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EXTENDED GENERALIZED MITTAG-LEFFLER FUNCTION APPLIED ON FRACTIONAL INTEGRAL INEQUALITIES

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Abstract. This paper presents several fractional generalizations and extensions of known integral inequalities. To obtain these, an extended generalized Mittag-Leffler function and its fractional integral operator are used.

1. Introduction

Inequalities which involve integrals of functions and their derivatives, whose study has a history of about one century, are of great importance in mathematics, with far-reaching applications in the theory of differential equations, approximations and probability, among others. They occupy a central position in mathematical analysis and its applications.

In recent years considerable interest in fractional calculus has been stimulated by the applications that this calculus finds in numerical analysis and different areas of physics and engineering. Fractional calculus made it possible to adopt a theoretical model on experimental data. There are several well known forms of the fractional operators (meaning fractional integral and fractional derivative) that have been studied extensively for their applications: Riemann-Liouville, Weyl, Erdély-Kober, Hadamard, Katugampola are just a few. Here we will need *the left-sided Riemann-Liouville fractional integral of order* $\sigma > 0$ defined as in [4] for $f \in L_1[a,b]$ by

$$J^{\sigma}_{a+} f(x) = \frac{1}{\Gamma(\sigma)} \int_a^x (x-t)^{\sigma-1} f(t)\, dt, \quad x \in (a,b]. \tag{1}$$

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Recall, by $L_p[a,b]$, $1 \le p < \infty$, the space of all Lebesgue measurable functions f for which $|f^p|$ is Lebesgue integrable on $[a,b]$ is denoted, and by $L_\infty[a,b]$ the set of all functions measurable and essentially bounded on $[a,b]$. Clearly, $L_\infty[a,b] \subset L_p[a,b]$ for all $p \ge 1$.

For this paper, we have been motivated with researches of integral inequalities by W. Liu et al. [5] and by Z. Dahmani [3], such as:

Theorem 1.1 ([5, Theorem 4]). *Let f, g be positive continuous functions on $[a,b]$ such that f is decreasing and g is increasing. Then the following inequality*

$$\frac{\int_a^x f^\beta(x)\,dx}{\int_a^x f^\gamma(x)\,dx} \ge \frac{\int_a^x g^\alpha(x) f^\beta(x)\,dx}{\int_a^x g^\alpha(x) f^\gamma(x)\,dx} \tag{2}$$

holds for every $\alpha > 0$ and $\beta \ge \gamma > 0$. If f is increasing, then (2) is reverse.

Theorem 1.2 ([3, Theorem 3.6]). *Let $(f_i)_{i=1,2,\dots,n}$ and g be positive continuous functions on $[a,b]$ such that $(f_i)_{i=1,2,\dots,n}$ are decreasing and g is increasing. Then the following inequality*

$$\frac{J^\sigma_{a+}\left[\prod_{i\neq s}^n f_i^{\gamma_i} f_s^\beta(x)\right] J^\sigma_{a+}\left[g^\alpha(x)\prod_{i=1}^n f_i^{\gamma_i}(x)\right]}{J^\sigma_{a+}\left[g^\alpha(x)\prod_{i\neq s}^n f_i^{\gamma_i} f_s^\beta\right] J^\sigma_{a+}\left[\prod_{i=1}^n f_i^{\gamma_i}(x)\right]} \ge 1 \tag{3}$$

holds for every $a < x \le b$, $\sigma > 0$, $\alpha > 0$, $\beta \ge \gamma_s > 0$, where s is a fixed integer in $\{1,2,\dots,n\}$.

The aim of this paper is to present corresponding results using our extended generalized Mittag-Leffler function $E^{\delta,c,q,r}_{\rho,\sigma,\tau}(z;p)$ (Definition 1) with its fractional integral operator $\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau} f$ (Definition 2) recently presented in [1]. The well known Mittag-Leffler function E_ρ is defined by the power series

$$E_\rho(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\rho n + 1)} \quad (z \in \mathbb{C}, \Re(\rho) > 0) \tag{4}$$

and it is a natural extension of the exponential, hyperbolic and trigonometric functions. This function and its generalizations appear as a solution of fractional order differential or integral equations (for instance, see [6, 7]). An extended and generalized Mittag-Leffler function is defined as follows:

Definition 1 ([1]). Let $\rho, \sigma, \tau, \delta, c \in \mathbb{C}$, $\Re(\rho), \Re(\sigma), \Re(\tau) > 0$, $\Re(c) > \Re(\delta) > 0$ with $p \ge 0$, $r > 0$ and $0 < q \le r + \Re(\rho)$. Then $E^{\delta,c,q,r}_{\rho,\sigma,\tau}(z;p)$ is defined by

$$E^{\delta,c,q,r}_{\rho,\sigma,\tau}(z;p) = \sum_{n=0}^{\infty} \frac{\mathrm{B}_p(\delta + nq, c - \delta)}{\mathrm{B}(\delta, c - \delta)} \frac{(c)_{nq}}{\Gamma(\rho n + \sigma)} \frac{z^n}{(\tau)_{nr}}. \tag{5}$$

Here $(c)_{nq}$ denotes the generalized Pochhammer symbol $(c)_{nq} = \frac{\Gamma(c+nq)}{\Gamma(c)}$ and B_p is an extension of the beta function

$$\mathrm{B}_p(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} e^{-\frac{p}{t(1-t)}}\, dt, \quad (\Re(x), \Re(y), \Re(p) > 0).$$

As shown in [1], the series is absolutely convergent for all values of z provided that $q < r + \Re(\rho)$. Moreover, if $q = r + \Re(\rho)$, then $E^{\delta,c,q,r}_{\rho,\sigma,\tau}(z;p)$ converges for $|z| < \frac{r^r\, \Re(\rho)^{\Re(\rho)}}{q^q}$.

