Commun. Korean Math. Soc. **35** (2020), No. 4, pp. 1203–1219
https://doi.org/10.4134/CKMS.c200055
pISSN: 1225-1763 / eISSN: 2234-3024

GENERIC LIGHTLIKE SUBMANIFOLDS OF AN INDEFINITE TRANS-SASAKIAN MANIFOLD WITH A NON-SYMMETRIC NON-METRIC CONNECTION OF TYPE (ℓ, m)

Chul Woo Lee and Jae Won Lee

Abstract. Jin [7] defined a new connection on semi-Riemannian manifolds, which is a non-symmetric and non-metric connection. He said that this connection is an $(\ell,\, m)$-*type connection*. Jin also studied lightlike hypersurfaces of an indefinite trans-Sasakian manifold with an (ℓ, m)-type connection in [7]. We study further the geometry of this subject. In this paper, we study generic lightlike submanifolds of an indefinite trans-Sasakian manifold endowed with an (ℓ, m)-type connection.

1. Introduction

The notion of (ℓ, m)-type connection on indefinite almost contact manifolds $\bar{M}$ was introduced by Jin [7]. Here we quote Jin's definition as follows:

A linear connection $\bar{\nabla}$ on $\bar{M}$ is called a *non-symmetric non-metric connection of type* $(\ell,\, m)$, and abbreviate it to $(\ell,\, m)$-*type connection*, if there exist smooth functions ℓ and m on $\bar{M}$ such that $\bar{\nabla}$ and its torsion tensor $\bar{T}$ satisfy

$$(1)\qquad (\bar{\nabla}_{\bar{X}}\bar{g})(\bar{Y},\bar{Z}) = -\,\ell\{\theta(\bar{Y})\bar{g}(\bar{X},\bar{Z})+\theta(\bar{Z})\bar{g}(\bar{X},\bar{Y})\} - m\{\theta(\bar{Y})\bar{g}(J\bar{X},\bar{Z})+\theta(\bar{Z})\bar{g}(J\bar{X},\bar{Y})\},$$

$$(2)\qquad \bar{T}(\bar{X},\bar{Y}) = \ell\{\theta(\bar{Y})\bar{X}-\theta(\bar{X})\bar{Y}\}+m\{\theta(\bar{Y})J\bar{X}-\theta(\bar{X})J\bar{Y}\},$$

where J is a $(1,1)$-type tensor field and θ is a 1-form associated with a smooth vector field ζ by $\theta(\bar{X}) = \bar{g}(\bar{X},\zeta)$. We set $(\ell, m) \neq (0,0)$ and we denote by $\bar{X}$, $\bar{Y}$ and $\bar{Z}$ the smooth vector fields on $\bar{M}$. Semi-symmetric non-metric connection and non-metric ϕ-symmetric connection are important examples of this connection such that (1) $(\ell,\, m) = (1,\, 0)$ and (2) $(\ell,\, m) = (0,\, 1)$, respectively.

Received February 15, 2020; Accepted March 26, 2020.
2010 *Mathematics Subject Classification.* Primary 53C25, 53C40, 53C50.
Key words and phrases. (ℓ, m)-type connection, generic lightlike submanifold, indefinite trans-Sasakian structure.
Chul Woo Lee was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (2018 R1D1A1B07040576).

Especially, in cases: (3) $(\ell, m) = (1, 0)$ in (1) and $(\ell, m) = (0, 1)$ in (2) (see [10] in details); (4) $(\ell, m) = (0, 0)$ in (1) and $(\ell, m) = (0, 1)$ in (2) and (5) $(\ell, m) = (0, 0)$ in (1) and $(\ell, m) = (1, 0)$ in (2), this connection $\bar\nabla$ reduce to quarter-symmetric non-metric connection, quarter-symmetric metric connection and semi-symmetric metric connection, respectively.

Remark 1.1. Denote by $\widetilde\nabla$ the Levi-Civita connection of a semi-Riemannian manifold $(\bar M, \bar g)$ with respect to the metric $\bar g$. It is known [7] that *a linear connection $\bar\nabla$ on $\bar M$ is an (ℓ, m)-type connection if and only if $\bar\nabla$ satisfies*

$$\bar\nabla_{\bar X}\bar Y = \widetilde\nabla_{\bar X}\bar Y + \theta(\bar Y)\{\ell\bar X + mJ\bar X\}. \tag{3}$$

A lightlike submanifold M of an indefinite almost contact manifold $\bar M$ is said to be *generic* if there exists a screen distribution $S(TM)$ on M such that

$$J(S(TM)^{\perp}) \subset S(TM), \tag{4}$$

where $S(TM)^{\perp}$ is the orthogonal complement of $S(TM)$ in the tangent bundle $T\bar M$ on $\bar M$, *i.e.*, $T\bar M = S(TM) \oplus_{orth} S(TM)^{\perp}$. The notion of generic lightlike submanifolds was introduced by Jin-Lee [8] and later, studied by several geometers [3, 5, 6, 9]. Its geometry is an extension of that of lightlike hypersurfaces and half lightlike submanifolds. Much of its theory will be immediately generalized in a formal way to general lightlike submanifolds.

The subject of study in this paper is generic lightlike submanifolds of an indefinite trans-Sasakian manifold $M = (\bar M, \zeta, \theta, J, \bar g)$ endowed with an (ℓ, m)-type connection subject to the following two conditions that (1) the tensor field J and the 1-form θ, defined by (1) and (2), are identical with the indefinite trans-Sasakian structure tensor J and the structure 1-form θ of $\bar M$, respectively, and (2) the structure vector field ζ of $\bar M$ is tangent to M.

2. (ℓ, m)-type connections

The notion of trans-Sasakian manifold $\bar M$, of type (α, β), was introduced by Oubina [11]. Sasakian, Kenmotsu and cosymplectic manifolds are important kinds of trans-Sasakian manifolds such that

$$\alpha = 1, \ \beta = 0; \qquad \alpha = 0, \ \beta = 1; \qquad \alpha = \beta = 0,$$

respectively. If $\bar M$ is a semi-Riemannian manifold with a trans-Sasakian structure of type (α, β), then $\bar M$ is called an indefinite trans-Sasakian manifold as follows:

An odd-dimensional semi-Riemannian manifold $(\bar M, \bar g)$ is called an *indefinite trans-Sasakian manifold* if there exist (1) a structure set $\{J, \zeta, \theta, \bar g\}$, where J is a $(1, 1)$-type tensor field, ζ is a vector field and θ is a 1-form such that

$$\begin{aligned} J^2\bar X = -\bar X + \theta(\bar X)\zeta, \quad \theta(\zeta) = 1, \quad \theta(\bar X) = \epsilon\,\bar g(\bar X, \zeta), \\ \theta\circ J = 0, \quad \bar g(J\bar X, J\bar Y) = \bar g(\bar X, \bar Y) - \epsilon\,\theta(\bar X)\theta(\bar Y), \end{aligned} \tag{5}$$

(2) a Levi-Civita connection $\widetilde\nabla$ and two smooth functions α and β such that

$$(\widetilde\nabla_{\bar X}J)\bar Y = \alpha\{\bar g(\bar X, \bar Y)\zeta - \epsilon\,\theta(\bar Y)\bar X\} + \beta\{\bar g(J\bar X, \bar Y)\zeta - \epsilon\,\theta(\bar Y)J\bar X\},$$

where ϵ denotes $\epsilon = 1$ or -1 according as ζ is spacelike or timelike respectively. $\{J, \zeta, \theta, \bar{g}\}$ is called an *indefinite trans-Sasakian structure of type* (α, β).

In the entire discussion of this article, we shall assume that the vector field ζ is a spacelike one, *i.e.*, $\epsilon = 1$, without loss of generality.

