_1, k_2, k_3 *satisfy the following equation*
$$24H_1H_2 - 22H_3 - 2H_1 = 0,$$
so that $k_1 = \sqrt{3}, k_2 = 0, k_3 = -\sqrt{3}$*. Therefore* M *is congruent to an open part of* $f^{-1}(0)$ *and a tube of radius* $\pi/6$ *around the standard embedding of a Cayley projective plane* $\mathbb{O}\boldsymbol{P}^2$ *into* $\mathbb{S}^{25}$*. Also* M *is a Cartan minimal hypersurface of dimension* 24*.*

(c) *Let* $k = 2$ *and* M *be* L_2*-biharmonic and* $H_3 = 0$*. Then we have the followings*:
- *If* $m = 1$*, then* k_1, k_2, k_3 *are* $k_1 = \sqrt{3}$*,* $k_2 = 0$*,* $k_3 = -\sqrt{3}$*. So* M *is congruent to an open part of* $f^{-1}(0)$ *and a tube of radius* $\pi/6$ *around the standard embedding of a real projective plane* $\mathbb{R}\boldsymbol{P}^2$ *into* $\mathbb{S}^4$*. Also* M *is a Cartan minimal hypersurface of dimension* 3*.*
- *If* $m = 2$*, then* k_1, k_2, k_3 *are* $k_1 = \sqrt{3}, k_2 = 0, k_3 = -\sqrt{3}$*. So* M *is congruent to an open part of* $f^{-1}(0)$ *and a tube of radius* $\pi/6$ *around the standard embedding of a complex projective plane* $\mathbb{C}\boldsymbol{P}^2$

into $\mathbb{S}^7$. Also M is a Cartan minimal hypersurface of dimension 6.

- *If $m = 4$, then k_1, k_2, k_3 are $k_1 = \sqrt{3}, k_2 = 0, k_3 = -\sqrt{3}$ or approximately $k_1 \approx 0.993, k_2 \approx -0.271, k_3 \approx -3.777$ or $k_1 \approx 3.777, k_2 \approx 0.271, k_3 \approx -0.993$. If $k_1 = \sqrt{3}, k_2 = 0, k_3 = -\sqrt{3}$, then M is congruent to an open part of $f^{-1}(0)$ and a tube of radius $\pi/6$ around the standard embedding of a quaternionic projective plane $\mathbb{H}\boldsymbol{P}^2$ into $\mathbb{S}^{13}$. Also M is a Cartan minimal hypersurface of dimension* 12. *If $k_1 \approx 0.993, k_2 \approx -0.271, k_3 \approx -3.777$ or $k_1 \approx 3.777, k_2 \approx 0.271, k_3 \approx -0.993$, then M is congruent to an open part of $f^{-1}(a)$ and a tube of radius θ around the standard embedding of a quaternionic projective plane $\mathbb{H}\boldsymbol{P}^2$ into $\mathbb{S}^{13}$ where $a \approx 0.713$ and $\theta \approx \pi/12.138$.*
- *If $m = 8$, then k_1, k_2, k_3 are $k_1 = \sqrt{3}, k_2 = 0, k_3 = -\sqrt{3}$ or approximately $k_1 \approx 1.189, k_2 \approx -0.177, k_3 \approx -2.757$ or $k_1 \approx 2.757, k_2 \approx 0.177, k_3 \approx -1.189$. If $k_1 = \sqrt{3}, k_2 = 0, k_3 = -\sqrt{3}$, then M is congruent to an open part of $f^{-1}(0)$ and a tube of radius $\pi/6$ around the standard embedding of a Cayley projective plane $\mathbb{O}\boldsymbol{P}^2$ into $\mathbb{S}^{25}$. Also M is a Cartan minimal hypersurface of dimension* 24. *If $k_1 \approx 1.189, k_2 \approx -0.177, k_3 \approx -2.757$ or $k_1 \approx 2.757, k_2 \approx 0.177, k_3 \approx -1.189$, then M is congruent to an open part of $f^{-1}(a)$ and a tube of radius θ around the standard embedding of a Cayley projective plane $\mathbb{O}\boldsymbol{P}^2$ into $\mathbb{S}^{25}$ where $a \approx 0.502$ and $\theta \approx \pi/9.028$.*

(d) *Let $k = 2$ and M be proper L_2-biharmonic. Then we have the followings*:

- *If $m = 1$, then k_1, k_2, k_3 satisfy the equation*

$$H_1 H_3 - H_2 = 0,$$

so that either $k_1 = 1, k_2 = \sqrt{3} - 2, k_3 = -\sqrt{3} - 2$ or $k_1 = 2 + \sqrt{3}, k_2 = 2 - \sqrt{3}, k_3 = -1$. Therefore M is congruent to an open part of $f^{-1}(\frac{\sqrt{2}}{2})$ and a tube of radius $\pi/12$ around the standard embedding of a real projective plane $\mathbb{R}\boldsymbol{P}^2$ into $\mathbb{S}^4$.

- *If $m = 2$, then k_1, k_2, k_3 satisfy the following equation*

$$2H_1 H_3 - H_4 - H_2 = 0,$$

so that there is no real solution for all k_1, k_2, k_3. Therefore there is no proper L_2-biharmonic hypersurface in $\mathbb{S}^7$ with three disjoint principal curvatures, and H_1 and H_2 are constants.

- *If $m = 4$, then k_1, k_2, k_3 satisfy the equation*

$$4H_1 H_3 - 3H_4 - H_2 = 0,$$

so that approximately either $k_1 \approx 1.083, k_2 \approx -0.225, k_3 \approx -3.213$ or $k_1 \approx 3.213, k_2 \approx 0.225, k_3 \approx -1.083$. Then M is congruent to *an open part of* $f^{-1}(a)$ *and a tube of radius* θ *around the standard embedding of a quaternionic projective plane* $\mathbb{H}\boldsymbol{P}^2$ *into* $\mathbb{S}^{13}$ *where* $a \approx 0.617$ *and* $\theta \approx \pi/10.411$.

- *If* $m = 8$, *then* k_1, k_2, k_3 *satisfy the following equation*

$$8H_1H_3 - 7H_4 - H_2 = 0,$$

so that there is no real solution for all k_1, k_2, k_3. *Therefore there is no proper* L_2*-biharmonic hypersurface in* $\mathbb{S}^{25}$ *with three disjoint principal curvatures, and* H_1 *and* H_2 *are constants.*

An immediate result of Theorem 1.2, we get the following classification of proper L_2-biharmonic hypersurfaces in space form $R^4(c)$ with three distinct principal curvatures and H_2 is constant.

Theorem 1.3. *Let* M^3 *be a connected, oriented isometrically immersed hypersurface in space form* $R^4(c)$ *with three distinct principal curvatures. If* M *is proper* L_2*-biharmonic and* H_2 *is constant, then* $c = 1$ *and* M *is congruent to an open part of* $f^{-1}(\frac{\sqrt{2}}{2})$ *and a tube of radius* $\pi/12$ *around the standard embedding of a real projective plane* $\mathbb{R}\boldsymbol{P}^2$ *into* $\mathbb{S}^4$ *and principal curvatures of* M *are* $2+\sqrt{3}, 2-\sqrt{3}, -1$.

2. Preliminaries

We recall the prerequisites from [3, 10, 11, 13, 23, 27]. Let $R^{n+1}(c)$ be the simply connected Riemannian space form of constant sectional curvature c which is the Euclidean space $\mathbb{R}^{n+1}$ for $c = 0$, and the Hyperbolic space $\mathbb{H}^{n+1}$, for $c = -1$, and the Euclidean sphere $\mathbb{S}^{n+1}$ for $c = +1$. Let $\varphi : M^n \to R^{n+1}(c)$ be a connected oriented hypersurface isometrically immersed into $R^{n+1}(c)$ with N as a unit normal vector field, ∇ and $\overline{\nabla}$ the Levi-Civita connections on M and $R^{n+1}(c)$, respectively. For simplicity we also denote the induced connection on the pullback bundle $\varphi^*TR^{n+1}(c)$ by $\overline{\nabla}$. Let X, Y be vector fields on M. We have the following formula for the shape operator of M,

$$\overline{\nabla}_X d\varphi(Y) = d\varphi(\nabla_X Y) + \langle SX, Y\rangle N,$$
$$d\varphi(SX) = -\overline{\nabla}_X N.$$

As it is known, the shape operator is a self-adjoint linear operator. Let $k_1, \dots, k_n$ be its eigenvalues which are called principal curvatures of M. Define $s_0 = 1$ and

$$s_k = \sum_{1 \le i_1 < \cdots < i_k \le n} k_{i_1} \cdots k_{i_k}. \tag{3}$$

The k-th mean curvature of M is defined by

$$\binom{n}{k} H_k = s_k.$$

For $k = 1$, $H_1 = \frac{1}{n}tr(S) = H$ is the mean curvature of M. For $k = 2$, the scalar curvature of M is $s = n(n-1)H_2$. In general, when k is odd, the sign of H_k depends on the chosen orientation and when k is even, H_k is an intrinsic geometric quantity.

