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ON MULTI SUBSPACE-HYPERCYCLIC OPERATORS

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Abstract. In this paper, we introduce and investigate multi subspace-hypercyclic operators and prove that multi-hypercyclic operators are multi subspace-hypercyclic. We show that if T is M-hypercyclic or multi M-hypercyclic, then T^n is multi M-hypercyclic for any natural number n and by using this result, make some examples of multi subspace-hypercyclic operators. We prove that multi M-hypercyclic operators have somewhere dense orbits in M. We show that analytic Toeplitz operators can not be multi subspace-hypercyclic. Also, we state a sufficient condition for coanalytic Toeplitz operators to be multi subspace-hypercyclic.

1. Introduction and preliminaries

Let H be an infinite-dimensional and separable Hilbert space. We denote the set of all linear continuous operators on H by $B(H)$. We say an operator $T \in B(H)$ is hypercyclic, if there is $x \in H$ such that $orb(T, x)$ is dense in H, where

$$orb(T, x) = \{x, Tx, T^2x, \ldots\}.$$

The concept of hypercyclicity is a notable matter in dynamical systems and studied by mathematicians for many years. One can read [4] and [5] to see some interesting material on this topic.

We say that T is multi-hypercyclic, if there is $\{x_1, x_2, \ldots, x_n\} \subseteq H$ such that $\bigcup_{i=1}^{n} orb(T, x_i)$ is dense in H.

Herrero in [6] conjectured that on a Hilbert space, multi-hypercyclic operators are hypercyclic. Peris in [14] proved that this conjecture is true even for multi-hypercyclic operators that are defined on an arbitrary locally convex space. Also, Miller proved in [12, Theorem 3] that if T is multi-hypercyclic, then T^n is multi-hypercyclic for any natural number n.

Subspace-hypercyclic operators were defined by Madore and Martinez-Avendano in [9]. We say an operator $T \in B(H)$ is subspace-hypercyclic with respect to a closed subspace M of H if there is $x \in H$ such that $orb(T, x) \cap M$ is dense in M. They proved in [9] that subspace-hypercyclic operators do not exist

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on finite-dimensional spaces. Bamerni, Kadets and Kilicman proved in [1] the following interesting theorem.

Theorem 1.1 ([1]). *Let A be a dense subset of a Hilbert space H. Then there exists a non-trivial closed subspace M of H such that $A \cap M$ is dense in M.*

By Theorem 1.1, they stated in [1] that any hypercyclic operator is subspace-hypercyclic. Also, one can discover in [2], [8], and [13], more knowledge about subspace-hypercyclic operators and related topics. Now it is natural to define multi subspace-hypercyclic operators as follows.

Definition. Let $T \in B(H)$ and let M be a closed and non-zero subspace of H. We say that T is multi subspace-hypercyclic with respect to M or multi M-hypercyclic if there exists $\{x_1, x_2, \ldots, x_n\} \subseteq H$ such that

$$\overline{\bigcup_{i=1}^{n}(orb(T, x_i) \cap M)} = M.$$

We say $F = \{x_1, x_2, \ldots, x_n\} \subseteq H$ is a minimal set for multi M-hypercyclicity for $T \in B(H)$ if for any $E \subset F$ we have

$$\overline{\bigcup_{x_i \in E}(orb(T, x_i) \cap M)} \neq M.$$

It is clear by the definition that subspace-hypercyclic operators are multi subspace-hypercyclic.

In this paper, we investigate multi subspace-hypercyclic operators. In Section 2, we prove that multi-hypercyclic operators are multi subspace-hypercyclic. We show that if T is M-hypercyclic or multi M-hypercyclic, then T^n is multi M-hypercyclic for any natural number n and by this, make examples of multi subspace-hypercyclic operators. We prove that multi M-hypercyclic operators have somewhere dense orbits in M. Also, we show that if T is a multi M-hypercyclic operator, then $ker(T^* - \lambda) \subseteq M^{\perp}$.

In Section 3, we show that analytic Toeplitz operators can not be multi subspace-hypercyclic. We state a sufficient condition for coanalytic Toeplitz operators to be multi subspace-hypercyclic.

