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ASYMPTOTIC EVALUATION OF $\int_0^\infty \left(\frac{\sin x}{x}\right)^n dx$

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Abstract. We consider the integral $\int_0^\infty \left(\frac{\sin x}{x}\right)^n dx$ as a function of the positive integer n. We show that there exists an asymptotic series in $\frac{1}{n}$ and compute the first terms of this series together with an explicit error bound.

1. Introduction and results

In this note we show the following.

Theorem 1.1. *There exists an asymptotic series*

$$\int_0^\infty \left(\frac{\sin x}{x}\right)^n dx \approx \frac{\pi}{2}\left(1+\sum_{\nu=1}^\infty \frac{a_\nu}{n^\nu}\right).$$

Here the series on the right is an asymptotic series in the sense of Poincaré, that is, we write $f(n) \approx \sum_{\nu=0}^\infty \frac{a_\nu}{n^\nu}$, if for every fixed k we have $f(n) = \sum_{\nu=0}^k \frac{a_\nu}{n^\nu} + \mathcal{O}(n^{-k-1})$. We do not know whether the series is converging for n sufficiently large, but we seriously doubt it. In fact, unless there is an unexpected amount of cancelation, the coefficients a_ν grow about as fast as $\nu!$.

Our main motivation for studying this integral is the fact that the intersection of the unit cube with a plane passing through the midpoint, which is orthogonal to a diagonal of the cube has $(n-1)$-dimensional measure equal to $\frac{2\sqrt{n}}{\pi}\int_0^\infty \left(\frac{\sin t}{t}\right)^n dt$. For an overview of this and related results we refer the reader to the work of Chakerian and Logothetti [3]. As the sum of n independent random variables uniformly distributed on $[0,1]$ equals the L^1-norm of a random point in the unit cube, these intersections occur naturally in probabilistic problems. For an example we refer the reader to the work of Silberstein [6].

For $n=1$ the integrand is the sinc-function, which plays a crucial rôle in signal processing, as witnessed by the importance of the Nyquist-Schannon sampling theorem, for a historical overview we refer the reader to [4]. After a suitable renormalisation, the function $\left(\frac{\sin x}{x}\right)^n$ is the Fourier transform of the

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B-spline of order n, which are also of importance in the theory of sampling. For an overview we refer the reader to the work of Butzer, Splettstößer and Stens [2], in particular Section 4.1.

The coefficients a_ν can be computed, and the error in the approximation can be bounded effectively. As an example, we compute the following.

Proposition 1.2. *For* $n \notin \{2,4,6,8,10\}$ *we have*

$$\int_0^\infty \left(\frac{\sin x}{x}\right)^n dx$$
$$= \sqrt{\frac{3\pi}{2n}}\left(1-\frac{3}{20n}-\frac{13}{1120n^2}+\frac{27}{3200n^3}+\frac{52791}{3942400n^4}+\widetilde{\mathcal{O}}\left(\frac{7.26\cdot 10^{-3}}{n^5}\right)\right).$$

Here $\widetilde{\mathcal{O}}$ denotes the effective Landau symbol, that is, the implied constant in the Landau symbol is of absolute value at most 1.

As an application we have the following, which answers a question by Schneider [5].

Proposition 1.3. *For all* $n \neq 4$ *we have*

$$2^{1-n}\cdot n\cdot\binom{n-1}{\lfloor (n-1)/2\rfloor} > \left(\frac{2}{\pi}\int_0^\infty \left(\frac{\sin x}{x}\right)^n dx\right)^{-1}. \tag{1}$$

Schneider [5] showed that (1) holds for all large n and stated that it probably holds for all $n \neq 4$. In particular, Proposition 1.3 shows that the condition "$n = 3$ or n sufficiently large" in [5, Theorem 2] can be replaced by $n \neq 4$.

All computations were performed using Mathematica 11.3.

2. Existence of an asymptotic series

We begin by restricting the relevant range of the integral.

Lemma 2.1. *We have*

$$0 \le \int_1^\infty \left(\frac{\sin x}{x}\right)^n dx \le e^{-n/6}.$$

Proof. For $n = 1, 2$ the claim follows by direct computation, hence, from now on we assume that $n \geq 3$. In particular, the integral converges absolutely.