Let $\bar{\nabla}$ be an (ℓ, m)-type connection on $(\bar{M}, \bar{g})$. By directed calculation from (3), (5) and the fact that $\theta(JY) = 0$, we obtain

$$\begin{aligned}(\bar{\nabla}_{\bar{X}} J)\bar{Y} = \alpha\{\bar{g}(\bar{X}, \bar{Y})\zeta - \theta(\bar{Y})\bar{X}\} + \beta\{\bar{g}(J\bar{X}, \bar{Y})\zeta - \theta(\bar{Y})J\bar{X}\} \\ - \theta(\bar{Y})\{\ell J\bar{X} - m\bar{X} + m\theta(\bar{X})\zeta\}.\end{aligned} \tag{6}$$

Replacing $\bar{Y}$ by ζ to (6) and using $J\zeta = 0$ and $\theta(\bar{\nabla}_X \zeta) = \ell\theta(X)$, we obtain

$$\bar{\nabla}_{\bar{X}}\zeta = (m - \alpha)J\bar{X} + (\ell + \beta)\bar{X} - \beta\theta(\bar{X})\zeta. \tag{7}$$

Let (M, g) be an m-dimensional lightlike submanifold of an indefinite trans-Sasakian manifold $(\bar{M}, \bar{g})$ of dimension $(m + n)$. Then the radical distribution $Rad(TM) = TM \cap TM^\perp$ on M is a subbundle of the tangent bundle TM and the normal bundle $TM^\perp$, of rank $r\,(1 \le r \le \min\{m, n\})$. In case $1 < r < \min\{m, n\}$, we say that M is an *r-lightlike submanifold* [3] of $\bar{M}$. In this case, there exist two complementary non-degenerate distributions $S(TM)$ and $S(TM^\perp)$ of $Rad(TM)$ in TM and $TM^\perp$, respectively, which are called the *screen distribution* and *co-screen distribution* of M such that

$$TM = Rad(TM) \oplus_{orth} S(TM), \; TM^\perp = Rad(TM) \oplus_{orth} S(TM^\perp),$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle E over M. Also denote by $(5)_i$ the i-th equation of (5). We use the same notations for any others. Let X, Y, Z and W be the vector fields on M, unless otherwise specified. We use the following range of indices:

$$i, j, k, \ldots \in \{1, \ldots, r\}, \qquad a, b, c, \ldots \in \{r + 1, \ldots, n\}.$$

Let $tr(TM)$ and $ltr(TM)$ be complementary vector bundles to TM in $T\bar{M}_{|M}$ and $TM^\perp$ in $S(TM)^\perp$ respectively and let $\{N_1, \ldots, N_r\}$ be a lightlike basis of $ltr(TM)_{|\mathcal{U}}$, where $\mathcal{U}$ is a coordinate neighborhood of M, such that

$$\bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0,$$

where $\{\xi_1, \ldots, \xi_r\}$ is a lightlike basis of $Rad(TM)_{|\mathcal{U}}$. In this case,

$$\begin{aligned}T\bar{M} &= TM \oplus tr(TM) = \{Rad(TM) \oplus tr(TM)\} \oplus_{orth} S(TM) \\ &= \{Rad(TM) \oplus ltr(TM)\} \oplus_{orth} S(TM) \oplus_{orth} S(TM^\perp),\end{aligned}$$

$$\{\xi_1, \ldots, \xi_r, \; N_1, \ldots, N_r, \; F_{r+1}, \ldots, F_m, \; E_{r+1}, \ldots, E_n\}$$

is a quasi-orthonormal field of frames of $\bar{M}$, where $\{F_{r+1}, \ldots, F_m\}$ is an orthonormal basis of $S(TM)$ and $\{E_{r+1}, \ldots, E_n\}$ is an orthonormal basis of $S(TM^\perp)$. Denote $\epsilon_a = \bar{g}(E_a, E_a)$. Then $\epsilon_a\delta_{ab} = \bar{g}(E_a, E_b)$.

Let P be the projection morphism of TM on $S(TM)$. Then the local Gauss-Weingarten formulae of M and $S(TM)$ are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + \sum_{i=1}^{r} h_i^{\ell}(X,Y)N_i + \sum_{a=r+1}^{n} h_a^s(X,Y)E_a, \tag{8}$$

$$\bar{\nabla}_X N_i = -A_{N_i}X + \sum_{j=1}^{r} \tau_{ij}(X)N_j + \sum_{a=r+1}^{n} \rho_{ia}(X)E_a, \tag{9}$$

$$\bar{\nabla}_X E_a = -A_{E_a}X + \sum_{i=1}^{r} \lambda_{ai}(X)N_i + \sum_{b=r+1}^{n} \sigma_{ab}(X)E_b; \tag{10}$$

$$\nabla_X PY = \nabla^*_X PY + \sum_{i=1}^{r} h_i^*(X,PY)\xi_i, \tag{11}$$

$$\nabla_X \xi_i = -A^*_{\xi_i}X - \sum_{j=1}^{r} \tau_{ji}(X)\xi_j, \tag{12}$$

where ∇ and ∇^* are induced linear connections on M and $S(TM)$ respectively, h_i^{ℓ} and h_a^s are called the *local second fundamental forms* on M, h_i^* are called the *local second fundamental forms* on $S(TM)$. A_{N_i}, A_{E_a} and $A^*_{\xi_i}$ are called the *shape operators*, and τ_{ij}, ρ_{ia}, λ_{ai} and σ_{ab} are 1-forms.

Let M be a generic lightlike submanifold of $\bar{M}$. From (4) we show that $J(Rad(TM))$, $J(ltr(TM))$ and $J(S(TM^{\perp}))$ are vector subbundles of $S(TM)$. Thus there exist two non-degenerate almost complex distributions H_o and H with respect to J, *i.e.*, $J(H_o) = H_o$ and $J(H) = H$, such that

$$S(TM) = \{J(Rad(TM)) \oplus J(ltr(TM))\} \oplus_{orth} J(S(TM^{\perp})) \oplus_{orth} H_o,$$
$$H = Rad(TM) \oplus_{orth} J(Rad(TM)) \oplus_{orth} H_o.$$

In this case, the tangent bundle TM on M is decomposed as follows:

$$TM = H \oplus J(ltr(TM)) \oplus_{orth} J(S(TM^{\perp})). \tag{13}$$

Consider local null vector fields U_i and V_i for each i, local non-null unit vector fields W_a for each a, and their 1-forms u_i, v_i and w_a defined by

$$U_i = -JN_i, \qquad V_i = -J\xi_i, \qquad W_a = -JE_a, \tag{14}$$

$$u_i(X) = g(X,V_i), \quad v_i(X) = g(X,U_i), \quad w_a(X) = \epsilon_a g(X,W_a). \tag{15}$$

Denote by S the projection morphism of TM on H and by F the tensor field of type $(1,1)$ globally defined on M by $F = J \circ S$. Then JX is expressed as

$$JX = FX + \sum_{i=1}^{r} u_i(X)N_i + \sum_{a=r+1}^{n} w_a(X)E_a. \tag{16}$$

Applying J to (16) and using $(5)_1$ and (14), we have

$$F^2X = -X + \theta(X)\zeta + \sum_{i=1}^{r} u_i(X)U_i + \sum_{a=r+1}^{n} w_a(X)W_a. \tag{17}$$