2. Some results and examples

In this section, we present some properties of multi subspace-hypercyclic operators. First, we show that multi-hypercyclic operators are multi subspace-hypercyclic.

Theorem 2.1. *Let $T \in B(H)$ be a multi-hypercyclic operator. Then T is multi subspace-hypercyclic with respect to a closed and non-trivial subspace M of H.*

Proof. Let $x_1, x_2, \ldots, x_n$ be elements in H such that $\bigcup_{1 \leq i \leq n} orb(T, x_i)$ is dense in H. By Theorem 1.1, there exists a closed and non-trivial subspace M of H

such that $(\bigcup_{1\leq i\leq n} orb(T,x_i))\cap M$ is dense in M. But

$$\overline{\bigcup_{1\leq i\leq n}(orb(T,x_i)\cap M)} = \overline{(\bigcup_{1\leq i\leq n} orb(T,x_i))\cap M}.$$

Hence T is multi subspace-hypercyclic with respect to M. ☐

In the next theorem, we show that multi subspace-hypercyclicity of T implies multi subspace-hypercyclicity of T^n.

Theorem 2.2. *Let $T \in B(H)$. If T is multi subspace-hypercyclic with respect to M, then T^n is multi subspace-hypercyclic with respect to M for any $n \in \mathbb{N}$.*

Proof. When $n = 1$ the proof is obvious. Consider that $n \geq 2$. By hypothesis, T is multi M-hypercyclic. So there is $x_1, x_2, \ldots, x_m$ in H such that $\bigcup_{1\leq i\leq m}(orb(T,x_i)\cap M)$ is dense in M. Let $y_{i,j} = T^j x_i$, where $1 \leq i \leq m$ and $1 \leq j \leq n-1$. Consider that

$$\bigcup_{1\leq i\leq m}(orb(T,x_i)\cap M) = \bigcup_{\substack{1\leq i\leq m\\ 0\leq j\leq n-1}}(orb(T^n,y_{i,j})\cap M). \tag{1}$$

The left side of (1) is dense in M. So the other side of (1) must be dense in M too. So T^n is multi subspace-hypercyclic with respect to M. ☐

Now by Theorem 2.1 and Theorem 2.2, we can conclude that if T is multi-hypercyclic, then T^n is multi subspace-hypercyclic for any natural number n.

Theorem 2.3. *Let $T \in B(H)$. Suppose that there exists $n \in \mathbb{N}$ such that T^n is multi subspace-hypercyclic with respect to M. Then T is multi subspace-hypercyclic with respect to M.*

Proof. Let n be a positive integer greater than or equal to 2 such that T^n is multi subspace-hypercyclic with respect to M. So there exist $x_1, x_2, \ldots, x_m$ in H such that

$$\overline{\bigcup_{1\leq i\leq m}(orb(T^n,x_i)\cap M)} = M. \tag{2}$$

But

$$orb(T^n,x_i)\cap M \subseteq orb(T,x_i)\cap M. \tag{3}$$

Now by (2) and (3), we conclude that $\bigcup_{1\leq i\leq m}(orb(T,x_i)\cap M)$ is dense in M. Therefore T is multi M-hypercyclic. ☐

It is said in Section 1 that subspace-hypercyclic operators are multi subspace-hypercyclic. In the next theorem, we show that if T is subspace-hypercycclic, then T^n is multi subspace-hypercyclic for any natural number n.

Theorem 2.4. *Let $T \in B(H)$ be an M-hypercyclic operator. Then T^n is multi M-hypercyclic for any $n \in \mathbb{N}$.*

Proof. It is clear when $n = 1$. Now let $n \geq 2$. Let x be an M-hypercyclic vector for T. Hence $\overline{orb(T,x) \cap M} = M$. But if we consider $x_1 := x$, $x_2 := Tx, \ldots$, $x_n := T^{n-1}x$, then

$$\begin{aligned}
&\bigcup_{j=1}^{n}(orb(T^n, T^{j-1}x) \cap M) \\
&= (orb(T^n, x) \cup orb(T^n, Tx) \cup \cdots \cup orb(T^n, T^{n-1}x)) \cap M \\
&= \{x, Tx, \ldots, T^{n-1}x, T^n x, T^{n+1}x, \ldots\} \cap M \\
&= orb(T,x) \cap M.
\end{aligned}$$

Therefore T^n is multi M-hypercyclic. □

By using Theorem 2.4, we can make some examples as follows.