Let M^n have three principal curvatures, k_1, k_2, k_3 with respective multiplicities m_1, m_2, m_3, $n = m_1 + m_2 + m_3$. Therefore we get by Equation (3),

$$s_k = \sum_{i,j} \binom{m_1}{i}\binom{m_2}{j}\binom{m_3}{k-i-j} k_1^i k_2^j k_3^{k-i-j}. \tag{4}$$

The Newton transformations $P_k : \mathcal{X}(M) \to \mathcal{X}(M)$ are defined inductively by $P_0 = I$ and

$$P_k = s_k I - S \circ P_{k-1},\ 1 \le k \le n.$$

Therefore

$$P_k = \sum_{l=0}^{k} (-1)^l s_{k-l} S^l. \tag{5}$$

From the Cayley-Hamilton theorem, one gets that $P_n = 0$. Each P_k is a self adjoint linear operator which commutes with S and the eigenvalues of P_k are given by

$$\mu_{k,i} = \sum_{1 \le i_1 < \cdots < i_k \le n,\, i_j \ne i} k_{i_1} \cdots k_{i_k}. \tag{6}$$

For $0 \le k \le n-1$, the second order linear differential operator $L_k : C^\infty(M) \to C^\infty(M)$ as the natural generalization of the Laplace operator for Euclidean hypersurfaces M, is defined by

$$L_k f = tr(P_k \circ \nabla^2 f), \tag{7}$$

where $\nabla^2 f$ is metrically equivalent to the Hessian of f and is defined by $\big\langle (\nabla^2 f)X, Y \big\rangle = \langle \nabla_X(\nabla f), Y \rangle$ for all vector fields X, Y on M, and ∇f is the gradient vector field of f. When $k = 0$, $L_0 = \Delta$.

We have the following properties of shape operator, curvature tensor and Newton transformation which they are used to prove other results of the paper. If X, Y, Z are tangent vector fields on M, then we have

$$\begin{aligned} R(X,Y)Z &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \\ &= c\left(\langle Z, Y\rangle X - \langle Z, X\rangle Y\right) + \langle SY, Z\rangle SX - \langle SX, Z\rangle SY, \end{aligned} \tag{8}$$

$$(\nabla_X S)Y = (\nabla_Y S)X, \quad \text{(Codazzi equation)}$$

$$tr(P_k) = (n-k)s_k. \tag{9}$$

We recall that a hypersurface M^n in $R^{n+1}(c)$ is said to be isoparametric if it has constant principal curvatures $k_1 > k_2 > \cdots > k_l$ with respective constant multiplicities $m_1, m_2, \ldots, m_l$, $n = m_1 + m_2 + \cdots + m_l$. It is known for $c = 0, -1$, isoparametric hypersurfaces has at most two principal curvatures. For $l = 3$

we have the following classification of isoparametric hypersurfaces in Euclidean sphere.

Theorem 2.1 (cf. [10, 11, 23])**.** *Let M^n be an isoparametric hypersurface in $\mathbb{S}^{n+1}$ with three constant principal curvatures $k_1 > k_2 > k_3$ and respective multiplicities m_1, m_2, m_3. Then we have the followings*:

I. *$m = m_1 = m_2 = m_3 = 2^q$, $n = 3 \cdot 2^q$, $q = 0, 1, 2, 3$, and there exists an angle θ, $0 < \theta < \pi/3$ such that*

$$k_1 = \cot\theta,\ k_2 = \cot(\theta + \frac{\pi}{3}) = \frac{k_1 - \sqrt{3}}{1 + \sqrt{3}k_1},\ k_3 = \cot(\theta + \frac{2\pi}{3}) = \frac{k_1 + \sqrt{3}}{1 - \sqrt{3}k_1}. \tag{10}$$

II. *In the ambient Euclidean space $\mathbb{R}^{n+2} \supset \mathbb{S}^{n+1}$, there is a homogeneous polynomial F of degree 3 over $\mathbb{R}^{n+2}$ whose the range of $f = F|_{\mathbb{S}^{n+1}}$ is $[-1, 1]$, the only critical values of f are ± 1 and for any $a \in (-1, 1)$, $f^{-1}(a)$ is an isoparametric hypersurface and is a tube around the two focal submanifolds $f^{-1}(1)$ and $f^{-1}(-1)$. For $a = \cos(3\theta)$, M is up to congruency an open part of $f^{-1}(a)$ and a tube of radius θ around the two focal submanifolds.*

III. *The two focal submanifolds are standard embedding of a projective plane $\mathbb{F}\boldsymbol{P}^2$ into $\mathbb{S}^{n+1}$ where $\mathbb{F}$ is the division algebra $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ (quaternions), $\mathbb{O}$ (Cayley numbers) corresponding to the principal multiplicity $m = 1, 2, 4,$ or 8.*

IV. *Let $\mathbb{F}$ be one of the division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$. Let $X, Y, Z \in \mathbb{F}$ and $a, b \in \mathbb{R}$. Then*

$$\begin{aligned} F = a^3 - 3ab^2 &+ \frac{3a}{2}(X\overline{X} + Y\overline{Y} - 2Z\overline{Z}) \\ &+ \frac{3\sqrt{3}b}{2}(X\overline{X} - Y\overline{Y}) + \frac{3\sqrt{3}}{2}(XYZ + \overline{XYZ}). \end{aligned}$$

Isoparametric hypersurfaces with three distinct principal curvatures are usually called Cartan hypersurfaces. When a Cartan hypersurface in $\mathbb{S}^{n+1}$ is minimal, it is congruent to one of the following hypersurfaces:

$$\begin{aligned} &M^3 = SO(3)/(\mathbb{Z}_2 + \mathbb{Z}_2) \to \mathbb{S}^4 \\ &M^6 = SU(3)/T^2 \to \mathbb{S}^7 \\ &M^{12} = Sp(3)/(Sp(1) \times Sp(1) \times Sp(1)) \to \mathbb{S}^{13} \\ &M^{24} = F_4/Spin(8) \to \mathbb{S}^{25} \end{aligned}$$

Principal curvatures of a Cartan minimal hypersurface are $\sqrt{3}, 0, -\sqrt{3}$.

3. Proof of main result

Before proving Theorem 1.2, we give an auxiliary Lemma for L_k-biharmonic hypersurface M in space form $R^{n+1}(c)$ which has three distinct principal curvatures and we show H_{k+1} is constant when $k = 1$ or 2 and $H_1, \ldots, H_k$ are constant. In its proof, we benefit from the techniques of [17–19, 21] but adapt them to our context. So our proof is much involved and quite different.

Lemma 3.1. *Let M^n be a connected, oriented isometrically immersed L_k-biharmonic hypersurface in space form $R^{n+1}(c)$. Suppose that M has three distinct principal curvatures and $k = 1$ or 2. If $H_1, \ldots, H_k$ are constant, then H_{k+1} is constant.*

Proof. We have $P_{k+1} = s_{k+1}I - S \circ P_k$. So by Equation (1) we get

$$P_{k+1}\nabla s_{k+1} = \frac{3}{2}s_{k+1}\nabla s_{k+1}. \tag{11}$$

Let s_{k+1} be non constant. We consider $\{e_i\}_{i=1}^n$ is a local orthonormal frame field on M which diagonalize S and P_{k+1} simultaneously and $e_1 = \frac{\nabla s_{k+1}}{|\nabla s_{k+1}|}$. We put

$$Se_i = \lambda_i e_i \text{ and } P_{k+1}e_i = \mu_{k+1,i}e_i, \quad i = 1, \ldots, n. \tag{12}$$

Then we have by Equations (11) and (12),

$$\mu_{k+1,1} = \frac{3}{2}s_{k+1}. \tag{13}$$

So we get by Equations (5) and (13),

$$\frac{3}{2}s_{k+1} = \sum_{l=0}^{k+1}(-1)^l s_{k+1-l}{\lambda_1}^l = s_{k+1} + \sum_{l=1}^{k+1}(-1)^l s_{k+1-l}{\lambda_1}^l.$$

Therefore

$$s_{k+1} = 2\sum_{l=1}^{k+1}(-1)^l s_{k+1-l}{\lambda_1}^l. \tag{14}$$

We have $\nabla s_{k+1} = \sum_{i=1}^n e_i(s_{k+1})e_i = |\nabla s_{k+1}|e_1$. Thus

$$e_1(s_{k+1}) \neq 0 \quad \text{and} \quad \forall i \neq 1 \; e_i(s_{k+1}) = 0. \tag{15}$$

By assumption $s_1, \ldots, s_k$ are constant, so by Equation (14) we get for every i,

$$e_i(s_{k+1}) = 2e_i(\lambda_1)\sum_{l=1}^{k+1}(-1)^l s_{k+1-l}\, l{\lambda_1}^{l-1}. \tag{16}$$

Since $e_1(s_{k+1}) \neq 0$, by Equation (16) we have $e_1(\lambda_1) \neq 0$. If $e_i(\lambda_1) \neq 0$ for some $i \neq 1$, then $e_i(s_{k+1}) = 0$ and Equation (16) imply that $\sum_{l=1}^{k+1}(-1)^l s_{k+1-l}\, l{\lambda_1}^{l-1}$ $= 0$. So this polynomial shows that λ_1 is constant which is a contradiction with $e_1(\lambda_1) \neq 0$. Thus λ_1 is non constant,

$$e_1(\lambda_1) \neq 0 \quad \text{and} \quad \forall i \neq 1 \; e_i(\lambda_1) = 0. \tag{17}$$