The lower bound is trivial for n even. For n odd the integral consists of a positive integral over the range $[1, \frac{\pi}{2}]$, and a sequence of ranges of the form $[2k\pi + \frac{\pi}{2}, (2k+2)\pi + \frac{\pi}{2}]$. Considering a single interval of this form we have

$$\int_{2k\pi+\pi/2}^{2(k+1)\pi+\pi/2} \left(\frac{\sin x}{x}\right)^n dx$$
$$= \int_{\pi/2}^{\pi}\left(\frac{1}{(2k\pi+t)^n}-\frac{1}{((2k+2)\pi-t)^n}-\frac{1}{((2k+1)\pi+t)^n}+\frac{1}{(2k+3)\pi-t)^n}\right)\sin^n t,$$

and the integrand is positive as $\frac{1}{t^n}$ is convex. For the upper bound we split the integral into the range $[1, \frac{\pi}{2}]$ and $[\frac{\pi}{2}, \infty]$. For the first range we have

$$\frac{\sin x}{x} \leq \sin 1 = 0.8414 \cdots < 0.8464 \cdots = e^{1/6},$$

hence, the contribution of this range is at most $(\frac{\pi}{2} - 1)e^{-n/6}$. For the second part we use $\sin x \leq 1$, and see that the integral is bounded by $\int_{\pi/2}^\infty \frac{dx}{x^n} = \frac{1}{n-1}\left(\frac{\pi}{2}\right)^{n-1}$. Our claim now follows. □

Lemma 2.2. *Let* $\sum_{k\geq 1} a_k x^k$ *be the Taylor series of* $\log \frac{\sin x}{x}$. *Then we have* $|a_k| < 0.517 \cdot 3^{-k}$, *and for* $|x| < \frac{\pi}{2}$ *and* K *even the error bound*

$$\left| \log \frac{\sin x}{x} - \sum_{k=1}^K a_n x^n \right| < 1.09 \left(\frac{x}{3}\right)^{K+2}.$$

Proof. Put $f(z) = \log \frac{\sin z}{z}$. Note that z is holomorphic for $|z| < \pi$ and symmetric. We have for $0 < r < \pi$

$$|a_k| = \left|\frac{f^{(k)}(0)}{n!}\right| = \frac{1}{2\pi}\left|\int\limits_{\partial B_r(0)} \frac{f(\zeta)}{(-\zeta)^{k+1}}\, d\zeta\right| \leq \frac{1}{2\pi r^{k+1}} \int\limits_{\partial B_r(0)} |f(\zeta)|\, d\zeta.$$

We choose $r = 3$ and compute the integral numerically to be $M = 9.733\ldots$, and obtain the claimed bound for $|a_k|$. We conclude that for $|x| \leq \frac{\pi}{2}$

$$\begin{aligned}\left|\log \frac{\sin x}{x} - \sum_{k=1}^K a_k x^k\right| = \left|\sum_{k=K+2}^\infty a_k x^k\right| &\leq 0.517\left(\frac{x}{3}\right)^{K+2} \sum_{k=0}^\infty \left(\frac{\pi}{6}\right)^k \\ &\leq 1.09\left(\frac{x}{3}\right)^{K+2}.\end{aligned}$$

□

Lemma 2.3. *For* $|x| \leq 1$ *we have*

$$\log\left(\frac{\sin x}{x}\right) \leq -\frac{x^2}{6}.$$

Proof. We compute the first coefficients of the Taylor series as

$$\log\left(\frac{\sin x}{x}\right) = -\frac{x^2}{6} - \frac{x^4}{180} - \frac{x^6}{2835} + \mathcal{O}(x^8).$$

By Lemma 2.2 we obtain

$$\log\left(\frac{\sin x}{x}\right) < -\frac{x^2}{6} - \frac{x^4}{180} - \frac{x^6}{2835} + 1.09\left(\frac{x}{3}\right)^8.$$

For $x \leq 1$ the second term always dominates the last one, and our claim follows. □