Example 2.5. Let T be a hypercyclic operator on a Hilbert space H. If we consider $T \oplus I : H \oplus H \to H \oplus H$, then $T \oplus I$ is subspace-hypercyclic with respect to $M := H \oplus \{0\}$. Now by Theorem 2.4, we can deduce that $(T \oplus I)^n = T^n \oplus I$ is multi M-hypercyclic for any $n \in \mathbb{N}$.

Example 2.6. Let $\lambda \in \mathbb{C}$ with $|\lambda| > 1$. Suppose that B is the backward shift on l^2. It is proved in [11, Corollary 2] that $T = \lambda B$ is subspace-hypercyclic with respect to any finite-codimensional subspace. So we can conclude from Theorem 2.4 that $T^n = \lambda^n B^n$ is multi M-hypercyclic for any finite-codimensional subspace M of l^2.

Corollary 2.7. *Let $T \in B(H)$ be an operator with this property that $T^n = I$ for some $n \in \mathbb{N}$. Then T can not be subspace-hypercyclic. Especially, if $T = T^{-1}$ then T can not be subspace-hypercyclic.*

Proof. Consider that $T^n = I$ for some $n \in \mathbb{N}$. Without loss of generality, we can assume that $n \geq 2$. Suppose on contrary that T is subspace-hypercyclic. By Theorem 2.4, T^n must be multi subspace-hypercyclic. But this is impossible since the identity operator can not be multi subspace-hypercyclic. □

We show in the following that multi subspace-hypercyclic operators have somewhere dense orbits. For this, first, note to the following lemma.

Lemma 2.8 ([14])**.** *Let X be a topological space and let $F_1, F_2, \ldots, F_n$ be a finite family of closed subsets of X such that $X = \bigcup_{j=1}^{n} F_i$. If $(F_1)^\circ = \phi$, then $X = \bigcup_{j=2}^{n} F_i$.*

Theorem 2.9. *Let $T \in B(H)$ be a multi M-hypercyclic operator. Then there exists $x \in H$ such that $orb(T,x) \cap M$ is somewhere dense in M.*

Proof. Let T be a multi M-hypercyclic operator with a minimal set $\{x_1, x_2, \ldots, x_n\}$ for multi M-hypercyclicity. So

$$\overline{\bigcup_{1 \leq i \leq n}(orb(T,x_i) \cap M)} = \bigcup_{1 \leq i \leq n} \overline{(orb(T,x_i) \cap M)} = M.$$

Suppose that there is a x_{i_0}, where $1 \leq i_0 \leq n$, such that $orb(T, x_{i_0}) \cap M$ is nowhere dense. That means $(\overline{orb(T, x_{i_0}) \cap M})^\circ = \phi$. Then by Lemma 2.8, $\bigcup_{\substack{1 \leq i \leq n \\ i \neq i_0}} \overline{(orb(T, x_i) \cap M)} = M$. But this contradicts to minimality of $\{x_1, x_2, \dots, x_n\}$. So $(\overline{orb(T, x_{i_0}) \cap M})^\circ \neq \phi$. Therefore $orb(T, x_{i_0}) \cap M$ is somewhere dense in M. Similarly, for any other x_i with $1 \leq i \leq n$, we can conclude that $orb(T, x_i) \cap M$ is somewhere dense in M. □

In the proof of Theorem 2.9, we can assume without loss of generality that $x_i \in M$ for any $1 \leq i \leq n$. So we can say that if $T \in B(H)$ is a multi M-hypercyclic operator, then there exists $x \in M$ such that $orb(T, x) \cap M$ is somewhere dense in M.

Bourdon and Feldman in [3] showed that for a locally convex space X and for $T \in B(X)$, somewhere density of $orb(T, x)$ in X implies density of $orb(T, x)$ in X. But this result is not true for subspace somewhere dense orbits. In [7] authors constructed an operator T on a Banach space X such that $Orb(T, x) \cap M$ is somewhere dense in M, but it is not everywhere dense in M.