Now we show that multiplicity of λ_1 is one. Let's $\nabla_{e_i}e_j = \sum_l \omega_{ij}^l e_l$. Then $\nabla_{e_l}\langle e_i, e_j\rangle = 0$ and the Codazzi equation $(\nabla_{e_i}S)e_j = (\nabla_{e_j}S)e_i$ give that

$$\omega_{li}^j = -\omega_{lj}^i, \tag{18}$$

$$e_i(\lambda_j) = (\lambda_i - \lambda_j)\omega_{ji}^j \quad i \neq j, \tag{19}$$

$$(\lambda_i - \lambda_j)\omega_{li}^j = (\lambda_l - \lambda_j)\omega_{il}^j \quad i \neq j \neq l. \tag{20}$$

If $\lambda_1 = \lambda_j$ for some $j \neq 1$, then by Equation (19) we get $e_1(\lambda_1) = e_1(\lambda_j) = (\lambda_1 - \lambda_j)\omega^j_{j1} = 0$ which is a contradiction with Equation (17). By assumption M has three distinct principal curvatures. Without loss of generality, We denote them by

$$\lambda_1,\ \lambda_2 = \cdots = \lambda_p = \alpha_2, \lambda_{p+1} = \cdots = \lambda_n = \beta_n. \tag{21}$$

Let's $n \geq 4$ and $p = n-1$ (for $n = 3$ or $p \leq n-2$, the proof is in similar way). By Equations (19) and (21) we have

$$e_2(\alpha_2) = \cdots = e_{n-1}(\alpha_2) = 0. \tag{22}$$

In the following we show that $e_n(\alpha_2) = 0$. We have by Equation (19), for $i \neq 1$, $e_i(\lambda_1) = (\lambda_i - \lambda_1)\omega^1_{1i} = 0$. So

$$\omega^1_{1i} = 0, \quad i = 1, \ldots, n. \tag{23}$$

We know by Equation (21),

$$\beta_n = s_1 - \lambda_1 - (n-2)\alpha_2. \tag{24}$$

Thus by Equations (17), (22) and (24) for $i = 2, \ldots, n-1$, $e_i(\beta_n) = 0$, and by Equations (19) and (21), $e_i(\beta_n) = e_i(\lambda_n) = (\lambda_i - \lambda_n)\omega^n_{ni} = 0$. So

$$\omega^n_{ni} = 0, \quad i = 2, \ldots, n. \tag{25}$$

By Equations (19) and (21), $\omega^n_{n1} = \frac{e_1(\lambda_n)}{\lambda_1 - \lambda_n} = \frac{e_1(\beta_n)}{\lambda_1 - \beta_n}$, and so by Equation (24)

$$\omega^n_{n1} = -\frac{e_1\left(\lambda_1 + (n-2)\alpha_2\right)}{2\lambda_1 + (n-2)\alpha_2 - s_1}. \tag{26}$$

By Equation (19), for $j = 2, \ldots, n-1$, we have $\omega^j_{j1} = \frac{e_1(\lambda_j)}{\lambda_1 - \lambda_j}$ and $\omega^j_{jn} = \frac{e_n(\lambda_j)}{\lambda_n - \lambda_j}$. So by Equation (21) we get

$$\omega^j_{j1} = \frac{e_1(\alpha_2)}{\lambda_1 - \alpha_2}, \qquad j = 2, \ldots, n-1, \tag{27}$$

$$\omega^j_{jn} = \frac{e_n(\alpha_2)}{s_1 - \lambda_1 - (n-2)\alpha_2}, \quad j = 2, \ldots, n-1. \tag{28}$$

For $j \neq l$ and $j, l = 2, \ldots, n-1$, we have by Equation (20), $(\lambda_1 - \lambda_j)\omega^j_{l1} = (\lambda_l - \lambda_j)\omega^j_{1l} = 0$ and $(\lambda_n - \lambda_j)\omega^j_{ln} = (\lambda_l - \lambda_j)\omega^j_{nl} = 0$. Thus

$$\omega^j_{l1} = \omega^j_{ln} = 0, \qquad j \neq l \text{ and } j, l = 2, \ldots, n-1. \tag{29}$$

For $i, j = 2, \ldots, n$, by Equation (17) we get, $[e_i, e_j](\lambda_1) = e_i e_j(\lambda_1) - e_j e_i(\lambda_1) = 0$ and so $[e_i, e_j](\lambda_1) = \sum_l (\omega^l_{ij} - \omega^l_{ji}) e_l(\lambda_1) = (\omega^1_{ij} - \omega^1_{ji}) e_1(\lambda_1) = 0$. Therefore

$$\omega^1_{ij} = \omega^1_{ji}, \quad i, j = 2, \ldots, n. \tag{30}$$

For $l = 2, \ldots, n-1$, by Equation (20), $(\lambda_n - \lambda_1)\omega^1_{ln} = (\lambda_l - \lambda_1)\omega^1_{nl}$ and $(\lambda_1 - \lambda_n)\omega^n_{l1} = (\lambda_l - \lambda_n)\omega^n_{1l}$. Therefore by Equations (18) and (30) we get

$$\omega^1_{ln} = \omega^1_{nl} = \omega^n_{1l} = 0, \quad l = 2, \ldots, n-1. \tag{31}$$

By Equations (18), (23) and (31) we get

$$\nabla_{e_1} e_1 = \nabla_{e_1} e_n = 0. \tag{32}$$

We have $\nabla_{e_i} e_1 = \sum_l \omega_{i1}^l e_l = -\sum_l \omega_{il}^1 e_l$, so Equations (26), (27), (29) and (31) imply that

$$\nabla_{e_i} e_1 = -\frac{e_1(\alpha_2)}{\lambda_1 - \alpha_2} e_i, \quad i = 2, \ldots, n-1, \tag{33}$$

$$\nabla_{e_n} e_1 = -\frac{e_1(\lambda_1 + (n-2)\alpha_2)}{2\lambda_1 + (n-2)\alpha_2 - s_1} e_n, \tag{34}$$

and we get by Equations (25) and (26),

$$\nabla_{e_n} e_n = \frac{e_1(\lambda_1 + (n-2)\alpha_2)}{2\lambda_1 + (n-2)\alpha_2 - s_1} e_1. \tag{35}$$

By Equations (28), (29) and (31), we get

$$\nabla_{e_i} e_n = \frac{e_n(\alpha_2)}{s_1 - \lambda_1 - (n-2)\alpha_2} e_i, \quad i = 2, \ldots, n-1. \tag{36}$$

Let's put

$$\alpha = -\frac{e_1(\alpha_2)}{\lambda_1 - \alpha_2}, \ \beta = -\frac{e_1(\lambda_1 + (n-2)\alpha_2)}{2\lambda_1 + (n-2)\alpha_2 - s_1}, \ \gamma = \frac{e_n(\alpha_2)}{s_1 - \lambda_1 - (n-2)\alpha_2}. \tag{37}$$

Now by Equations (27), (28) and (37),

$$\nabla_{e_i} e_i = \alpha e_1 + \sum_{\substack{l=2,\ldots,n-1 \\ l \neq i}} \omega_{ii}^l e_l - \gamma e_n. \tag{38}$$

Then by Equations (8), (12), (21), (23), (25), (31), (32), (33), (34), (35), (36) and (37) we get that

$$R(e_1, e_2)e_1 = \left(-e_1(\alpha) + \alpha^2\right) e_2 = -(c + \lambda_1\alpha_2)e_2.$$

Therefore

$$e_1(\alpha) = c + \lambda_1\alpha_2 + \alpha^2. \tag{39}$$

We have

$$R(e_1, e_n)e_1 = \left(e_1(\beta) + \beta^2\right) e_n = -(c + \lambda_1\beta_n)e_n. \tag{40}$$

Therefore by Equations (24) and (40),

$$e_1(\beta) = -(c + \lambda_1\beta_n + \beta^2) = -(c + \lambda_1(s_1 - \lambda_1 - (n-2)\alpha_2) + \beta^2). \tag{41}$$

We have

$$R(e_3, e_n)e_1 = \left(e_n(\alpha) + \frac{(\alpha+\beta)e_n(\alpha_2)}{s_1 - \lambda_1 - (n-2)\alpha_2}\right) e_3 + e_3(\beta)e_n = 0.$$

So

$$e_n(\alpha) = -\frac{(\alpha+\beta)e_n(\alpha_2)}{s_1 - \lambda_1 - (n-2)\alpha_2}. \tag{42}$$

We have

$$R(e_n, e_2)e_n = \left(e_n(\gamma) - \alpha\beta + \gamma^2\right) e_2 = -(c + \beta_n \alpha_2)e_2. \tag{43}$$

Therefore by Equations (24) and (43),

$$e_n(\gamma) - \alpha\beta + \gamma^2 = -(c + (s_1 - \lambda_1 - (n-2)\alpha_2)\alpha_2). \tag{44}$$

We have by Equations (6) and (5) we have

$$\mu_{k,1} = \binom{n-2}{k}\alpha_2^k + \binom{n-2}{k-1}\beta_n\alpha_2^{k-1}, \tag{45}$$

$$\mu_{k,1} = \sum_{l=0}^{k}(-1)^l s_{k-l}\lambda_1^l, \tag{46}$$

$$\mu_{k,n} = \binom{n-2}{k}\alpha_2^k + \binom{n-2}{k-1}\lambda_1\alpha_2^{k-1}, \tag{47}$$

$$\mu_{k,n} = \sum_{l=0}^{k}(-1)^l s_{k-l}\beta_n^l = \sum_{l=0}^{k}(-1)^l s_{k-l}(s_1 - \lambda_1 - (n-2)\alpha_2)^l. \tag{48}$$