By Lemma 2.3 we obtain

$$\int_{n^{-1/3}}^{1}\left(\frac{\sin x}{x}\right)^n\,dx\le\int_{n^{-1/3}}^{1}\exp\left(-\frac{nx^2}{6}\right)\,dx<e^{-n^{1/3}/6}.$$

We now compute

$$\begin{aligned}\int_0^{n^{-1/3}}\left(\frac{\sin x}{x}\right)^n\,dx &= \int_0^{n^{-1/3}}\exp\left(n\log\left(\frac{\sin x}{x}\right)\right)\,dx\\ &= \int_0^{n^{-1/3}}\exp\left(-\frac{nx^2}{6}+n\sum_{k=2}^{K}a_{2k}x^{2k}+\mathcal{O}\left(nx^{2K+2}\right)\right)\,dx\\ &= \int_0^{n^{-1/3}}\exp\left(-\frac{nx^2}{6}+n\sum_{k=2}^{K}a_{2k}x^{2k}\right)\,dx\\ &\quad+\int_0^{n^{-1/3}}\exp\left(-\frac{nx^2}{6}\right)\left(e^{\mathcal{O}(nx^{2K+2})}-1\right)\,dx\\ &= \int_0^{\frac{n^{1/6}}{\sqrt{3}}}\exp\left(-\frac{u^2}{2}+\sum_{k=2}^{K}a_{2k}\frac{(3u^2)^k}{n^{k-1}}\right)\,\frac{du}{\sqrt{n/3}}\\ &\quad+\int_0^{\frac{n^{1/6}}{\sqrt{3}}}e^{-\frac{u^2}{2}}\left(e^{\mathcal{O}(n^{-K}u^{2K+2})}-1\right)\,\frac{du}{\sqrt{n/3}}.\end{aligned}$$

We first estimate the second integral. For $u<n^{1/6}$, the second factor of the integrand is $\mathcal{O}(n^{-\frac{2}{3}K+\frac{1}{3}})$, thus the integral is bounded by $\int_0^\infty e^{-\frac{u^2}{2}}\,\mathcal{O}(n^{-\frac{2}{3}K})\,du=\mathcal{O}(n^{-\frac{2}{3}K})$.

The first integral contributes to the main term. We have

$$\exp\left(\sum_{k=2}^{K}a_{2k}\frac{(3u^2)^k}{n^{k-1}}\right)=1+\sum_{k=2}^{\infty}\sum_{\frac{k}{2}\le\ell\le k-1}b_{k,\ell}^{(K)}\frac{u^{2k}}{n^\ell},$$

where

$$\begin{aligned}b_{k,\ell}^{(K)}=\frac{3^k}{(k-\ell)!}\sum_{\substack{\kappa_1+\dots+\kappa_{k-\ell}=k\\ 2\le\kappa_i\le K}}\prod_{i=1}^{k-\ell}a_{2\kappa_i} &\le \frac{0.517^{k-\ell}}{3^k(k-\ell)!}\sum_{\substack{\kappa_1+\dots+\kappa_{k-\ell}=k\\ 2\le\kappa_i\le K}}1\\ &\le \frac{0.517^{k-\ell}}{3^k(k-\ell)!}\binom{\ell-1}{k-\ell-1}.\end{aligned}$$

Here we have used the fact that

$$\sum_{\substack{\kappa_1+\dots+\kappa_{k-\ell}=k\\ 2\le\kappa_i\le K}}1\le\sum_{\substack{\kappa_1+\dots+\kappa_{k-\ell}=k\\ 2\le\kappa_i}}1=\binom{\ell-1}{k-\ell-1}.$$

To bound sums involving $b_{k,\ell}$, the following is helpful.