By above note and this fact that multi-hypercyclic operators are hypercyclic, the following question arises:

Question. Let T be multi M-hypercyclic. Can we conclude that T is M-hypercyclic?

Theorem 2.10. *Let $T \in B(H)$ be a multi M-hypercyclic operator. Then $ker(T^{*^n} - \lambda) \subseteq M^\perp$ for any $n \in \mathbb{N}$.*

Proof. Let T be a multi M-hypercyclic operator, where M is a closed and non-trivial subspace of H. First, we prove the case $n = 1$. Suppose that $\varphi \in ker(T^* - \lambda)$. We show that $\varphi = 0$ on M. Let $\{x_1, x_2, \dots, x_n\}$ be a minimal set for multi M-hypercyclicity of T. So $\overline{\bigcup_{1 \leq i \leq n} (orb(T, x_i) \cap M)} = M$. By Theorem 2.9, $orb(T, x_1) \cap M$ is somewhere dense in M. Without loss of generality, we can assume that $x_1 \in orb(T, x_1) \cap M$ and $0 \in orb(T, x_1) \cap M$. So there exist increasing sequences $\{n_k\}$ and $\{m_k\}$ of natural numbers such that $T^{n_k} x_1 \to x_1$ and $T^{m_k} x_1 \to 0$, where $T^{n_k} x_1, T^{m_k} x_1 \in M$ for any k. Hence

$$\varphi(T^{n_k} x_1) \to \varphi(x_1) \quad \text{and} \quad \varphi(T^{m_k} x_1) \to 0.$$

But $\varphi \in ker(T^* - \lambda)$. So $\varphi(T x_1) = \lambda(\varphi x_1)$. Therefore

$$\lambda^{n_k} \varphi(x_1) \to \varphi(x_1) \quad \text{and} \quad \lambda^{m_k} \varphi(x_1) \to 0.$$

Hence $\varphi(x_1) = 0$ and therefore

$$orb(T, x_1) \cap M \subseteq ker(\varphi) \cap M.$$

But $\overline{orb(T, x_1) \cap M}$ contains an open set in M. Hence $\overline{ker(\varphi) \cap M} = ker(\varphi) \cap M$ contains an open set in M. But this indicates that $\varphi|_M = 0$. Hence $\varphi \in M^\perp$.

Now by Theorem 2.3 and note to this fact that $(T^*)^n = (T^n)^*$, we can infer that $ker(T^{*n} - \lambda) \subseteq M^\perp$ for $n \geq 2$. □

3. Multi subspace-hypercyclicity of Toeplitz operators

In this section, we peruse the multi subspace-hypercyclicity of Toeplitz operators. Let us recall some preliminaries.

Definition ([10, Definition 1.1.1])**.** The Hardy-Hilbert space consists of analytic functions such that having power series representation with square-summable complex coefficients and it is denoted by $\mathbf{H}^2$. That means

$$\mathbf{H}^2 = \{f : f(z) = \Sigma_{n=0}^{\infty} a_n z^n \text{ and } \Sigma_{n=0}^{\infty} |a_n|^2 < \infty\}.$$

It is not hard to see that $\mathbf{H}^2$ is a Hilbert space and any function in $\mathbf{H}^2$ is analytic on the unit disk ([10]).

The symbol $\mathbb{D}$ is used to indicate the open unit disk in $\mathbb{C}$. That is $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Also, The symbol S^1 is used to indicate the unit circle in $\mathbb{C}$. That is $S^1 = \{z \in \mathbb{C} : |z| = 1\}$.

Let $L^2 = L^2(S^1)$ be the Hilbert space of square-integrable functions on S^1 with respect to Lebesque measure, normalized so that the measure of the entire circle is 1. The Hardy-Hilbert space is a subspace of L^2 [10, p. 5].

The space H^∞ consists of all functions that are bounded and analytic on the $\mathbb{D}$. When $f \in H^\infty$, we define $||f||_\infty = \sup\{|f(z)| : z \in \mathbb{D}\}$. It is proved that $H^2 \cap L^\infty = H^\infty$, where $L^\infty = L^\infty(S^1)$ is the set of essentially bounded functions on S^1(one can see [11] and [10, Definition 1.1.23]).