Also by Equation (4), we have

$$s_r = \binom{n-2}{r-1}\lambda_1\alpha_2^{r-1} + \binom{n-2}{r}\alpha_2^r + \binom{n-2}{r-1}\alpha_2^{r-1}\beta_n + \binom{n-2}{r-2}\lambda_1\alpha_2^{r-2}\beta_n. \tag{49}$$

We have by Equation (7),

$$L_k s_{k+1} = \sum_{i=0}^{n}\mu_{k,i}\left(e_ie_i(s_{k+1}) - (\nabla_{e_i}e_i)(s_{k+1})\right). \tag{50}$$

Thus we get by Equations (15), (32), (35), (38) and (50),

$$L_k s_{k+1} = \mu_{k,1}e_1e_1(s_{k+1}) - \Big(\sum_{i=2}^{n-1}\mu_{k,i}(\alpha e_1 + \sum_{\substack{l=2,\dots,n-1\\ l\neq i}}\omega_{ii}^l e_l - \gamma e_n)(s_{k+1})\Big) + \beta\mu_{k,n}e_1(s_{k+1}). \tag{51}$$

Then Equations (15) and (51) imply that

$$L_k s_{k+1} = \mu_{k,1}e_1e_1(s_{k+1}) - \alpha\Big(\sum_{i=2}^{n-1}\mu_{k,i}e_1(s_{k+1})\Big) + \beta\mu_{k,n}e_1(s_{k+1}). \tag{52}$$

We know $\sum_{i=2}^{n-1}\mu_{k,i} = tr(P_k) - \mu_{k,1} - \mu_{k,n}$ and by Equation (9), $\sum_{i=2}^{n-1}\mu_{k,i} = (n-k)s_k - \mu_{k,1} - \mu_{k,n}$. So by Equations (2) and (52) we get that

$$\begin{aligned}\mu_{k,1}\left(e_1e_1(s_{k+1}) + \alpha e_1(s_{k+1})\right) + \left((\alpha+\beta)\mu_{k,n} - \alpha(n-k)s_k\right)e_1(s_{k+1})\\ = s_{k+1}\left(s_1s_{k+1} - (k+2)s_{k+2} - c(n-k)s_k\right).\end{aligned} \tag{53}$$

Now we show that $e_ie_1(s_{k+1}) = 0$ and $e_ie_1(\lambda_1) = 0$ for every $i = 2, \dots, n$. We know by Equation (15) for every $i = 2, \dots, n$, $[e_i, e_1](s_{k+1}) = e_ie_1(s_{k+1}) - e_1e_i(s_{k+1}) = e_ie_1(s_{k+1})$. On the other hand by Equations (15), (23) and (29),

$[e_i, e_1](s_{k+1}) = (\nabla_{e_i} e_1 - \nabla_{e_1} e_i)(s_{k+1}) = \sum_{l=1}^n (\omega_{i1}^l - \omega_{1i}^l) e_l(s_{k+1}) = 0$. Therefore we get

$$e_i e_1(s_{k+1}) = 0, \quad i = 2, \ldots, n, \tag{54}$$

and in the similar way

$$e_i e_1(\lambda_1) = 0, \quad i = 2, \ldots, n. \tag{55}$$

By Equation (37), we have $(\lambda_1 - \alpha_2)\alpha = -e_1(\alpha_2)$ and $(2\lambda_1 + (n-2)\alpha_2 - s_1)\beta = -e_1(\lambda_1 + (n-2)\alpha_2)$. Differentiating these equations in direction of e_n and by Equations (17) and (55), and constancy of s_1 we get

$$-e_n(\alpha_2)\alpha + (\lambda_1 - \alpha_2)e_n(\alpha) = -e_n e_1(\alpha_2), \tag{56}$$

$$\beta(n-2)e_n(\alpha_2) + (2\lambda_1 + (n-2)\alpha_2 - s_1)e_n(\beta) = -(n-2)e_n e_1(\alpha_2). \tag{57}$$

So by eliminating $e_n e_1(\alpha_2)$ from Equations (56) and (57) we get

$$\begin{aligned} &(n-2)\big(-e_n(\alpha_2)\alpha + (\lambda_1 - \alpha_2)e_n(\alpha)\big) \\ &= (n-2)\beta e_n(\alpha_2) + (2\lambda_1 + (n-2)\alpha_2 - s_1)e_n(\beta). \end{aligned} \tag{58}$$

By substituting $e_n(\alpha)$ of Equation (42) in Equation (58) we get

$$e_n(\beta) = \frac{(n-2)(\alpha+\beta)(n\alpha_2 - s_1)e_n(\alpha_2)}{(2\lambda_1 + (n-2)\alpha_2 - s_1)(s_1 - \lambda_1 - (n-1)\alpha_2)}. \tag{59}$$

By Equation (24),

$$e_n(\beta_n) = -(n-2)e_n(\alpha_2). \tag{60}$$

Thus by Equations (45) and (60), we have

$$e_n(\mu_{k,1}) = (k-1)(s_1 - \lambda_1 - (n-1)\alpha_2)\binom{n-2}{k-1}\alpha_2^{k-2}e_n(\alpha_2). \tag{61}$$

Differentiating of Equation (53) in direction of e_n and use of Equation (54) we get

$$\begin{aligned} &e_n(\mu_{k,1})\left(e_1 e_1(s_{k+1}) + \alpha e_1(s_{k+1})\right) + \mu_{k,1} e_n(\alpha) e_1(s_{k+1}) \\ &+ e_1(s_{k+1})\big(e_n(\mu_{k,n})(\alpha+\beta) + \mu_{k,n}(e_n(\beta) + e_n(\alpha))\big) \\ = &-(k+2)s_{k+1}e_n(s_{k+2}). \end{aligned} \tag{62}$$

Differentiating of Equation (47) in direction of e_n we get

$$e_n(\mu_{k,n}) = ((n-k-1)\alpha_2 + (k-1)\lambda_1)\binom{n-2}{k-1}\alpha_2^{k-2}e_n(\alpha_2). \tag{63}$$

By Equations (17) and (46) we get

$$e_i(\mu_{k,1}) = 0, \quad i = 2, \ldots, n. \tag{64}$$

Now for showing that $e_n(\alpha_2) = 0$ we consider two cases:

Case 1: If $k = 1$, then by Equations (62) and (64) we have

$$e_1(s_2)\big(e_n(\mu_{1,n})(\beta+\alpha) + \mu_{1,n}(e_n(\beta) + e_n(\alpha)) + \mu_{1,1}e_n(\alpha)\big) = -3s_2 e_n(s_3). \tag{65}$$

By Equation (63) we have

$$e_n(\mu_{1,n}) = (n-2)e_n(\alpha_2). \tag{66}$$

By Equation (6) we have

$$\mu_{1,1} = s_1 - \lambda_1, \tag{67}$$

$$\mu_{1,n} = \lambda_1 + (n-2)\alpha_2. \tag{68}$$

Now by Equations (42), (59), (65), (66), (67) and (68) we get

$$\begin{aligned} & e_1(s_2)e_n(\alpha_2)\Bigg[(\beta+\alpha)\Big[(n-2)+(\lambda_1+(n-2)\alpha_2) \\ & \times \Big[\frac{(n-2)(n\alpha_2-s_1)}{(2\lambda_1+(n-2)\alpha_2-s_1)(s_1-\lambda_1-(n-1)\alpha_2)} - \frac{1}{s_1-\lambda_1-(n-1)\alpha_2}\Big]\Big] \\ & + \frac{s_1-\lambda_1}{s_1-\lambda_1-(n-1)\alpha_2}\Bigg] = -3s_2 e_n(s_3). \end{aligned} \tag{69}$$

We have by Equation (49),

$$s_3 = \binom{n-2}{2}\lambda_1\alpha_2^2 + \binom{n-2}{3}\alpha_2^3 + \binom{n-2}{2}\alpha_2^2\beta_n + (n-2)\lambda_1\alpha_2\beta_n. \tag{70}$$

Differentiating of Equation (70) in direction of e_n and using Equation (60) we get

$$\begin{aligned} e_n(s_3) = e_n(\alpha_2)\Big[& 2\binom{n-2}{2}\lambda_1\alpha_2 + 3\binom{n-2}{3}\alpha_2^2 \\ & + 2\binom{n-2}{2}(s_1-\lambda_1-(n-2)\alpha_2)\alpha_2 - (n-2)\binom{n-2}{2}\alpha_2^2 \\ & + (n-2)(s_1-\lambda_1-(n-2)\alpha_2)\lambda_1 - (n-2)^2\lambda_1\alpha_2\Big]. \end{aligned} \tag{71}$$

Let $e_n(\alpha_2) \neq 0$. So using Equation (71) and dividing Equation (69) by $e_n(\alpha_2)$, we have