Lemma 2.4. *We have* $\sum_{k/2\le\ell\le k-1}\binom{\ell-1}{k-\ell-1} = F_{k-1}$*, where* F_n *denotes the* n*-th Fibonacci number.*

Proof. We prove our claim by induction on k. For $k \le 3$ the statement is immediate. Note that $\sum_{k/2\le\ell\le k-1}\binom{\ell-1}{k-\ell-1}$ equals the number of possibilities to write k as a sum of integers $\kappa_i \ge 2$. Each such sum either ends with the summand 2, or it ends with a summand > 2. The number of sums of the first kind equals the number of representations of $k-2$, as we can delete the last summand. The number of sums of the second kind equals the number of representations of $k-1$, as we can reduce the last summand by 1. Hence our claim follows. □

Using this notation the first integral is

$$\begin{aligned}
&\int_0^{\frac{n^{1/6}}{\sqrt{3}}} \exp\left(-\frac{u^2}{2} + \sum_{k=2}^{K} a_{2k}\frac{(3u^2)^k}{n^{k-1}}\right)\,\frac{du}{\sqrt{n/3}} \\
&= \int_0^{\frac{n^{1/6}}{\sqrt{3}}} \exp\left(-\frac{u^2}{2}\right)\left(1 + \sum_{k=2}^{\infty}\sum_{\frac{k}{2}\le\ell\le k-1} b_{k,\ell}^{(K)}\frac{u^{2k}}{n^\ell}\right)\,du \\
&= \int_0^{\frac{n^{1/6}}{\sqrt{3}}} \exp\left(-\frac{u^2}{2}\right)\left(1 + \sum_{k=2}^{K}\sum_{\frac{k}{2}\le\ell\le k-1} b_{k,\ell}^{(K)}\frac{u^{2k}}{n^\ell}\right)\,du \\
&\quad + \mathcal{O}\left(n^{-K/6}\int_0^{\frac{n^{1/6}}{\sqrt{3}}} \exp\left(-\frac{u^2}{2}\right)\left(\sum_{k=K+1}^{\infty}\sum_{\frac{k}{2}\le\ell\le k-1} b_{k,\ell}^{(K)}\right)\,du\right).
\end{aligned}$$

We have

$$\begin{aligned}
\sum_{k=K+1}^{\infty}\sum_{\frac{k}{2}\le\ell\le k-1} b_{k,\ell}^{(K)} &\le \sum_{k=K+1}^{\infty}\sum_{\frac{k}{2}\le\ell\le k-1}\frac{0.517^{k-\ell}}{3^k(k-\ell)!}\binom{\ell-1}{k-\ell-1} \\
&\le \sum_{k=K+1}\frac{1}{3^k}\sum_{\frac{k}{2}\le\ell\le k-1}\binom{\ell-1}{k-\ell-1} = \sum_{k=K+1}^{\infty}\frac{F_{k-1}}{3^k} < 1.
\end{aligned}$$

As both the sum and the integral in the error term converge absolutely, the error is $\mathcal{O}(n^{-K/6})$. For the main term we interchange sum and integral, and extend the integral to $[0,\infty)$ to obtain

$$\begin{aligned}
&\int_0^{\frac{n^{1/6}}{\sqrt{3}}} \exp\left(-\frac{u^2}{2}\right)\left(1 + \sum_{k=2}^{K}\sum_{\frac{k}{2}\le\ell\le k-1} b_{k,\ell}^{(K)}\frac{u^{2k}}{n^\ell}\right)\,du \\
&= \int_0^{\frac{n^{1/6}}{\sqrt{3}}} \exp\left(-\frac{u^2}{2}\right)\,du + \sum_{k=2}^{K}\sum_{\frac{k}{2}\le\ell\le k-1}\frac{b_{k,\ell}^{(K)}}{n^\ell}\int_0^{\frac{n^{1/6}}{\sqrt{3}}} \exp\left(-\frac{u^2}{2}\right)u^{2k}\,du
\end{aligned}$$

$$= \int_0^\infty \exp\left(-\frac{u^2}{2}\right)\,du + \sum_{k=2}^K \sum_{\frac{k}{2}\le \ell\le k-1} \frac{b_{k,\ell}^{(K)}}{n^\ell} \int_0^\infty \exp\left(-\frac{u^2}{2}\right) u^{2k}\,du$$
$$+ \mathcal{O}\left(n^{\frac{2K}{3}} e^{-n^{1/3}}\right)$$
$$= \sqrt{\frac{\pi}{2}}\left(1 + \sum_{\ell=1}^{K-1} \frac{1}{n^\ell} \sum_{k=\ell+1}^{2\ell} b_{k,\ell}^K (2k-1)!!\right) + \mathcal{O}\left(n^{\frac{2K}{3}} e^{-n^{1/3}}\right),$$

where in the last step we used the fact that the moments of the normal distribution are given by

$$\int_0^\infty e^{-\frac{u^2}{2}} u^{2k}\,du = (2k-1)!!\sqrt{\frac{\pi}{2}},$$

where $(2n+1)!! = 1\cdot 3\cdot 5\cdots(2n-1)(2n+1)$ is the double factorial, see e.g. [1, 7.4.4].