3. Structure equations

Let $\bar{M}$ be an indefinite trans-Sasakian manifold with an (ℓ, m)-type connection $\bar{\nabla}$. In the following, we shall assume that ζ is tangent to M. Călin [2] proved that if ζ is tangent to M, then it belongs to $S(TM)$ which we assumed in this paper. Using (1), (2), (8) and (16), we see that

$$\begin{aligned}(\nabla_X g)(Y,Z) = &\sum_{i=1}^{r}\{h_i^\ell(X,Y)\eta_i(Z) + h_i^\ell(X,Z)\eta_i(Y)\} \\ &- \ell\{\theta(Y)g(X,Z) + \theta(Z)g(X,Y)\} \\ &- m\{\theta(Y)\bar{g}(JX,Z) + \theta(Z)\bar{g}(JX,Y)\},\end{aligned} \tag{18}$$

$$T(X,Y) = \ell\{\theta(Y)X - \theta(X)Y\} + m\{\theta(Y)FX - \theta(X)FY\}, \tag{19}$$

$$h_i^\ell(X,Y) - h_i^\ell(Y,X) = m\{\theta(Y)u_i(X) - \theta(X)u_i(Y)\}, \tag{20}$$

$$h_a^s(X,Y) - h_a^s(Y,X) = m\{\theta(Y)w_a(X) - \theta(X)w_a(Y)\}, \tag{21}$$

for all i and a, where η_i's are 1-forms such that $\eta_i(X) = \bar{g}(X, N_i)$.

Theorem 3.1. *Let M be a lightlike hypersurface of an indefinite trans-Sasakian manifold $\bar{M}$ with an (ℓ, m)-type connection subject such that ζ is tangent to M. Then either h_i^ℓ or h_a^s is symmetric if and only if $m = 0$.*

Proof. (1) If $m = 0$, then h_i^ℓ are symmetric by (20). Conversely, if h_i^ℓ is symmetric, then, taking $X = \zeta$ and $Y = U_i$ to (20), we have $m = 0$.

(2) If $m = 0$, then h_a^s are symmetric by (21). Conversely, if h_a^s is symmetric, then, taking $X = \zeta$ and $Y = W_a$ to (21), we have $m = 0$. □

From the facts that $h_i^\ell(X,Y) = \bar{g}(\bar{\nabla}_X Y, \xi_i)$ and $\epsilon_a h_a^s(X,Y) = \bar{g}(\bar{\nabla}_X Y, E_a)$, we know that h_i^ℓ and h_a^s are independent of the choice of $S(TM)$. Applying $\bar{\nabla}_X$ to $g(\xi_i, \xi_j) = 0$, $\bar{g}(\xi_i, E_a) = 0$, $\bar{g}(N_i, N_j) = 0$, $\bar{g}(N_i, E_a) = 0$ and $\bar{g}(E_a, E_b) = \epsilon\delta_{ab}$ by turns and using (1) and (8) $\sim$ (10), we obtain

$$\begin{aligned}&h_i^\ell(X,\xi_j) + h_j^\ell(X,\xi_i) = 0, \qquad h_a^s(X,\xi_i) = -\epsilon_a\lambda_{ai}(X), \\ &\eta_j(A_{N_i}X) + \eta_i(A_{N_j}X) = 0, \qquad \eta_i(A_{E_a}X) = \epsilon_a\rho_{ia}(X), \\ &\epsilon_b\sigma_{ab} + \epsilon_a\sigma_{ba} = 0; \quad h_i^\ell(X,\xi_i) = 0, \ h_i^\ell(\xi_j,\xi_k) = 0, \ A^*_{\xi_i}\xi_i = 0.\end{aligned} \tag{22}$$

The local second fundamental forms are related to their shape operators by

$$h_i^\ell(X,Y) = g(A^*_{\xi_i}X, Y) + m\theta(Y)u_i(X) - \sum_{k=1}^{r} h_k^\ell(X,\xi_i)\eta_k(Y), \tag{23}$$

$$\epsilon_a h_a^s(X,Y) = g(A_{E_a}X, Y) + \epsilon_a m\theta(Y)w_a(X) - \sum_{k=1}^{r}\lambda_{ak}(X)\eta_k(Y), \tag{24}$$

$$h_i^*(X, PY) = g(A_{N_i}X, PY) + \{\ell\eta_i(X) + mv_i(X)\}\theta(PY). \tag{25}$$

Replacing Y by ζ to (8) and using (7), (16), (23) and (24), we have

$$\nabla_X\zeta = (m-\alpha)FX + (\ell+\beta)X - \beta\theta(X)\zeta, \tag{26}$$

$$\theta(A^*_{\xi_i}X) = -\alpha u_i(X), \qquad h_i^\ell(X,\zeta) = (m-\alpha)u_i(X), \tag{27}$$

$$\theta(A_{E_a}X) = -\epsilon_a\alpha w_a(X), \quad h_a^s(X,\zeta) = (m-\alpha)w_a(X). \tag{28}$$

Applying $\bar\nabla_X$ to $\bar g(\zeta, N_i) = 0$ and using (7), (9) and (25), we have

$$\begin{aligned} &\theta(A_{N_i}X) = -\alpha v_i(X) + \beta\eta_i(X),\\ &h_i^*(X,\zeta) = (\ell+\beta)\eta_i(X) + (m-\alpha)v_i(X). \end{aligned} \tag{29}$$

Applying $\bar\nabla_X$ to $(14)_{1,2,3}$ and (16) by turns and using (6), (8) $\sim$ (12), (14) $\sim$ (16) and (23) $\sim$ (25), we have

$$\begin{aligned} &h_j^\ell(X,U_i) = h_i^*(X,V_j), \qquad \epsilon_a h_i^*(X,W_a) = h_a^s(X,U_i),\\ &h_j^\ell(X,V_i) = h_i^\ell(X,V_j), \qquad \epsilon_a h_i^\ell(X,W_a) = h_a^s(X,V_i),\\ &\epsilon_b h_b^s(X,W_a) = \epsilon_a h_a^s(X,W_b), \end{aligned} \tag{30}$$

$$\begin{aligned} \nabla_X U_i \;=\; & F(A_{N_i}X) + \sum_{j=1}^r \tau_{ij}(X)U_j + \sum_{a=r+1}^n \rho_{ia}(X)W_a\\ & - \{\alpha\eta_i(X) + \beta v_i(X)\}\zeta, \end{aligned} \tag{31}$$

$$\begin{aligned} \nabla_X V_i \;=\; & F(A^*_{\xi_i}X) - \sum_{j=1}^r \tau_{ji}(X)V_j + \sum_{j=1}^r h_j^\ell(X,\xi_i)U_j\\ & - \sum_{a=r+1}^n \epsilon_a\lambda_{ai}(X)W_a - \beta u_i(X)\zeta, \end{aligned} \tag{32}$$

$$\begin{aligned} \nabla_X W_a \;=\; & F(A_{E_a}X) + \sum_{i=1}^r \lambda_{ai}(X)U_i + \sum_{b=r+1}^n \sigma_{ab}(X)W_b\\ & - \epsilon_a\beta w_a(X)\zeta, \end{aligned} \tag{33}$$

$$\begin{aligned} (\nabla_X F)(Y) \;=\; & \sum_{i=1}^r u_i(Y)A_{N_i}X + \sum_{a=r+1}^n w_a(Y)A_{E_a}X\\ & - \sum_{i=1}^r h_i^\ell(X,Y)U_i - \sum_{a=r+1}^n h_a^s(X,Y)W_a\\ & + \{\alpha g(X,Y) + \beta\bar g(JX,Y) - \theta(X)\theta(Y)\}\zeta\\ & + (m-\alpha)\theta(Y)X - (\ell+\beta)\theta(Y)FX, \end{aligned} \tag{34}$$

$$\begin{aligned} (\nabla_X u_i)(Y) = & -\sum_{j=1}^r u_j(Y)\tau_{ji}(X) - \sum_{a=r+1}^n w_a(Y)\lambda_{ai}(X)\\ & - h_i^\ell(X,FY) - (\ell+\beta)\theta(Y)u_i(X), \end{aligned} \tag{35}$$

$$(\nabla_X v_i)(Y) = \sum_{j=1}^{r} v_j(Y)\tau_{ij}(X) + \sum_{a=r+1}^{n} \epsilon_a w_a(Y)\rho_{ia}(X) + \sum_{j=r+1}^{r} u_j(Y)\eta_i(A_{N_j}X) - g(A_{N_i}X, FY) + (m-\alpha)\theta(Y)\eta_i(X) - (\ell+\beta)\theta(Y)v_i(X). \tag{36}$$