(72)

$$\begin{aligned} & e_1(s_2)(\beta+\alpha)\left[(n-2) + (\lambda_1+(n-2)\alpha_2)\left[\tfrac{(n-2)(n\alpha_2-s_1)}{(2\lambda_1+(n-2)\alpha_2-s_1)(s_1-\lambda_1-(n-1)\alpha_2)}\right.\right. \\ & \left.\left. -\tfrac{1}{s_1-\lambda_1-(n-1)\alpha_2}\right] + \tfrac{s_1-\lambda_1}{s_1-\lambda_1-(n-1)\alpha_2}\right] \\ = & -3s_2\left[2\binom{n-2}{2}\lambda_1\alpha_2+3\binom{n-2}{3}\alpha_2^2+2\binom{n-2}{2}(s_1-\lambda_1-(n-2)\alpha_2)\alpha_2\right. \\ & \left.-(n-2)\binom{n-2}{2}\alpha_2^2+(n-2)(s_1-\lambda_1-(n-2)\alpha_2)\lambda_1-(n-2)^2\lambda_1\alpha_2\right]. \end{aligned}$$

Then differentiating of Equation (72) in direction of e_n and using Equations (42) and (59) we get

(73)

$$\begin{aligned}
&e_1(s_2)\Big[\tfrac{(\beta+\alpha)e_n(\alpha_2)}{s_1-\lambda_1-(n-1)\alpha_2}(-1+\tfrac{(n-2)(n\alpha_2-s_1)}{2\lambda_1+(n-2)\alpha_2-s_1})\,[n-2+(\lambda_1+(n-2)\alpha_2)\\
&\times\Big[\tfrac{(n-2)(n\alpha_2-s_1)}{(2\lambda_1+(n-2)\alpha_2-s_1)(s_1-\lambda_1-(n-1)\alpha_2)}-\tfrac{1}{s_1-\lambda_1-(n-1)\alpha_2}\Big]+\tfrac{s_1-\lambda_1}{s_1-\lambda_1-(n-1)\alpha_2}\Big]\\
&+(\beta+\alpha)\Big[(n-2)e_n(\alpha_2)\Big[\tfrac{(n-2)(n\alpha_2-s_1)}{(2\lambda_1+(n-2)\alpha_2-s_1)(s_1-\lambda_1-(n-1)\alpha_2)}-\tfrac{1}{s_1-\lambda_1-(n-1)\alpha_2}\Big]\\
&+(\lambda_1+(n-2)\alpha_2)\,[[n(n-2)e_n(\alpha_2)(2\lambda_1+(n-2)\alpha_2-s_1)(s_1-\lambda_1-(n-1)\alpha_2)\\
&-(n-2)(n\alpha_2-s_1)\,[(n-2)e_n(\alpha_2)(s_1-\lambda_1-(n-1)\alpha_2)\\
&-(n-1)(2\lambda_1+(n-2)\alpha_2-s_1)e_n(\alpha_2)]]\\
&\times\tfrac{1}{(2\lambda_1+(n-2)\alpha_2-s_1)^2(s_1-\lambda_1-(n-1)\alpha_2)^2}-\tfrac{(n-1)e_n(\alpha_2)}{(s_1-\lambda_1-(n-1)\alpha_2)^2}\Big]\\
&+\tfrac{(n-1)(s_1-\lambda_1)e_n(\alpha_2)}{(s_1-\lambda_1-(n-1)\alpha_2)^2}\Big]\Big]\\
=&\ -3s_2e_n(\alpha_2)\left[2\binom{n-2}{2}s_1-2(n-2)^2\lambda_1+\alpha_2\left[6\binom{n-2}{3}-6(n-2)\binom{n-2}{2}\right]\right].
\end{aligned}$$

Let's divide Equation (73) by $e_n(\alpha_2)$ and then substitute $e_1(s_2)$ of Equation (72). So coefficients $\beta+\alpha$ and s_2 are eliminated. Thus we get that α_2 should satisfy of a polynomial of degree 7 which its coefficients of functions of λ_1. So α_2 is a function of λ_1. Then by Equation (17), we get $e_n(\alpha_2)=0$ which is contradiction.

Case 2: If $k=2$, then by Equations (61), (64) and $\beta_n-\alpha_2=s_1-\lambda_1-(n-1)\alpha_2\neq 0$ we have $e_n(\alpha_2)=0$.

Therefore by Case 1 and Case 2, we have

$$e_2(\alpha_2)=\cdots=e_n(\alpha_2)=0. \tag{74}$$

Now by Equations (37) and (44),

$$\alpha\beta=c+(s_1-\lambda_1-(n-2)\alpha_2)\alpha_2. \tag{75}$$

By Equation (37) we have

$$e_1(\lambda_1)=-\beta(2\lambda_1+(n-2)\alpha_2-s_1)-(n-2)\alpha(\alpha_2-\lambda_1). \tag{76}$$

Differentiating Equation (76) in direction e_1 and by use of Equations (16), (37), (39), (41) and (75) we get

$$\begin{aligned}
e_1e_1(\lambda_1)=&\ (c+\lambda_1(s_1-\lambda_1-(n-2)\alpha_2))(2\lambda_1+(n-2)\alpha_2-s_1)\\
&+\left(-\beta+\frac{(n-2)\alpha}{2}\right)\frac{e_1(s_{k+1})}{\sum_{l=1}^{k+1}(-1)^l s_{k+1-l}l{\lambda_1}^{l-1}}\\
&-(n-2)(c+(s_1-\lambda_1-(n-2)\alpha_2)\alpha_2)(\alpha_2-\lambda_1)\\
&-(n-2)(c+\lambda_1\alpha_2)(\alpha_2-\lambda_1)
\end{aligned} \tag{77}$$

$$+\beta^2(2\lambda_1+(n-2)\alpha_2-s_1)-2(n-2)\alpha^2(\alpha_2-\lambda_1).$$

We rewrite the last term of Equation (77). We have by Equations (16), (37), (75) and (76),

$$\begin{aligned}
(78)\qquad & \beta^2(2\lambda_1+(n-2)\alpha_2-s_1)-2(n-2)\alpha^2(\alpha_2-\lambda_1)\\
&= -\beta(e_1(\lambda_1)+(n-2)e_1(\alpha_2)-2(n-2)\alpha e_1(\alpha_2)\\
&= (2\alpha-\beta)\frac{e_1(s_{k+1})}{2\sum_{l=1}^{k+1}(-1)^l s_{k+1-l}l{\lambda_1}^{l-1}}\\
&\quad -(n-2)(c+\alpha_2(s_1-\lambda_1-(n-2)\alpha_2))(\alpha_2-\lambda_1)\\
&\quad +2(c+\alpha_2(s_1-\lambda_1-(n-2)\alpha_2))(2\lambda_1+(n-2)\alpha_2-s_1).
\end{aligned}$$

So substituting Equation (78) in equation (77), we get

$$\begin{aligned}
(79)\qquad e_1e_1(\lambda_1) &= (n\alpha-3\beta)\frac{e_1(s_{k+1})}{2\sum_{l=1}^{k+1}(-1)^l s_{k+1-l}l{\lambda_1}^{l-1}}\\
&\quad +(c+\lambda_1(s_1-\lambda_1-(n-2)\alpha_2))(n\lambda_1-s_1)\\
&\quad -(n-2)(c+\lambda_1\alpha_2)(\alpha_2-\lambda_1).
\end{aligned}$$

Differentiating of Equation (16) in direction e_1 and use of Equations (76) and (79) we obtain

$$\begin{aligned}
(80)\quad e_1e_1(s_{k+1}) &= 2e_1e_1(\lambda_1)\sum_{l=1}^{k+1}(-1)^l s_{k+1-l}l{\lambda_1}^{l-1}\\
&\quad +2(e_1(\lambda_1))^2\sum_{l=2}^{k+1}(-1)^l s_{k+1-l}l(l-1){\lambda_1}^{l-2}\\
&= 2e_1e_1(\lambda_1)\sum_{l=1}^{k+1}(-1)^l s_{k+1-l}l{\lambda_1}^{l-1}\\
&\quad +(-\beta(2\lambda_1+(n-2)\alpha_2-s_1)-(n-2)\alpha(\alpha_2-\lambda_1))\\
&\quad \times(\frac{e_1(s_{k+1})}{\sum_{l=1}^{k+1}(-1)^l s_{k+1-l}l{\lambda_1}^{l-1}})\sum_{l=2}^{k+1}(-1)^l s_{k+1-l}l(l-1){\lambda_1}^{l-2}\\
&= \frac{e_1(s_{k+1})}{\sum_{l=1}^{k+1}(-1)^l s_{k+1-l}l{\lambda_1}^{l-1}}((n\alpha-3\beta)\sum_{l=1}^{k+1}(-1)^l s_{k+1-l}l{\lambda_1}^{l-1}\\
&\quad +(-\beta(2\lambda_1+(n-2)\alpha_2-s_1)-(n-2)\alpha(\alpha_2-\lambda_1))\\
&\quad \times\sum_{l=2}^{k+1}(-1)^l s_{k+1-l}l(l-1){\lambda_1}^{l-2})\\
&\quad +2((c+\lambda_1(s_1-\lambda_1-(n-2)\alpha_2))(n\lambda_1-s_1)
\end{aligned}$$

$$- (n-2)(c+\lambda_1\alpha_2)(\alpha_2-\lambda_1)) \sum_{l=1}^{k+1}(-1)^l s_{k+1-l} l {\lambda_1}^{l-1}.$$