The existence of an asymptotic series now follows.

3. Explicite computations

The explicit computation is similar to the asymptotic approach. However, a term of magnitude $e^{-n^{1/3}/6}$, which is negligible for large n, would complicate matters a lot for medium n. On the other hand, we can easily compute the Taylor series of $\log\left(\frac{\sin x}{x}\right)$, and the real coefficients are significantly smaller than Lemma 2.2 predicts. Therefore it is advantageous to choose different parameters. In particular we will compute higher order terms even if they are negligible for large n. Explicitly computing the Taylor series up to x^{10} and estimating the remainder using Lemma 2.2, we see that for $x \le \frac{\pi}{2}$ we have

$$0 \ge \log\frac{\sin x}{x} - P(x) \ge -1.4\cdot 10^{-12} x^{22},$$

where $P(x) = \sum_{k=1}^{10} a_k x^{2k}$. Combining this estimate with Lemma 2.3 and the fact that for $\delta > 0$

$$\min(1, e^\delta - 1) \le \delta \max_{\delta\in[0,\log 2} \frac{e^\delta - 1}{\delta} = \frac{\delta}{\log 2}$$

we obtain

$$\int_0^{\pi/2} \left(\frac{\sin x}{x}\right)^n dx = \int_0^{\pi/2} \exp\left(nP(x)\right)\,dx$$
$$+ \widetilde{\mathcal{O}}\left(2.1\cdot 10^{-12} n \int_0^{\pi/2} e^{-n\frac{x^2}{6}} x^{22}\,dx\right)$$
$$= \int_0^{\pi/2} \exp\left(nP(x)\right)\,dx$$

$$+\widetilde{\mathcal{O}}\left(2.1\cdot 10^{-12}3^{23/2}n^{-23/2}\sqrt{\frac{\pi}{2}}21!!\right)$$
$$=\int_0^{\pi/2}\exp\left(nP(x)\right)\,dx+\widetilde{\mathcal{O}}\left(11104n^{-23/2}\right).$$

We have

$$\int_0^{\pi/2}\exp\left(nP(x)\right)\,dx=\sqrt{\frac{3}{n}}\int_0^{\frac{\pi\sqrt{n}}{2\sqrt{3}}}e^{-\frac{u^2}{2}}\left(1+\sum_{k\geq 2}\sum_{\frac{k}{2}\leq\ell\leq k-1}\frac{b_{k,\ell}^{(10)}}{n^\ell}u^{2k}\right)\,du.$$

Extracting the terms with $\ell\leq 10$ and extending the integral to $[0,\infty)$ we obtain

$$\sqrt{\frac{3\pi}{2n}}\left(1-\frac{3}{20n}-\frac{13}{1120n^2}+\frac{27}{3200n^3}+\frac{52791}{3942400n^4}+\frac{482427}{66560000n^5}\right.$$
$$-\frac{124996631}{10035200000n^6}-\frac{5270328789}{136478720000n^7}-\frac{7479063506161}{268461670400000n^8}$$
$$\left.+\frac{6921977624613}{56518246400000n^9}+\frac{263185409650739509946 7}{1028632084480000000n^{10}}\right).$$