Theorem 3.2. *There exist no generic lightlike submanifolds of an indefinite trans-Sasakian manifold with an (ℓ, m)-type connection subject such that ζ is tangent to M and F satisfies the following equation*:

$$(\nabla_X F)Y = (\nabla_Y F)X, \qquad \forall\, X,\, Y \in \Gamma(TM).$$

Proof. Let $(\nabla_X F)Y = (\nabla_Y F)X$. Using (20), (21) and (34), we obtain

$$\sum_{i=1}^{r}\{u_i(Y)A_{N_i}X - u_i(X)A_{N_i}Y\} + \sum_{a=r+1}^{n}\{w_a(Y)A_{E_a}X - w_a(X)A_{E_a}Y\} - 2\beta\bar{g}(X, JY)\zeta + \{\theta(X)u_i(Y) - \theta(Y)u_i(X)\}U_i + \{\theta(X)w_a(Y) - \theta(Y)w_a(X)\}W_a + (m-\alpha)\{\theta(Y)X - \theta(X)Y\} - (\ell+\beta)\{\theta(Y)FX - \theta(X)FY\} = 0. \tag{37}$$

Taking the scalar product with ζ and using $(28)_1$ and $(29)_1$, we have

$$\alpha\sum_{i=1}^{r}\{u_i(Y)v_i(X) - u_i(X)v_i(Y)\} = \beta\sum_{i=1}^{r}\{u_i(Y)\eta_i(X) - u_i(X)\eta_i(Y)\} - 2\beta\bar{g}(X, JY).$$

Taking $X = V_j$, $Y = U_j$ and $X = \xi_j$, $Y = U_j$ to this equation by turns, we obtain $\alpha = 0$ and $\beta = 0$, respectively. Taking $X = \xi_j$ to (37), we have

$$\theta(X)\{m\xi_i + \ell V_i\} + \sum_{j=1}^{r} u_j(X)A_{N_j}\xi_i + \sum_{a=r+1}^{n} w_a(X)A_{E_a}\xi_i = 0.$$

Taking $X = \zeta$ to this, we have $m\xi_i + \ell V_i = 0$. It follows that $\ell = m = 0$. It is a contradiction to $(\ell, m) \neq (0,\, 0)$. Thus we have our theorem. □

Corollary 3.3. *There exist no generic lightlike submanifolds of an indefinite trans-Sasakian manifold with an (ℓ, m)-type connection subject such that ζ is tangent to M and F is parallel with respect to the connection ∇.*

Theorem 3.4. *Let M be a generic lightlike submanifold of an indefinite trans-Sasakian manifold $\bar{M}$ with an (ℓ, m)-type connection such that ζ is tangent to M. If U_i's or V_i's are parallel with respect to ∇, then $\tau_{ij} = 0$ and $\alpha = \beta = 0$, i.e., $\bar{M}$ is an indefinite cosymplectic manifold.*

Proof. (1) If U_i is parallel with respect to ∇, then, taking the scalar product with ζ, V_j, W_a, U_j and N_j to (31) such that $\nabla_X U_i = 0$ respectively, we get

$$(38)\qquad \alpha = \beta = 0, \quad \tau_{ij} = 0, \quad \rho_{ia} = 0, \quad \eta_j(A_{N_i} X) = 0, \quad h_i^*(X, U_j) = 0.$$

As $\alpha = \beta = 0$, $\bar{M}$ is an indefinite cosymplectic manifold.

(2) If V_i is parallel with respect to ∇, then, taking the scalar product with ζ, U_j, V_j, W_a and N_j to (32) with $\nabla_X V_i = 0$ respectively, we get

$$(39)\qquad \beta = 0, \quad \tau_{ji} = 0, \quad h_j^\ell(X, \xi_i) = 0, \quad \lambda_{ai} = 0, \quad h_i^\ell(X, U_j) = 0.$$

As $h_i^\ell(X, U_j) = 0$, we get $h_i^\ell(\zeta, U_j) = 0$. Taking $X = U_j$ and $Y = \zeta$ to (20), we get $h_i^\ell(U_j, \zeta) = m\delta_{ij}$. On the other hand, replacing X by U to $(27)_2$, we have $h_i^\ell(U_j, \zeta) = (m - \alpha)\delta_{ij}$. It follows that $\alpha = 0$. Since $\alpha = \beta = 0$, $\bar{M}$ is an indefinite cosymplectic manifold. □

4. Indefinite generalized Sasakian space forms

Definition. An indefinite trans-Sasakian manifold $\bar{M}$ is said to be an *indefinite generalized Sasakian space form* and denote it by $\bar{M}(f_1, f_2, f_3)$ if there exist three smooth functions f_1, f_2 and f_3 on $\bar{M}$ such that

$$(40)\qquad \begin{aligned} \widetilde{R}(\bar{X}, \bar{Y})\bar{Z} = {} & f_1\{\bar{g}(\bar{Y}, \bar{Z})\bar{X} - \bar{g}(\bar{X}, \bar{Z})\bar{Y}\} \\ & + f_2\{\bar{g}(\bar{X}, J\bar{Z})J\bar{Y} - \bar{g}(\bar{Y}, J\bar{Z})J\bar{X} + 2\bar{g}(\bar{X}, J\bar{Y})J\bar{Z}\} \\ & + f_3\{\theta(\bar{X})\theta(\bar{Z})\bar{Y} - \theta(\bar{Y})\theta(\bar{Z})\bar{X} \\ & \quad + \bar{g}(\bar{X}, \bar{Z})\theta(\bar{Y})\zeta - \bar{g}(\bar{Y}, \bar{Z})\theta(\bar{X})\zeta\}, \end{aligned}$$

where $\widetilde{R}$ is the curvature tensor of the Levi-Civita connection $\bar{\nabla}$.

The notion of generalized Sasakian space form was introduced by Alegre et al. [1], while the indefinite generalized Sasakian space forms were introduced by Jin [4]. Sasakian space form, Kenmotsu space form and cosymplectic space form are important kinds of generalized Sasakian space forms such that

$$f_1 = \tfrac{c+3}{4},\ f_2 = f_3 = \tfrac{c-1}{4}; \quad f_1 = \tfrac{c-3}{4},\ f_2 = f_3 = \tfrac{c+1}{4}; \quad f_1 = f_2 = f_3 = \tfrac{c}{4}$$

respectively, where c is a constant J-sectional curvature of each space forms.