Let $F_i = F_i(\lambda_1^{\max_1}\alpha_2^{\min_2}, \lambda_1^{\min_1}\alpha_2^{\max_2})$'s be polynomials in term of λ_1 and α_2 of degree $\max_1 + \min_2 = \min_1 + \max_2$ where $\max_j$ and $\min_j$ show the maximum and minimum power of its base. So by use of this notation and by Equation (80) we get

$$\begin{aligned}(81)\qquad & e_1e_1(s_{k+1}) \\ &= \frac{e_1(s_{k+1})}{F_1(\lambda_1^k)}\left[(n\alpha-3\beta)F_1(\lambda_1^k)\right. \\ &\quad \left.+(-\beta F_2(\lambda_1,\alpha_2)-(n-2)\alpha F_3(\lambda_1,\alpha_2))F_4(\lambda_1^{k-1})\right] \\ &\quad + \left[2F_5(\lambda_1^3,\lambda_1^2\alpha_2)-2(n-2)F_6(\lambda_1^2\alpha_2,\lambda_1\alpha_2^2)\right] \\ &= \frac{e_1(s_{k+1})}{F_1(\lambda_1^k)}\left[\alpha F_7(\lambda_1^k,\lambda_1^{k-1}\alpha_2)+\beta F_8(\lambda_1^k,\lambda_1^{k-1}\alpha_2)\right]+F_9(\lambda_1^3,\lambda_1\alpha_2^2).\end{aligned}$$

Now by Equations (46), (48), (49), (53) and (81) we have

$$\begin{aligned} & F_{10}(\lambda_1^k)\left[\frac{e_1(s_{k+1})}{F_1(\lambda_1^k)}\left[\alpha F_7(\lambda_1^k,\lambda_1^{k-1}\alpha_2)+\beta F_8(\lambda_1^k,\lambda_1^{k-1}\alpha_2)\right]\right. \\ &\quad \left.+F_9(\lambda_1^3,\lambda_1\alpha_2^2)+\alpha e_1(s_{k+1})\right] \\ &\quad + e_1(s_{k+1})\left[F_{11}(\lambda_1^k,\alpha_2^k)(\alpha+\beta)-\alpha(n-k)s_k\right] \\ &= F_{12}(\lambda_1^2\alpha_2^{k-1},\alpha_2^{k+1})\left[F_{13}(\lambda_1^2\alpha_2^{k-1},\alpha_2^{k+1})+F_{14}(\lambda_1^2\alpha_2^k,\alpha_2^{k+2})\right].\end{aligned}$$

Therefore

$$\begin{aligned}(82)\qquad & e_1(s_{k+1})\left[\alpha F_{15}(\lambda_1^{2k},\lambda_1^k\alpha_2^k)+\beta F_{16}(\lambda_1^{2k},\lambda_1^k\alpha_2^k)\right] \\ &= F_{17}(\lambda_1^{k+4}\alpha_2^{2k-1},\lambda_1^k\alpha_2^{2k+3})+F_{18}(\lambda_1^{2k+3},\lambda_1^{2k+1}\alpha_2^2).\end{aligned}$$

Differentiating of Equation (82) in direction e_1,

$$\begin{aligned}(83)\qquad & e_1e_1(s_{k+1})\left[\alpha F_{15}+\beta F_{16}\right] \\ &\quad + e_1(s_{k+1})\left[e_1(\alpha)F_{15}+e_1(\beta)F_{16}+\alpha\left[e_1(\lambda_1)\frac{\partial F_{15}}{\partial\lambda_1}+e_1(\alpha_2)\frac{\partial F_{15}}{\partial\alpha_2}\right]\right. \\ &\quad \left.+\beta\left[e_1(\lambda_1)\frac{\partial F_{16}}{\partial\lambda_1}+e_1(\alpha_2)\frac{\partial F_{16}}{\partial\alpha_2}\right]\right] \\ &= e_1(\lambda_1)\frac{\partial F_{17}}{\partial\lambda_1}+e_1(\alpha_2)\frac{\partial F_{17}}{\partial\alpha_2}+e_1(\lambda_1)\frac{\partial F_{18}}{\partial\lambda_1}+e_1(\alpha_2)\frac{\partial F_{18}}{\partial\alpha_2},\end{aligned}$$

and by use of Equations (37), (39), (41), (75), (76) and (83) we get

$$\begin{aligned}(84)\qquad & e_1e_1(s_{k+1})\left[\alpha F_{15}(\lambda_1^{2k},\lambda_1^k\alpha_2^k)+\beta F_{16}(\lambda_1^{2k},\lambda_1^k\alpha_2^k)\right] \\ &\quad + e_1(s_{k+1})\left[\alpha^2F_{19}(\lambda_1^{2k},\lambda_1^{k-1}\alpha_2^{k+1})+\beta^2F_{20}(\lambda_1^{2k},\lambda_1^{k-1}\alpha_2^{k+1})\right. \\ &\quad \left.+F_{21}(\lambda_1^{2k+2},\lambda_1^{k-1}\alpha_2^{k+3})\right]\end{aligned}$$

$$= \alpha \left[F_{22}(\lambda_1^{k+5}\alpha_2^{2k-2}, \lambda_1^{k-1}\alpha_2^{2k+4}) + F_{23}(\lambda_1^{2k+3}, \lambda_1^{2k}\alpha_2^3)\right] + \beta \left[F_{24}(\lambda_1^{k+4}\alpha_2^{2k-1}, \lambda_1^{k-1}\alpha_2^{2k+4}) + F_{25}(\lambda_1^{2k+3}, \lambda_1^{2k}\alpha_2^3)\right].$$

By substituting Equation (81) in Equation (84) and multiplying in $F_1(\lambda_1^k)$ we get

(85)
$$e_1(s_{k+1}) \left[\alpha^2 F_{26}(\lambda_1^{3k}, \lambda_1^{2k-1}\alpha_2^{k+1}) + \beta^2 F_{27}(\lambda_1^{3k}, \lambda_1^{2k-1}\alpha_2^{k+1}) + F_{28}(\lambda_1^{3k+2}, \lambda_1^{2k-1}\alpha_2^{k+3})\right] = \alpha \left[F_{29}(\lambda_1^{2k+5}\alpha_2^{2k-2}, \lambda_1^{2k-1}\alpha_2^{2k+4}) + F_{30}(\lambda_1^{3k+3}, \lambda_1^{2k+1}\alpha_2^{k+2})\right] + \beta \left[F_{31}(\lambda_1^{2k+4}\alpha_2^{2k-1}, \lambda_1^{2k-1}\alpha_2^{2k+4}) + F_{32}(\lambda_1^{3k+3}, \lambda_1^{2k+1}\alpha_2^{k+2})\right].$$

Now we compute two terms

$$e_1(s_{k+1})\alpha^2 F_{26}(\lambda_1^{3k}, \lambda_1^{2k-1}\alpha_2^{k+1}) \text{ and } e_1(s_{k+1})\beta^2 F_{27}(\lambda_1^{3k}, \lambda_1^{2k-1}\alpha_2^{k+1})$$

of Equation (85). By Equation (82) we get

(86)
$$e_1(s_{k+1})\alpha^2 F_{26}(\lambda_1^{3k}, \lambda_1^{2k-1}\alpha_2^{k+1}) = \frac{1}{F_{15}(\lambda_1^{2k}, \lambda_1^k\alpha_2^k)} \left[\alpha \left[F_{33}(\lambda_1^{4k+4}\alpha_2^{2k-1}, \lambda_1^{3k-1}\alpha_2^{3k+4}) + F_{34}(\lambda_1^{5k+3}, \lambda_1^{4k}\alpha_2^{k+3})\right] + e_1(s_{k+1}) F_{35}(\lambda_1^{5k+1}\alpha_2, \lambda_1^{3k-1}\alpha_2^{2k+3})\right],$$

(87)
$$e_1(s_{k+1})\beta^2 F_{27}(\lambda_1^{3k}, \lambda_1^{2k-1}\alpha_2^{k+1}) = \frac{1}{F_{16}(\lambda_1^{2k}, \lambda_1^k\alpha_2^k)} \left[\beta \left[F_{36}(\lambda_1^{4k+4}\alpha_2^{2k-1}, \lambda_1^{3k-1}\alpha_2^{3k+4}) + F_{37}(\lambda_1^{5k+3}, \lambda_1^{4k}\alpha_2^{k+3})\right] + e_1(s_{k+1}) F_{38}(\lambda_1^{5k+1}\alpha_2, \lambda_1^{3k-1}\alpha_2^{2k+3})\right].$$

Substituting Equations (86) and (87) in Equation (85) we get

(88)
$$e_1(s_{k+1}) F_{39}(\lambda_1^{7k+2}, \lambda_1^{4k-1}\alpha_2^{3k+3}) = \alpha \left[F_{40}(\lambda_1^{6k+5}\alpha_2^{2k-2}, \lambda_1^{4k-1}\alpha_2^{4k+4}) + F_{41}(\lambda_1^{7k+3}, \lambda_1^{4k+1}\alpha_2^{3k+2})\right] + \beta \left[F_{42}(\lambda_1^{6k+4}\alpha_2^{2k-1}, \lambda_1^{4k-1}\alpha_2^{4k+4}) + F_{43}(\lambda_1^{7k+3}, \lambda_1^{4k+1}\alpha_2^{3k+2})\right].$$