We have

$$\left(\frac{u+1}{u}\right)^{20}e^{-\frac{(u+1)^2}{2}+\frac{u^2}{2}}\leq e^{\frac{20}{u}-u-\frac{1}{2}},$$

hence, for $u_0>5$ and $k\leq 20$ we obtain

$$\int_{u_0}^\infty e^{-\frac{u^2}{2}}u^k\,du\leq\frac{e^{-\frac{u_0^2}{2}}u_0^k}{1-e^{-3/2}},$$

and we conclude that the error introduced in the extension of the integral is bounded by

$$\sqrt{\frac{3}{n}}\frac{e^{-\frac{\pi n}{12}}}{1-e^{-3/2}}\left(1+\sum_{2\leq k\leq 10}\sum_{\frac{k}{2}\leq\ell\leq k-1}\frac{|b_{k,\ell}^{(10)}|}{n^\ell}\left(\frac{\pi^2 n}{12}\right)^k\right)$$
$$\leq 2.23\cdot e^{-\frac{\pi n}{12}}\left(4.04\cdot 10^{-2}n+8.14\cdot 10^{-4}n^2+1.1\cdot 10^{-5}n^3\right.$$
$$\left.+1.04\cdot 10^{-7}n^4+3.69\cdot 10^{-10}n^5\right)$$
$$\leq 1.59\cdot 10^{-9}n^5e^{-\frac{\pi n}{12}}<n^{-\frac{23}{2}},$$

provided that $n\geq 400$. We conclude that for $n\geq 400$ we have

$$(2)\quad\int_0^\infty\left(\frac{\sin x}{x}\right)^n\,dx$$
$$=\sqrt{\frac{3\pi}{2n}}\left(1-\frac{3}{20n}-\frac{13}{1120n^2}+\frac{27}{3200n^3}+\frac{52791}{3942400n^4}+\widetilde{\mathcal{O}}\left(\frac{7.25\cdot 10^{-3}}{n^5}\right)\right)$$
$$+\widetilde{\mathcal{O}}(e^{-n/6})+\widetilde{\mathcal{O}}\left(\sqrt{\frac{3}{n}}\int_0^{\frac{\pi\sqrt{n}}{2\sqrt{3}}}e^{-\frac{u^2}{2}}\sum_{k\geq 2}\sum_{\max(11,\frac{k}{2})\leq\ell\leq k-1}\frac{b_{k,\ell}^{(10)}}{n^\ell}u^{2k}\,du\right).$$

Next we bound the contribution of a summand with ℓ large. As $e^{-\frac{u^2}{2}}u^{2k}$ increases for $u<\sqrt{2k}$ and decreases for $u>2k$, we have the bound

$$\int_0^{\frac{\pi\sqrt{n}}{2\sqrt{3}}} e^{-\frac{u^2}{2}}u^{2k}\,du \leq \begin{cases} (2k-1)!!, & k\leq \frac{\pi^2 n}{24},\\ \left(\frac{\pi\sqrt{n}}{2\sqrt{3}}\right)^{2k+1} e^{-\frac{\pi^2 n}{24}}, & k>\frac{\pi^2 n}{24}.\end{cases}$$

Hence, the contribution of the range $k>\frac{\pi^2 n}{24}$ is at most

$$\begin{aligned}&\sum_{k>\frac{\pi^2 n}{24}} \left(\frac{\pi\sqrt{n}}{2\sqrt{3}}\right)^{2k+1} e^{-\frac{\pi^2 n}{24}} \sum_{k/2\leq\ell\leq k-1}\frac{|b_{k,\ell}|}{n^\ell}\\ &\leq \sqrt{n}e^{-\frac{\pi^2 n}{24}}\sum_{k>\frac{\pi^2 n}{24}}\sum_{k/2\leq\ell\leq k-1}\frac{0.907^{2k+1}(0.517n)^{k-\ell}}{3^k(k-\ell)!}\binom{\ell-1}{k-\ell-1}\\ &\leq \sqrt{n}e^{-\frac{\pi^2 n}{24}+0.517n}\sum_{k>\frac{\pi^2 n}{24}}\frac{0.907^{2k+1}F_{k-1}}{3^k} < 1.64\sqrt{n}\cdot 0.796^n.\end{aligned}$$