Denote by $\bar{R}$ the curvature tensors of the non-metric ϕ-symmetric connection $\bar{\nabla}$ on $\bar{M}$. By directed calculations from (2), (3) and (5), we see that

$$(41)\qquad \begin{aligned} \bar{R}(\bar{X}, \bar{Y})\bar{Z} = {} & \widetilde{R}(\bar{X}, \bar{Y})\bar{Z} \\ & + (\bar{\nabla}_{\bar{X}}\theta)(\bar{Z})\{\ell\bar{Y} + mJ\bar{Y}\} - (\bar{\nabla}_{\bar{Y}}\theta)(\bar{Z})\{\ell\bar{X} + mJ\bar{X}\} \\ & + \theta(\bar{Z})\{(\bar{X}\ell)\bar{Y} - (\bar{Y}\ell)\bar{X} + (\bar{X}m)J\bar{Y} - (\bar{Y}m)J\bar{X} \\ & \quad - m\alpha[\theta(\bar{Y})\bar{X} - \theta(\bar{X})\bar{Y}] - m\beta[\theta(\bar{Y})J\bar{X} - \theta(\bar{X})J\bar{Y}] \\ & \quad - 2m\beta\bar{g}(\bar{X}, J\bar{Y})\zeta\}. \end{aligned}$$

Taking the scalar product with ξ_i and N_i to (41) by turns and, then denote by R and R^* the curvature tensors of the induced linear connections ∇ and

∇^* on M and $S(TM)$ respectively. Using the Gauss-Weingarten formulae, we obtain Gauss equations for M and $S(TM)$, respectively:

$$
\begin{aligned}
\bar{R}(X,Y)Z = R(X,Y)Z
&+ \sum_{i=1}^{r}\{h_i^{\ell}(X,Z)A_{N_i}Y - h_i^{\ell}(Y,Z)A_{N_i}X\}\\
&+ \sum_{a=r+1}^{n}\{h_a^{s}(X,Z)A_{E_a}Y - h_a^{s}(Y,Z)A_{E_a}X\}\\
&\quad + \sum_{i=1}^{r}\{(\nabla_X h_i^{\ell})(Y,Z) - (\nabla_Y h_i^{\ell})(X,Z)\\
&\quad + \sum_{j=1}^{r}[\tau_{ji}(X)h_j^{\ell}(Y,Z) - \tau_{ji}(Y)h_j^{\ell}(X,Z)]\\
&\quad + \sum_{a=r+1}^{n}[\lambda_{ai}(X)h_a^{s}(Y,Z) - \lambda_{ai}(Y)h_a^{s}(X,Z)]\\
&\quad - \ell[\theta(X)h_i^{\ell}(Y,Z) - \theta(Y)h_i^{\ell}(X,Z)]\\
&\quad - m[\theta(X)h_i^{\ell}(FY,Z) - \theta(Y)h_i^{\ell}(FX,Z)]\}N_i\\
&+ \sum_{a=r+1}^{n}\{(\nabla_X h_a^{s})(Y,Z) - (\nabla_Y h_a^{s})(X,Z)\\
&\quad + \sum_{i=1}^{r}[\rho_{ia}(X)h_i^{\ell}(Y,Z) - \rho_{ia}(Y)h_i^{\ell}(X,Z)]\\
&\quad + \sum_{b=r+1}^{n}[\sigma_{ba}(X)h_b^{s}(Y,Z) - \sigma_{ba}(Y)h_b^{s}(X,Z)]\\
&\quad - \ell[\theta(X)h_a^{s}(Y,Z) - \theta(Y)h_a^{s}(X,Z)]\\
&\quad - m[\theta(X)h_a^{s}(FY,Z) - \theta(Y)h_a^{s}(FX,Z)]\}E_a,
\end{aligned}
\tag{42}
$$

$$
\begin{aligned}
R(X,Y)PZ = R^*(X,Y)PZ
&+ \sum_{i=1}^{r}\{h_i^{*}(X,PZ)A^*_{\xi_i}Y - h_i^{*}(Y,PZ)A_{\xi_i}X\}\\
&+ \sum_{i=1}^{r}\{(\nabla_X h_i^{*})(Y,PZ) - (\nabla_Y h_i^{*})(X,PZ)\\
&\quad + \sum_{k=1}^{r}[\tau_{ik}(Y)h_k^{*}(X,PZ) - \tau_{ik}(X)h_k^{*}(Y,PZ)]\\
&\quad - \ell[\theta(X)h_i^{*}(Y,PZ) - \theta(Y)h_i^{*}(FX,PZ)]\\
&\quad - m[\theta(X)h_i^{*}(FY,PZ) - \theta(Y)h_i^{*}(FX,PZ)]\}\xi_i.
\end{aligned}
\tag{43}
$$

substituting (42) and (40) and using $(22)_4$ and (43), we get

$$
\begin{aligned}
(44)\qquad & (\nabla_X h_i^\ell)(Y,Z) - (\nabla_Y h_i^\ell)(X,Z) \\
& + \sum_{j=1}^{r}\{\tau_{ji}(X)h_j^\ell(Y,Z) - \tau_{ji}(Y)h_j^\ell(X,Z)\} \\
& + \sum_{a=r+1}^{n}\{\lambda_{ai}(X)h_a^s(Y,Z) - \lambda_{ai}(Y)h_a^s(X,Z)\} \\
& - \ell\{\theta(X)h_i^\ell(Y,Z) - \theta(Y)h_i^\ell(X,Z)\} \\
& - m\{\theta(X)h_i^\ell(FY,Z) - \theta(Y)h_i^\ell(FX,Z)\} \\
& - m\{(\bar\nabla_X\theta)(Z)u_i(Y) - (\bar\nabla_Y\theta)(Z)u_i(X)\} \\
& - \theta(Z)\{[Xm + m\beta\theta(X)]u_i(Y) - [Ym + m\beta\theta(Y)]u_i(X)\} \\
= {} & f_2\{u_i(Y)\bar g(X,JZ) - u_i(X)\bar g(Y,JZ) + 2u_i(Z)\bar g(X,JY)\},
\end{aligned}
$$

$$
\begin{aligned}
(45)\qquad & (\nabla_X h_i^*)(Y,PZ) - (\nabla_Y h_i^*)(X,PZ) \\
& - \sum_{j=1}^{r}\{\tau_{ij}(X)h_j^*(Y,PZ) - \tau_{ij}(Y)h_j^*(X,PZ)\} \\
& - \sum_{a=r+1}^{n}\epsilon_a\{\rho_{ia}(X)h_a^s(Y,PZ) - \rho_{ia}(Y)h_a^s(X,PZ)\} \\
& + \sum_{j=1}^{r}\{h_j^\ell(X,PZ)\eta_i(A_{N_j}Y) - h_j^\ell(Y,PZ)\eta_i(A_{N_j}X)\} \\
& - \ell\{\theta(X)h_i^*(Y,PZ) - \theta(Y)h_i^*(X,PZ)\} \\
& - m\{\theta(X)h_i^*(FY,PZ) - \theta(Y)h_i^*(FX,PZ)\} \\
& - (\bar\nabla_X\theta)(PZ)\{\ell\eta_i(Y) + mv_i(Y)\} + (\bar\nabla_Y\theta)(PZ)\{\ell\eta_i(X) + mv_i(X)\} \\
& - \theta(PZ)\{[X\ell + m\alpha\theta(X)]\eta_i(Y) - [Y\ell + m\alpha\theta(Y)]\eta_i(X) \\
& + [Xm + m\beta\theta(X)]v_i(Y) - [Ym + m\beta\theta(Y)]v_i(X)\} \\
= {} & f_1\{g(Y,PZ)\eta_i(X) - g(X,PZ)\eta_i(Y)\} \\
& + f_2\{v_i(Y)\bar g(X,JPZ) - v_i(X)\bar g(Y,JPZ) + 2v_i(PZ)\bar g(X,JY)\} \\
& + f_3\{\theta(X)\eta_i(Y) - \theta(Y)\eta_i(X)\}\theta(PZ).
\end{aligned}
$$

Theorem 4.1. *Let M be a generic lightlike submanifold of an indefinite generalized Sasakian space form $\bar M(f_1, f_2, f_3)$ with an (ℓ, m)-type connection such that ζ is tangent to M. Then the functions α, β, f_1, f_2 and f_3 satisfy*