By Equations (16) and (76) we have

(89)
$$e_1(s_{k+1}) = \alpha F_{44}(\lambda_1^{k+1}, \lambda_1^k\alpha_2) + \beta F_{45}(\lambda_1^{k+1}, \lambda_1^k\alpha_2).$$

So by Equations (88) and (89) we have

(90)
$$\alpha F_{46}(\lambda_1^{8k+3}, \lambda_1^{4k-1}\alpha_2^{4k+4}) + \beta F_{47}(\lambda_1^{8k+3}, \lambda_1^{4k-1}\alpha_2^{4k+4}) = 0.$$

Now by Equations (75), (82) and (89),

(91)
$$\alpha^2 F_{48}(\lambda_1^{3k+1}, \lambda_1^{2k}\alpha_2^{k+1}) + \beta^2 F_{49}(\lambda_1^{3k+1}, \lambda_1^{2k}\alpha_2^{k+1}) = F_{50}(\lambda_1^{3k+2}\alpha_2, \lambda_1^k\alpha_2^{2k+3}).$$

By multiplying Equation (90) in α and β and by use of Equation (75) we get

$$\alpha^2 = \frac{F_{51}(\lambda_1^{8k+4}\alpha_2, \lambda_1^{4k-1}\alpha_2^{4k+6})}{F_{46}(\lambda_1^{8k+3}, \lambda_1^{4k-1}\alpha_2^{4k+4})}, \tag{92}$$

$$\beta^2 = \frac{F_{52}(\lambda_1^{8k+4}\alpha_2, \lambda_1^{4k-1}\alpha_2^{4k+6})}{F_{47}(\lambda_1^{8k+3}, \lambda_1^{4k-1}\alpha_2^{4k+4})}. \tag{93}$$

Now by substituting Equations (92) and (93) in Equation (91) we get

$$F_{53}(\lambda_1^{19k+8}\alpha_2, \lambda_1^{9k-2}\alpha_2^{10k+11}) = 0. \tag{94}$$

In the following we show that $e_1(\alpha_2) \neq 0$. Let $e_1(\alpha_2) = 0$ then by Equation (74), α_2 is constant and by Equation (37), $\alpha = 0$. Therefore by Equation (39), $c + \lambda_1\alpha_2 = 0$ and by differentiating we get $e_1(\lambda_1)\alpha_2 = 0$ and so by Equation (17), $\alpha_2 = 0$. If $k = 2$, then by hypothesis s_1 and s_2 is constant. Since $\alpha_2 = 0$, by Equations (4) and (21), $s_1 = \lambda_1 + \beta_n$, $s_2 = \lambda_1\beta_n$. Then differentiating in direction of e_1 we get $e_1(\lambda_1) + e_1(\beta_n) = 0$ and $e_1(\lambda_1)\beta_n + \lambda_1 e_1(\beta_n) = 0$, so $e_1(\lambda_1)(\beta_n - \lambda_1) = 0$. Therefore $\beta_n = \lambda_1$ which is a contradiction. If $k = 1$, we have s_1 is constant. Since $\alpha = \alpha_2 = 0$, by Equation (90), $\beta F_{47}(\lambda_1^{11}) = 0$. If $\beta \neq 0$, then $F_{47}(\lambda_1^{11}) = 0$, so λ_1 is constant which contradicts Equation (17). Thus $\beta = 0$ and by Equation (76), $e_1(\lambda_1) = 0$ which contradicts Equation (17). Finally by Equation (74) we have

$$e_1(\alpha_2) \neq 0 \quad \text{and} \quad e_2(\alpha_2) = \cdots = e_n(\alpha_2) = 0. \tag{95}$$

Now assume that $\gamma(t)$ be integral curve of e_1 that $\gamma(t_0) = p$ which $p \in M$ and $t_0 \in I$. By Equations (17) and (95), we have in some neighborhood of t_0, $\lambda_1 = \lambda_1(t)$ and $\alpha_2 = \alpha_2(t)$, and so $t = t(\alpha_2)$ and $\lambda_1 = \lambda_1(\alpha_2)$. Therefore by Equations (37), (76) and (90) we have

$$\frac{\mathrm{d}\lambda_1}{\mathrm{d}\alpha_2} = \frac{\mathrm{d}\lambda_1}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\alpha_2} = \frac{e_1(\lambda_1)}{e_1(\alpha_2)} = \frac{F_{54}(\lambda_1^{8k+4}, \lambda_1^{4k-1}\alpha_2^{4k+5})}{F_{55}(\lambda_1^{8k+4}, \lambda_1^{4k-1}\alpha_2^{4k+5})}. \tag{96}$$

Now differentiating of Equation (94) relative to α_2 and using Equation (96) we get

$$F_{56}(\lambda_1^{27k+12}, \lambda_1^{13k-4}\alpha_2^{14k+16}) = 0. \tag{97}$$

Now rewriting polynomials (94) and (97) in term of α_2 we get

$$\sum_{i=0}^{10k+11} f_i(\lambda_1)\alpha_2^i = 0, \tag{98}$$

$$\sum_{i=0}^{14k+16} g_i(\lambda_1)\alpha_2^i = 0, \tag{99}$$

where $f_i(\lambda_1)$ and $g_i(\lambda_1)$ are polynomials in term of λ_1. By multiplying equation (98) in $g_{14k+16}(\lambda_1)\alpha_2^{4k+5}$ and Equation (99) in $f_{10k+11}(\lambda_1)$ and subtracting them we get a polynomial in term of α_2 of degree $14k + 15$. Then by this new polynomial and Equation (98), similarly we get a polynomial of degree $14k+14$.

By continuing this method, finally we omit α_2 and we earn a polynomial in term of λ_1 with constant coefficients. So λ_1 should be constant which is a contradiction. Therefore s_{k+1} is constant. □

Proof of Theorem 1.2. Let k_1, k_2, k_3 be principal curvatures of M, respectively with multiplicities m_1, m_2, m_3, $n = m_1 + m_2 + m_3$. Suppose that $\{e_i\}_{i=1}^n$ is a local orthonormal frame field on M which are the eigenvectors of the shape operator S of M with respect to the globally chosen unit normal vector field N and $Se_i = k_1e_i \ i \leq m_1$, $Se_i = k_2e_i \ m_1 < i \leq m_1 + m_2$, $Se_i = k_3e_i \ m_1 + m_2 < i \leq n$.

Case 1. Let $k = 1$. By hypothesis s_1 is constant and by Lemma 3.1, s_2 is constant. Let $s_2 = 0$. If multiplicities of principal curvatures are greater than one, equations $Se_i = k_1e_i \ i \leq m_1$, $Se_i = k_2e_i \ m_1 < i \leq m_1 + m_2$ and $Se_i = k_3e_i \ m_1 + m_2 < i \leq n$ together with the Codazzi equation, $(\nabla_{e_i}S)e_j = (\nabla_{e_j}S)e_i$, imply that

$$\nabla_{e_i}k_1 = 0, \qquad i \leq m_1, \tag{100}$$

$$\nabla_{e_i}k_2 = 0, \qquad m_1 < i \leq m_1 + m_2, \tag{101}$$

$$\nabla_{e_i}k_3 = 0, \qquad m_1 + m_2 < i \leq n. \tag{102}$$

Since s_1 is constant and $s_2 = 0$, by Equation (4) we get that $k_2 = g_1(k_1)$ and $k_3 = g_2(k_1)$ where g_1 and g_2 are some smooth functions. So for every i we have

$$\nabla_{e_i}k_2 = g_1'(k_1)\nabla_{e_i}k_1. \tag{103}$$

We have

$$s_1 = m_1k_1 + m_2k_2 + m_3k_3. \tag{104}$$

Thus we have by Equations (102) and (103),

$$(m_1 + m_2g_1'(k_1))\nabla_{e_i}k_1 = 0 \quad m_1 + m_2 < i \leq n. \tag{105}$$

If for some i, $m_1 + m_2 < i \leq n$, $\nabla_{e_i}k_1 \neq 0$, then by Equation (105), $g_1'(k_1) = -\frac{m_1}{m_2}$. So $k_2 = g_1(k_1) = -\frac{m_1}{m_2}k_1 + C$ where C is a constant. Therefore by Equation (104), k_3 is constant. Now by equation $s_2 = 0$, we get that k_1 should satisfy a polynomial. Therefore k_1 is constant which is a contradiction. Thus for every i, $m_1 + m_2 < i \leq n$, $\nabla_{e_i}k_1 = 0$, and together with Equations (100), (101) and (103), we get k_2 is constant. In a similar way we get k_3 and so k_1 is constant. Therefore M is an isoparametric hypersurface. If $s_2 \neq 0$, By Equation (2), we have

$$s_1s_2 - 3s_3 - c(n-1)s_1 = 0. \tag{106}$$

Since s_1 and s_2 are constant, Equation (106) implies that s_3 is constant. Because M has three principal curvatures, we get that all principal curvatures are constant. So M is an isoparametric hypersurface.