The contribution of the summands belonging to a single $k\leq\frac{\pi^2 n}{24}$ is at most

$$\begin{aligned}(2k-1)!!\sum_{\frac{k}{2}\leq\ell\leq k-1}\frac{b_{k,\ell}}{n^\ell} &\leq \frac{(2k)!}{2^k k!}\sum_{\frac{k}{2}\leq\ell\leq k-1}\frac{0.517^{k-\ell}}{3^k(k-\ell)!n^\ell}\binom{\ell-1}{k-\ell-1}\\ &\leq \frac{F_{k-1}(2k)!}{6^k k!}\sum_{\frac{k}{2}\leq\ell\leq k-1}\frac{0.517^{k-\ell}}{(k-\ell)!n^\ell}\\ &\leq 2\frac{F_{k-1}(2k)!}{6^k k!}\cdot\frac{0.517^{k/2}}{(k/2)!n^{k/2}} < 0.479^k\left(\frac{k}{n}\right)^{k/2}.\end{aligned}$$

Hence, the sum over all $k\geq K$ is bounded by $1.45\cdot 0.479^K\left(\frac{K}{n}\right)^{K/2}$. We take $K=35$.

It remains to bound the range $\ell\geq 11$, $k\leq 34$. Here we obtain by direct computation

$$\sum_{k\leq 34}\sum_{\max(11,\frac{k}{2})\leq\ell\leq k-1}\frac{b_{k,\ell}}{n^\ell}\leq\frac{5.5\cdot 10^7}{n^{11}},$$

provided that $n>400$. We see that the last two error terms in (2) are bounded by

$$e^{-n/6}+1.45\cdot 0.479^{35}\left(\frac{35}{n}\right)^{\frac{35}{2}}+\frac{5.5\cdot 10^7}{n^{11}},$$

which for $n\geq 400$ is bounded by $\frac{10^{-5}}{n^5}$, hence our claim follows in this range.

Finally for $n\leq 400$ we check Proposition 1.2 directly using the following result due to Chakerian and Logothetti [3].

Lemma 3.1. *We have*

$$\int_0^\infty \left(\frac{\sin x}{x}\right)^n\ dx = \frac{\pi}{2^n(n-1)!}\sum_{j=0}^{\lfloor n/2\rfloor}(-1)^j\binom{n}{j}(n-2j)^{n-1}.$$

4. Proof of Proposition 1.3

Note first that for all $n \geq 1$ we have that the n^{-3} term in the asymptotic series in Proposition 1.2 dominates the error term, hence, for all n the right hand side of (1) is at most

$$\sqrt{\frac{\pi n}{6}}\left(1-\frac{3}{20n}-\frac{13}{1120n^2}\right)^{-1}.$$

To estimate the left hand side we use Stirling's formula in the following form, see [1, 6.1.38].

Lemma 4.1. *We have*

$$n! = \left(\frac{n}{e}\right)^n\sqrt{2\pi n}e^{\widetilde{\mathcal{O}}\left(\frac{1}{12n}\right)}.$$

We conclude that

$$\binom{2m}{m} = \frac{2m!}{m!^2} = \frac{2^{2m}}{\sqrt{\pi m}}e^{\widetilde{\mathcal{O}}\left(\frac{1}{6m}\right)}$$

and

$$\binom{2m+1}{m} = \frac{1}{2}\binom{2m+2}{m+1} = \frac{2^{2m+1}}{\sqrt{\pi(m+1)}}e^{\widetilde{\mathcal{O}}\left(\frac{1}{6(m+1)}\right)}.$$

We obtain that the left hand side of (1) is

$$\frac{n}{\sqrt{\pi\lfloor n/2\rfloor}}e^{\widetilde{\mathcal{O}}\left(\frac{1}{3n}\right)} \geq \sqrt{\frac{2n}{\pi}}\left(1-\frac{1}{3n}\right).$$

Hence, to prove Proposition 1.3 it suffices to check that Proposition 1.2 holds, and that

$$\sqrt{\frac{2n}{\pi}}\left(1-\frac{1}{3n}\right) > \sqrt{\frac{\pi n}{6}}\left(1-\frac{3}{20n}-\frac{13}{1120n^2}\right)^{-1},$$

that is,

$$\left(1-\frac{1}{3n}\right)\left(1-\frac{3}{20n}-\frac{13}{1120n^2}\right) > \sqrt{\frac{\pi^2}{12}} = 0.9068\ldots,$$

and we see that this inequality holds for $n \geq 6$. We conclude that Proposition 1.3 holds for $n \notin \{1,2,3,4,5,6,8,10\}$, and by direct inspection we find that it holds for all $n \neq 4$.

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