(1) *α is a constant on M,*
(2) *$\alpha\beta = 0$, and*
(3) *$f_1 - f_2 = \alpha^2 - \beta^2$ and $f_1 - f_3 = \alpha^2 - \beta^2 - \zeta\beta$.*

Proof. Applying $\bar{\nabla}_X$ to $\theta(U_i) = 0$ and $\theta(V_i) = 0$ by turns and using (8), (31), (32) and the facts that $g(FX, \zeta) = 0$ and $\zeta \in \Gamma(S(TM))$, we get

$$(\bar{\nabla}_X\theta)(U_i) = \alpha\eta_i(X) + \beta v_i(X), \qquad (\bar{\nabla}_X\theta)(V_i) = \beta u_i(X). \tag{46}$$

Applying ∇_X to $(30)_1$: $h^\ell_j(Y, U_i) = h^*_i(Y, V_j)$ and using (5), (16), (23), (25), $(27)_2$, $(29)_2$, $(30)_{1,2,4}$, (31) and (32), we obtain

$$\begin{aligned}(\nabla_X h^\ell_j)(Y, U_i) = {} & (\nabla_X h^*_i)(Y, V_j) \\ & - \sum_{k=1}^{r}\{\tau_{kj}(X)h^\ell_k(Y, U_i) + \tau_{ik}(X)h^*_k(Y, V_j)\} \\ & - \sum_{a=r+1}^{n}\{\lambda_{aj}(X)h^s_a(Y, U_i) + \epsilon_a\rho_{ia}(X)h^s_a(Y, V_j)\} \\ & + \sum_{k=1}^{r}\{h^*_i(Y, U_k)h^\ell_k(X, \xi_j) + h^*_i(X, U_k)h^\ell_k(Y, \xi_j)\} \\ & - g(A^*_{\xi_j}X, F(A_{N_i}Y)) - g(A^*_{\xi_j}Y, F(A_{N_i}X)) \\ & - \sum_{k=1}^{r} h^\ell_j(X, V_k)\eta_k(A_{N_i}Y) \\ & + \beta(m-\alpha)\{u_j(Y)v_i(X) - u_j(X)v_i(Y)\} \\ & + \alpha(m-\alpha)u_j(Y)\eta_i(X) - \beta(\ell+\beta)u_j(X)\eta_i(Y).\end{aligned}$$

Substituting this and (29) into the modified equation (44) which is change i with j and Z with U_i, and using $(22)_3$, $(30)_3$ and $(46)_1$, we have

$$\begin{aligned} & (\nabla_X h^*_i)(Y, V_j) - (\nabla_Y h^*_i)(X, V_j) \\ & - \sum_{k=1}^{r}\{\tau_{ik}(X)h^*_k(Y, V_j) - \tau_{ik}(Y)h^*_k(X, V_j)\} \\ & - \sum_{a=r+1}^{n}\epsilon_a\{\rho_{ia}(X)h^s_a(Y, V_j) - \rho_{ia}(Y)h^s_a(X, V_j)\} \\ & + \sum_{k=1}^{r}\{h^\ell_k(X, V_j)\eta_i(A_{N_k}Y) - h^\ell_k(Y, V_j)\eta_i(A_{N_k}X)\} \\ & - \theta(X)h^*_i(FY, V_j) + \theta(Y)h^*_i(FX, V_j) \\ & + \beta(m-2\alpha)\{u_j(Y)v_i(X) - u_j(X)v_i(Y)\} \\ & + (\ell\beta - \alpha^2 + \beta^2)\{u_j(Y)\eta_i(X) - u_j(X)\eta_i(Y)\} \\ = {} & f_2\{u_j(Y)\eta_i(X) - u_j(X)\eta_i(Y) + 2\delta_{ij}\bar{g}(X, JY)\}.\end{aligned}$$

Comparing this with (45) such that $PZ = V_j$ and using $(46)_2$, we obtain

$$\begin{aligned} & \{f_1 - f_2 - \alpha^2 + \beta^2\}\{u_j(Y)\eta_i(X) - u_j(X)\eta_i(Y)\} \\ = {} & 2\alpha\beta\{u_j(Y)v_i(X) - u(_jX)v_i(Y)\}.\end{aligned}$$

Taking $Y = U_j$, $X = \xi_i$ and $Y = U_j$, $X = V_i$ to this by turns, we have

$$f_1 - f_2 = \alpha^2 - \beta^2, \qquad \alpha\beta = 0.$$

Applying $\bar{\nabla}_X$ to $\theta(\zeta) = 1$ and using (7) and the fact: $\theta \circ J = 0$, we get

$$(\bar{\nabla}_X \theta)(\zeta) = -\ell\theta(X). \tag{47}$$

Applying $\bar{\nabla}_X$ to $\eta_i(Y) = \bar{g}(Y, N_i)$ and using (1) and (9), we have

$$\begin{aligned}(\nabla_X \eta_i)(Y) = & -g(A_{N_i}X, Y) + \sum_{j=1}^{r} \tau_{ij}(X)\eta_j(Y) \\ & - \{\ell\eta_i(X) + mv_i(X)\}\theta(Y).\end{aligned} \tag{48}$$

Applying ∇_X to $h_i^*(Y, \zeta) = (\ell + \beta)\eta_i(Y) + (m - \alpha)v_i(Y)$ and using (25), (26), (36), (48) and the fact that $\alpha\beta = 0$, we get

$$\begin{aligned}(\nabla_X h_i^*)(Y, \zeta) = & X(\ell + \beta)\eta_i(Y) + X(m - \alpha)v_i(Y) \\ & + (\ell + \beta)\{-g(A_{N_i}X, Y) - g(A_{N_i}Y, X) + \sum_{j=1}^{r} \tau_{ij}(X)\eta_j(Y) \\ & + \beta\theta(X)\eta_i(Y) - \ell[\theta(Y)\eta_i(X) + \theta(X)\eta_i(Y)] \\ & - m[\theta(Y)v_i(X) + \theta(X)v_i(Y)]\} \\ & + (m - \alpha)\{-g(A_{N_i}X, FY) - g(A_{N_i}Y, FX) \\ & + \sum_{j=1}^{r} v_j(Y)\tau_{ij}(X) + \sum_{a=r+1}^{n} \epsilon_a w_a(Y)\rho_{ia}(X) \\ & + \sum_{j=1}^{r} u_j(Y)\eta_i(A_{N_j}X) + (m - \alpha)\theta(Y)\eta_i(X) \\ & + \beta\theta(X)v_i(Y) - (\ell + \beta)\theta(Y)v_i(X)\}.\end{aligned}$$

Substituting this and $(29)_2$ into (45) with $PZ = \zeta$ and using (47), we get

$$\begin{aligned} & \{X\beta + (f_1 - f_3 - \alpha^2 + \beta^2)\theta(X)\}\eta_i(Y) \\ & - \{Y\beta + (f_1 - f_3 - \alpha^2 + \beta^2)\theta(Y)\}\eta_i(X) \\ = & (X\alpha)v_i(Y) - (Y\alpha)v_i(X).\end{aligned}$$