Case 2. Let $k = 2$. By hypothesis s_1 and s_2 is constant and by Lemma 3.1, s_3 is constant. Because M has three principal curvatures, we get that all principal curvatures are constant. So M is an isoparametric hypersurface. We know for $c = 0, -1$, isoparametric hypersurfaces has at most two principal curvatures, So by Case 1, we get $s_2 = 0$ and at least one of the multiplicities of principal curvatures is one, and by Case 2, there is not L_2-biharmonic hypersurface with three disjoint principal curvatures, and s_1 and s_2 is constant. In the rest, we assume that $c = 1$. By Theorem 2.1, an isoparametric hypersurface with three constant principal curvature $k_1 > k_2 > k_3$ in $\mathbb{S}^{n+1}$ have the multiplicities: $m = 1, 2, 4$ and 8. Therefore we have the following equations:

If $m = 1$, then by Equation (4), we have

$$s_1 = k_1 + k_2 + k_3, \quad s_2 = k_1k_2 + k_1k_3 + k_2k_3, \quad s_3 = k_1k_2k_3. \tag{107}$$

If $m = 2$, then by Equation (4), we have

$$s_1 = 2(k_1 + k_2 + k_3), \tag{108}$$

$$s_2 = 4k_1k_2 + 4k_1k_3 + k_1^2 + k_2^2 + 4k_2k_3 + k_3^2, \tag{109}$$

$$s_3 = 8k_1k_2k_3 + 2k_1^2k_2 + 2k_1k_3^2 + 2k_1^2k_3 + 2k_2k_3^2 + 2k_2^2k_3 + 2k_1k_2^2, \tag{110}$$

$$s_4 = k_2^2k_3^2 + 4k_1k_2k_3^2 + 4k_1k_2^2k_3 + k_1^2k_3^2 + 4k_1^2k_2k_3 + k_1^2k_2^2. \tag{111}$$

If $m = 4$, then by Equation (4), we have

$$s_1 = 4(k_1 + k_2 + k_3), \tag{112}$$

$$s_2 = 6k_3^2 + 16k_2k_3 + 6k_2^2 + 16k_1k_3 + 16k_1k_2 + 6k_1^2, \tag{113}$$

$$\begin{aligned} s_3 = {} & 4k_3^3 + 24k_2k_3^2 + 24k_2^2k_3 + 4k_2^3 + 24k_1k_3^2 + 64k_1k_2k_3 \\ & + 24k_1k_2^2 + 24k_1^2k_3 + 24k_1^2k_2 + 4k_1^3, \end{aligned} \tag{114}$$

$$\begin{aligned} s_4 = {} & k_3^4 + 16k_2k_3^3 + 36k_2^2k_3^2 + 16k_2^3k_3 + k_2^4 + 16k_1k_3^3 \\ & + 96k_1k_2k_3^2 + 96k_1k_2^2k_3 + 16k_1k_2^3 + 36k_1^2k_3^2 \\ & + 96k_1^2k_2k_3 + 36k_1^2k_2^2 + 16k_1^3k_3 + 16k_1^3k_2 + k_1^4. \end{aligned} \tag{115}$$

If $m = 8$, then by Equation (4), we have

$$s_1 = 8(k_1 + k_2 + k_3), \tag{116}$$

$$s_2 = 28k_3^2 + 64k_2k_3 + 28k_2^2 + 64k_1k_3 + 64k_1k_2 + 28k_1^2, \tag{117}$$

$$\begin{aligned} s_3 = {} & 56k_3^3 + 224k_2k_3^2 + 224k_2^2k_3 + 56k_2^3 + 224k_1k_3^2 + 512k_1k_2k_3 \\ & + 224k_1k_2^2 + 56k_1^3 + 224k_1^2k_3 + 224k_1^2k_2, \end{aligned} \tag{118}$$

$$\begin{aligned} s_4 = {} & 170k_1^4 + 448k_1^3k_2 + 448k_1^3k_3 + 784k_1^2k_2^2 + 1792k_1^2k_2k_3 \\ & + 784k_1^2k_3^2 + 1792k_1k_2^2k_3 + 448k_1k_2^3 + 1792k_1k_2k_3^2 + 448k_1k_3^3 \\ & + 170k_2^4 + 448k_2^3k_3 + 784k_2^2k_3^2 + 448k_2k_3^3 + 170k_3^4. \end{aligned} \tag{119}$$

Let $k = 1$. If $s_2 = 0$ and multiplicities of principal curvatures are greater than one, and or $s_2 \neq 0$, by Case1, M is an isoparametric hypersurface with three constant principal curvature $k_1 > k_2 > k_3$.

If $s_2 = 0$ and multiplicities of principal curvatures are greater than one, we have the following:

If $m = 2$, by Equations (10) and (109), we get $k_1 \approx 3.286, k_2 \approx 0.232, k_3 \approx -1.069$ or $k_1 \approx 1.069, k_2 \approx -0.232, k_3 \approx -3.286$.

If $m = 4$, by Equations (10) and (113), we get $k_1 \approx 2.527, k_2 \approx 0.147, k_3 \approx -1.261$, or $k_1 \approx 1.261, k_2 \approx -0.147, k_3 \approx -2.527$.

If $m = 8$, by Equations (10) and (117), we get $k_1 \approx 2.216, k_2 \approx 0.1, k_3 \approx -1.39$ or $k_1 \approx 1.39, k_2 \approx -0.1, k_3 \approx -2.216$.

If $s_2 \neq 0$, by Case1, M is an isoparametric hypersurface with three constant principal curvature $k_1 > k_2 > k_3$ and so the multiplicities: $m = 1, 2, 4$ and 8. So we have the following:

If $m = 1$ by Equations (10), (106) and (107), we get $k_1 = \sqrt{3}$, $k_2 = 0$ and $k_3 = -\sqrt{3}$.

If $m = 2$ by Equations (10), (106), (108), (109) and (110), we get either $k_1 = \sqrt{3}, k_2 = 0, k_3 = -\sqrt{3}$ or $k_1 \approx 1.369, k_2 \approx -0.107, k_3 \approx -2.261$ or $k_1 \approx 2.261, k_2 \approx 0.107, k_3 \approx -1.369$.

If $m = 4$ by Equations (10), (106), (112), (113) and (114), we get $k_1 = \sqrt{3}, k_2 = 0, k_3 = -\sqrt{3}$.

If $m = 8$ by Equations (10), (106), (116),(117) and(118), we get $k_1 = \sqrt{3}, k_2 = 0, k_3 = -\sqrt{3}$.

Let $k = 2$. By Case 2, M is an isoparametric hypersurface. So the multiplicities of constant principal curvatures $k_1 > k_2 > k_3$ is $m = 1, 2, 4$ and 8.

If $s_3 = 0$, then we have the following:

If $m = 1$, by Equations (10) and (107), we get $k_1 = \sqrt{3}$, $k_2 = 0$ and $k_3 = -\sqrt{3}$.

If $m = 2$ by Equations (10) and (110), we get $k_1 = \sqrt{3}, k_2 = 0, k_3 = -\sqrt{3}$.

If $m = 4$ by Equations (10) and (114), we get $k_1 = \sqrt{3}, k_2 = 0, k_3 = -\sqrt{3}$ or $k_1 \approx 0.993, k_2 \approx -0.271, k_3 \approx -3.777$ or $k_1 \approx 3.777, k_2 \approx 0.271, k_3 \approx -0.993$.

If $m = 8$ by Equations (10) and (118), we get $k_1 = \sqrt{3}, k_2 = 0, k_3 = -\sqrt{3}$ or $k_1 \approx 1.189, k_2 \approx -0.177, k_3 \approx -2.757$ or $k_1 \approx 2.757, k_2 \approx 0.177, k_3 \approx -1.189$.

If $s_3 \neq 0$, then by Equation (2), we have

$$s_1 s_3 - 4s_4 - (n-2)s_2 = 0. \tag{120}$$

If $m = 1$, by Equations (10), (107) and (120), we get either $k_1 = 1, k_2 = \sqrt{3} - 2, k_3 = -\sqrt{3} - 2$ or $k_1 = 2 + \sqrt{3}, k_2 = 2 - \sqrt{3}, k_3 = -1$.

If $m = 2$, by Equations (10), (108), (109), (110), (111) and (120), we get that there is not real solution for all k_1, k_2, k_3. So there is not proper L_2-biharmonic hypersurface in $\mathbb{S}^7$ with three disjoint principal curvatures, and s_1 and s_2 is constant.

If $m = 4$, by Equations (10), (112), (113), (114), (115) and (120), we get either $k_1 \approx 1.083, k_2 \approx -0.225, k_3 \approx -3.213$ or $k_1 \approx 3.213, k_2 \approx 0.225, k_3 \approx -1.083$.

If $m = 8$, by Equations (10), (116), (117), (118), (119) and (120), we get that there is not real solution for all k_1, k_2, k_3. So there is not proper L_2-biharmonic hypersurface in $\mathbb{S}^{25}$ with three disjoint principal curvatures, and s_1 and s_2 is constant. Summarizing all of above and Theorem 2.1, we get the result. □

Proof of Theorem 1.3. We have $P_3 = s_3 I - S \circ P_2$. Since $P_3 = 0$, Equation (1) implies that $3s_3 \nabla s_3 = 0$. Thus $\nabla s_3^2 = 0$, and so s_3 is constant. By assumption s_2 is constant and $s_3 \neq 0$, and so by Equation (2), s_1 is constant. Now by Theorem 1.2, we get the result. □

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