Definition ([10])**.** Let $\phi \in L^\infty$. We denote the Toeplitz operator with symbol ϕ by T_ϕ and for any $f \in \mathbf{H}^2$ we define it by $T_\phi f = P\phi f$, where P is the orthogonal projection of L^2 onto $\mathbf{H}^2$.

If $\phi \in H^\infty$, we say that T_ϕ is an analytic Toeplitz operator. We say T_ϕ is a coanalytic Toeplitz operator if T_ϕ^* is analytic.

Note that if $\phi \in H^\infty$, then for any $f \in \mathbf{H}^2$ we have $T_\phi f = P\phi f = \phi f$. Also, for $\phi \in H^\infty$ since $T_\phi^* = T_{\overline{\phi}}$, we can conclude that T_ϕ is coanalytic if and only if $\overline{\phi} \in H^\infty$.

It is established in [9] that an analytic Toeplitz operator can not be subspace-hypercyclic. In the next theorem, we extend their conclusion and we show that an analytic Toeplitz operator can not be multi subspace-hypercyclic. First, recall that for any $\lambda \in \mathbb{D}$, reproducing kernel $k_\lambda : \mathbb{D} \to \mathbb{C}$ defined by:

$$k_\lambda(z) = \Sigma_{n=0}^{\infty} \overline{\lambda}^n z^n = \frac{1}{1 - \overline{\lambda} z}.$$

It is proved that $k_\lambda \in \mathbf{H}^2$ and if $\Delta \subseteq \mathbb{D}$ be a set with an accumulation point in $\mathbb{D}$, then $span\{k_\lambda : \lambda \in \Delta\}$ is dense in $\mathbf{H}^2$ (one can see [5, Proposition 4.38 and Lemma 4.39]).

Theorem 3.1. *An analytic Toeplitz operator can not be multi subspace-hypercyclic with respect to a closed and non-trivial subspace M of $\mathbf{H}^2$.*

Proof. Let T_ϕ be an analytic Toeplitz operator. Suppose on contrary that T_ϕ is multi subspace-hypercyclic with respect to a closed and non-trivial subspace M of $\mathbf{H}^2$. Let $\lambda \in \mathbb{D}$ and let k_λ be the reproducing kernel of λ. So $k_\lambda \in \ker(T^*_\phi - \overline{\phi(\lambda)})$. But $\ker(T^*_\phi - \overline{\phi(\lambda)}) \subseteq M^\perp$. Therefore $k_\lambda \in M^\perp$. So

$$\overline{span\{k_\lambda : \lambda \in \mathbb{D}\}} \subseteq M^\perp.$$

That means $\mathbf{H}^2 \subseteq M^\perp$ and this is a contradiction. So T_ϕ can not be multi M-hypercyclic. □

Godefroy and Shapiro in [4] state a sufficient condition for hypercyclicity of coanalytic Toeplitz operators as follows.

Theorem 3.2 ([4])**.** *Let $\phi \in H^\infty$ be non-constant. If $\phi(\mathbb{D}) \cap S^1$ is non-empty, then T^*_ϕ is hypercyclic.*

By Theorem 3.2, we can express a sufficient condition for multi subspace-hypercyclicity of powers of a coanalytic Toeplitz operator.

Corollary 3.3. *Let $\phi \in H^\infty$ be non-constant. If $\phi(\mathbb{D}) \cap S^1$ is non-empty, then T^{*n}_ϕ is multi subspace-hypercyclic for any $n \in \mathbb{N}$.*

Proof. By hypothesis, $\phi(\mathbb{D}) \cap S^1$ is non-empty. So by Theorem 3.2, T^*_ϕ is hypercyclic. By Theorem 1.1, we can find a non-trivial and closed subspace M of $\mathbf{H}^2$ such that T^*_ϕ is M-hypercyclic. Now we can conclude by Theorem 2.4 that T^{*n}_ϕ is multi M-hypercyclic for any $n \in \mathbb{N}$. □

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