Taking $X = \zeta$, $Y = \xi_i$ and $X = U_j$, $Y = V_i$ to this by turns, we have

$$f_1 - f_3 = \alpha^2 - \beta^2 - \zeta\beta, \qquad U_j\alpha = 0.$$

Applying ∇_X to $h_i^\ell(Y, \zeta) = (m - \alpha)u_i(Y)$ and using (26) and (35), we get

$$\begin{aligned}(\nabla_X h_i^\ell)(Y, \zeta) = & X(m - \alpha)u_i(Y) - (\ell + \beta)h_i^\ell(Y, X) \\ & - (m - \alpha)\{\sum_{j=1}^{r} u_j(Y)\tau_{ji}(X) + \sum_{a=r+1}^{n} \epsilon_a w_a(Y)\lambda_{ai}(X) \\ & + h_i^\ell(X, FY) + h_i^\ell(Y, FX) + \ell\theta(Y)u_i(X)\end{aligned}$$

$$+ \ \beta[\theta(Y)u_i(X) - \theta(X)u_i(Y)]\}.$$

Substituting this into (44) with $Z = \zeta$ and using (20) and (47), we have

$$(X\alpha)u_i(Y) = (Y\alpha)u_i(X).$$

Taking $Y = U_i$ to this result and using the fact that $U_i\alpha = 0$, we have $X\alpha = 0$. Therefore α is a constant. This completes the proof of the theorem. □

Definition. (1) A screen distribution $S(TM)$ is said to be *totally umbilical* [3] in M if there exist smooth functions γ_i on a neighborhood $\mathcal{U}$ such that

$$h_i^*(X, PY) = \gamma_i g(X, PY).$$

In case $\gamma_i = 0$, we say that $S(TM)$ is *totally geodesic* in M.

(2) An r-lightlike submanifold M is said to be *screen conformal* [4] if there exist non-vanishing smooth functions φ_i on $\mathcal{U}$ such that

$$h_i^*(X, PY) = \varphi_i h_i^\ell(X, PY). \tag{49}$$

Theorem 4.2. *Let M be a generic lightlike submanifold of an indefinite generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ with an (ℓ, m)-type connection subject such that ζ is tangent to M. If*

(1) *$S(TM)$ is totally umbilical, or*

(2) *M is screen conformal,*

then $\bar{M}(f_1, f_2, f_3)$ is an indefinite β-Kenmotsu manifold with a semi-symmetric non-metric connection such that

$$\alpha = m = 0, \quad \beta = -\ell \neq 0, \qquad f_1 = -\beta^2, \quad f_2 = 0, \quad f_3 = -\zeta\beta.$$

Proof. (1) If $S(TM)$ is totally umbilical, then $(29)_2$ is reduced to

$$\gamma_i\theta(X) = (\ell + \beta)\eta_i(X) + (m - \alpha)v_i(X).$$

Taking $X = \zeta$, $X = \xi_i$ and $X = V_i$ to this equation by turns, we have

$$\gamma_i = 0, \qquad \ell = -\beta, \qquad m = \alpha, \tag{50}$$

respectively. As $\gamma_i = 0$, we obtain $h_i^* = 0$. Thus. from $(30)_{1,2}$, we have

$$h_j^\ell(X, U_i) = 0, \qquad h_a^s(X, U_i) = 0. \tag{51}$$

Replacing Y by U_j to (20) and using $(50)_1$ and the result: $m = \alpha$, we get

$$h_i^\ell(U_j, X) = \alpha\theta(X)\delta_{ij}.$$

Taking $X = \zeta$ to this and using $(27)_2$ such that $m = \alpha$, we have $\alpha = 0$.

As $\alpha = m = 0$ and $\beta = -\ell \neq 0$, $\bar{M}$ is an indefinite β-Kenmotsu manifold with a semi-symmetric non-metric connection and $f_1 + \beta^2 = f_2$ by Theorem 4.1. Taking $PZ = U_j$ to (45) and using $(46)_1$, (50) and (51), we have

$$f_2\{[v_j(Y)\eta_i(X) - v_j(X)\eta_i(Y)] + [v_i(Y)\eta_j(X) - v_i(X)\eta_j(Y)]\} = 0.$$

Taking $X = \xi_i$ and $Y = U_j$, we get $f_2 = 0$. Thus $f_1 = -\beta^2$ and $f_3 = -\zeta\beta$.

(2) If M is screen conformal, then, from $(28)_2$, $(29)_2$ and (49), we have

$$(\ell+\beta)\eta_i(X)+(m-\alpha)v_i(X)=\varphi_i(m-\alpha)u_i(X)\}.$$

Taking $X=\xi_i$ and $X=V_i$ to this equation by turns, we see that $\ell=-\beta$ and $m=\alpha$, respectively. As $\alpha\beta=0$, it follows that

$$\ell m=\ell\alpha=m\beta=0,\qquad \ell\beta=-\beta^2,\qquad m\alpha=\alpha^2. \tag{52}$$

Denote by μ_i the r-th vector fields on $S(TM)$ such that $\mu_i=U_i-\varphi_iV_i$. Using $(30)_{1,2,3,4}$ and (49), we see that

$$\begin{aligned} &h^\ell_j(X,\mu_i)=0, && h^s_a(X,\mu_i)=0,\\ &g(\mu_i,\mu_j)=-(\varphi_j+\varphi_i)\delta_{ij}, && J\mu_i=N_i-\varphi_i\xi_i. \end{aligned} \tag{53}$$

Applying ∇_X to $\mu_i=U_i-\varphi_iV_i$ and then, taking the scalar product with ζ to the resulting equation and using (31) and (32), we obtain

$$g(\nabla_X\mu_i,\zeta)=-\left\{\alpha\eta_i(X)+\beta v_i(X)-\varphi_i\beta u_i(X)\right\}.$$

Applying $\bar\nabla_X$ to $\theta(\mu_i)=0$ and using (8) and the last equation, we get

$$(\bar\nabla_X\theta)(\mu_i)=\alpha\eta_i(X)+\beta v_i(X)-\varphi_i\beta u_i(X). \tag{54}$$

Applying ∇_Y to (49), we have

$$(\nabla_X h^*_i)(Y,PZ)=(X\varphi_i)h^\ell_i(Y,PZ)+\varphi_i(\nabla_X h^\ell_i)(Y,PZ).$$

Substituting this and (49) into (45) and using (44), we have

$$\begin{aligned}
&\sum_{j=1}^r\{(X\varphi_i)\delta_{ij}-\varphi_i\tau_{ji}(X)-\varphi_j\tau_{ij}(X)-\eta_i(A_{N_j}X)\}h^\ell_j(Y,PZ)\\
&-\sum_{j=1}^r\{(Y\varphi_i)\delta_{ij}-\varphi_i\tau_{ji}(Y)-\varphi_j\tau_{ij}(Y)-\eta_i(A_{N_j}Y)\}h^\ell_j(X,PZ)\\
&-\sum_{a=r+1}^n\{\epsilon_a\rho_{ia}(X)+\varphi_i\lambda_{ai}(X)\}h^s_a(Y,PZ)\\
&+\sum_{a=r+1}^n\{\epsilon_a\rho_{ia}(Y)+\varphi_i\lambda_{ai}(Y)\}h^s_a(X,PZ)\\
&-(\bar\nabla_X\theta)(PZ)\{\ell\eta_i(Y)+mv_i(Y)-\varphi_i mu_i(Y)\}\\
&+(\bar\nabla_Y\theta)(PZ)\{\ell\eta_i(X)+mv_i(X)-\varphi_i mu_i(X)\}\\
&-\theta(PZ)\{[X\ell+\alpha^2\theta(X)]\eta_i(Y)-[Y\ell+\alpha^2\theta(Y)]\eta_i(X)\\
&+(Xm)g(\mu_i,Y)-(Ym)g(\mu_i,X)\}\\
=\;&f_1\{g(Y,PZ)\eta_i(X)-g(X,PZ)\eta_i(Y)\}\\
&+f_2\{g(\mu_i,Y)\bar g(X,JPZ)-g(\mu_i,X)\bar g(Y,JPZ)+2g(\mu_i,PZ)\bar g(X,JY)\}\\
&+f_3\{\theta(X)\eta_i(Y)-\theta(Y)\eta_i(X)\}\theta(PZ).
\end{aligned}$$

Replacing PZ by μ_j to this and using (46) and (52) $\sim$ (54), we obtain

$$\begin{aligned}(f_1+\beta^2)&\{v_j(Y)\eta_i(X)-v_j(X)\eta_i(Y)\}\\ -\ &\varphi_j(f_1+\beta^2)\{u_j(Y)\eta_i(X)-u_j(X)\eta_i(Y)\}\\ +\ &(f_2+\alpha^2)\{v_i(Y)\eta_j(X)-v_i(X)\eta_j(Y)\}\\ -\ &\varphi_i(f_2+\alpha^2)\{u_i(Y)\eta_j(X)-u_i(X)\eta_j(Y)\}\\ =\ &2f_2\delta_{ij}(\varphi_j+\varphi_i)\bar{g}(X,JY).\end{aligned} \tag{55}$$

Taking $X=\xi_i$ and $Y=V_j$ to this equation, we obtain

$$f_1+f_2=-(\alpha^2+\beta^2).$$

From this result and Theorem 5.1, we see that $\alpha=0$. As $\alpha=m=0$ and $\beta=-\ell\neq 0$, $\bar{M}(f_1,f_2,f_3)$ is an indefinite β-Kenmotsu manifold with a semi-symmetric non-metric connection. Taking $X=\xi_j$ and $Y=U_j$ to the modified equation (55) which is change j with i, we obtain $f_2\varphi_i=0$. As all φ_i are non-vanishing functions, we get $f_2=0$. Thus $f_1=-\beta^2$ and $f_3=-\zeta\beta$. ☐

Theorem 4.3. *Let M be a generic lightlike submanifold of an indefinite generalized Sasakian space form $\bar{M}(f_1,f_2,f_3)$ with an (ℓ,m)-type connection such that ζ is tangent to M. If U_i's or V_i's are parallel with respect to ∇, then $\bar{M}(f_1,f_2,f_3)$ is a flat manifold with an indefinite cosymplectic structure*;

$$\alpha=\beta=0,\qquad f_1=f_2=f_3=0.$$

Proof. (1) If U_i's are parallel with respect to the connection ∇, then we have the equations of (38). As $\alpha=\beta=0$, we get $f_1=f_2=f_3$ by Theorem 4.1. Applying ∇_Y to $(38)_5$ and using the fact that $\nabla_Y U_j=0$, we obtain

$$(\nabla_X h_i^*)(Y,U_j)=0.$$

Substituting this equation and (38) into (45) with $PZ=U_j$, we have

$$f_1\{v_j(Y)\eta_i(X)-v_j(X)\eta_i(Y)\}+f_2\{v_i(Y)\eta_j(X)-v_i(X)\eta_j(Y)\}=0.$$

Taking $X=\xi_i$ and $Y=V_j$ to this equation, we get $f_1+f_2=0$. Thus we see that $f_1=f_2=f_3=0$ and $\bar{M}$ is flat.

(2) If V_i's are parallel with respect to the connection ∇, then we have the equations in (39). As $\alpha=\beta=0$, $f_1=f_2=f_3$ by Theorem 4.1. Taking $Y=\xi_j$ and $Y=U_j$ to (20) by turns and using $(39)_{3,5}$, we have

$$h_i^\ell(\xi_j,X)=0,\qquad h_i^\ell(U_j,X)=m\theta(X)\delta_{ij}.$$

Using these two equations and (30), we see that

$$\begin{aligned}&h_k^\ell(\xi_i,V_j)=0,\qquad h_a^s(\xi_i,V_j)=\epsilon_a h_j^\ell(\xi_i,W_a)=0,\\ &h_k^\ell(U_j,V_j)=0,\qquad h_a^s(U_j,V_j)=\epsilon_a h_j^\ell(U_j,W_a)=0.\end{aligned} \tag{56}$$

From $(30)_1$ and $(39)_5$ and using the fact that $\nabla_X V_i=0$, we have

$$h_i^*(Y,V_j)=0.$$

Applying ∇_X to this equation and using the fact that $\nabla_X V_j = 0$, we have

$$(\nabla_X h_i^*)(Y, V_j) = 0.$$

Substituting the last two equations into (45) such that $PZ = V_j$, we get

$$\begin{aligned} &\sum_{a=r+1}^{n} \epsilon_a \{\rho_{ia}(Y) h_a^s(X, V_j) - \rho_{ia}(X) h_a^s(Y, V_j)\} \\ &+ \sum_{k=1}^{r} \{h_k^\ell(X, V_j)\eta_i(A_{N_k} Y) - h_k^\ell(Y, V_j)\eta_i(A_{N_k} X)\} \\ &= f_1\{u_j(Y)\eta_i(X) - u_j(X)\eta_i(Y)\} + 2f_2\delta_{ij}\bar{g}(X, JY). \end{aligned}$$

Taking $X = \xi_i$ and $Y = U_j$ to this equation and using (56), we obtain $f_1 + 2f_2 = 0$. It follows that $f_1 = f_2 = f_3 = 0$ and $\bar{M}$ is flat. □

References

[1] P. Alegre, D. E. Blair, and A. Carriazo, *Generalized Sasakian-space-forms*, Israel J. Math. **141** (2004), 157–183. https://doi.org/10.1007/BF02772217

[2] C. Călin, *Contributions to geometry of CR-submanifold*, Thesis, University of Iasi (Romania), 1998.

[3] K. L. Duggal and D. J. Jin, *Generic lightlike submanifolds of an indefinite Sasakian manifold*, Int. Electron. J. Geom. **5** (2012), no. 1, 108–119.

[4] D. H. Jin, *Indefinite trans-Sasakian manifold admitting an ascreen lightlike hypersurface*, Honam Math. J. **35** (2013), no. 4, 657–666. https://doi.org/10.5831/HMJ.2013.35.4.657

[5] ______, *Indefinite generalized Sasakian space form admitting a generic lightlike submanifold*, Bull. Korean Math. Soc. **51** (2014), no. 6, 1711–1726. https://doi.org/10.4134/BKMS.2014.51.6.1711

[6] ______, *Generic lightlike submanifolds of an indefinite trans-Sasakian manifold of a quasi-constant curvature*, Appl. Math. Sci., **9** (2015), no. 60, 2985–2997.

[7] ______, *Lightlike hypersurfaces of an indefinite trans-Sasakian manifold with an (ℓ, m)-type connection*, J. Korean Math. Soc. **55** (2018), no. 5, 1075–1089. https://doi.org/10.4134/JKMS.j170538

[8] D. H. Jin and J. W. Lee, *Generic lightlike submanifolds of an indefinite cosymplectic manifold*, Math. Probl. Eng. **2011** (2011), Art. ID 610986, 16 pp. https://doi.org/10.1155/2011/610986

[9] ______, *Generic lightlike submanifolds of an indefinite Kaehler manifold*, Inter. J. Pure and Appl. Math. **101** (2015), no. 4, 543–560.

[10] ______, *Generic lightlike submanifolds of an indefinite trans-Sasakian manifold with a non-metric ϕ-symmetric connection*, Ann. Math. Inform. **47** (2017), 85–107.

[11] J. A. Oubiña, *New classes of almost contact metric structures*, Publ. Math. Debrecen **32** (1985), no. 3-4, 187–193.

Chul Woo Lee
Department of Mathematics
Kyungpook National University
Daegu 41566, Korea
Email address: mathisu@knu.ac.kr

Jae Won Lee
Department of Mathematics Education and RINS
Gyeongsang National University
Jinju 38066, Korea
Email address: leejaew@gnu.ac.kr