Definition 2 ([1])**.** Let $w, \rho, \sigma, \tau, \delta, c \in \mathbb{C}$, $\Re(\rho), \Re(\sigma), \Re(\tau) > 0$, $\Re(c) > \Re(\delta) > 0$ with $p \geq 0$, $r > 0$ and $0 < q \leq r + \Re(\rho)$. Let $f \in L_1[a,b]$ and $x \in [a,b]$. Then the generalized fractional integral operator $\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau} f$ is defined by

$$(6) \qquad \left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau} f\right)(x;p) \;=\; \int_a^x (x-t)^{\sigma-1} E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(x-t)^\rho; p) f(t)\, dt.$$

If we apply different parameter choices in Definition 1 and Definition 2, then the corresponding known generalizations of Mittag-Leffler function and its fractional integral operator can be deduced, for instance those defined by Prabhakar ([8]), Rahman et al. ([9]), Salim-Faraj ([10]), Shukla-Prajapati ([11]), Srivastava-Tomovski ([12]). For more details see [1] and references therein.

Here we also mention a different way of extending the Mittag-Leffler function, done in papers [2] and [7], where generalized Mittag-Leffler function is defined by

$$E^{\gamma,c;\lambda,\rho}_{\alpha,\beta}(z;p) \;=\; \sum_{n=0}^{\infty} \frac{\mathrm{B}^{\lambda,\rho}_p(\gamma+n, c-\gamma)}{\mathrm{B}(\gamma, c-\gamma)} \frac{(c)_n}{\Gamma(\alpha n+\beta)} \frac{z^n}{n!},$$

using

$$\mathrm{B}^{\lambda,\rho}_p(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, {}_1F_1\left[\lambda; \rho, -\frac{p}{t(1-t)}\right] dt$$

and $p \in \mathbb{R}^+_0, \Re(c) > \Re(\gamma) > 0; \Re(\alpha) > 0; \rho \in \mathbb{C} \setminus \mathbb{Z}^-_0$. In these two papers, certain useful properties and formulas are investigated (such as integral representation, Mellin transform, recurrence relation, and derivative formulas). Also, image formulas associated with the fractional calculus operators with Appell function in the kernel and Caputo-type fractional differential operators involving Srivastava polynomials and extended Mittag-Leffler function are established.

For the purpose of this paper, we will use a simplified notation

$$\boldsymbol{E}(z;p) \;:=\; E^{\delta,c,q,r}_{\rho,\sigma,\tau}(z;p)$$

and

$$(\boldsymbol{\varepsilon} f)(x;p) \;:=\; \left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau} f\right)(x;p).$$

Remark 1.3. Setting $p = w = 0$ in (5) and (6) we obtain

$$\boldsymbol{E}(z;0) \;=\; \sum_{n=0}^{\infty} \frac{(\delta)_{nq}}{\Gamma(\rho n+\sigma)} \frac{z^n}{(\tau)_{nr}} \tag{7}$$

and

$$(\boldsymbol{\varepsilon} f)(x;0) \;=\frac{1}{\Gamma(\sigma)} \int_a^x (x-t)^{\sigma-1} f(t)\, dt. \tag{8}$$

One can see that $(\boldsymbol{\varepsilon} f)(x;0)$ is actually the left-sided Riemann-Liouville fractional integral J^{σ}_{a+} of order σ, as defined in (1). Also,

$$\boldsymbol{E}(0;p) = \frac{\mathrm{B}_p(\delta, c-\delta)}{\mathrm{B}(\delta, c-\delta)} \frac{1}{\Gamma(\sigma)}, \tag{9}$$

$$\boldsymbol{E}(0;0) = \frac{1}{\Gamma(\sigma)}. \tag{10}$$

We will use all these calculations in the proof of our results.

The right-sided versions of all inequalities in this paper can be established using the right-sided fractional integral operator

$$\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma,\tau} f\right)(x;p) = \int_x^b (t-x)^{\sigma-1} E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(t-x)^{\rho};p) f(t)\, dt$$

and proved analogously.

2. Generalizations of certain integral inequalities involving Mittag-Leffler function

In this section we present certain integral inequalities using our extended generalized Mittag-Leffler function $\boldsymbol{E}$ with the corresponding fractional integral operator $\boldsymbol{\varepsilon}$ (in real domain). For proving our inequalities, we follow similar methods as in the paper by W. Liu et al. ([5]), which we supplement with the necessary steps to obtain generalized results.

Further extensions of these results are given in the following section.

Theorem 2.1. *Let $w \in \mathbb{R}$, $\rho, \sigma, \tau > 0$, $c > \delta > 0$ with $p \geq 0$, $r > 0$ and $0 < q \leq r + \rho$. Let $\alpha > 0$, $\beta \geq \gamma > 0$ and $x \in [a,b]$. Let f, g be positive continuous functions, monotonic in the opposite sense with $f \in L_\beta[a,b]$ and $g \in L_\alpha[a,b]$. Then the following inequality holds*

$$\frac{(\boldsymbol{\varepsilon} f^{\beta})(x;p)}{(\boldsymbol{\varepsilon} f^{\gamma})(x;p)} \;\geq\; \frac{(\boldsymbol{\varepsilon}(g^{\alpha} f^{\beta}))(x;p)}{(\boldsymbol{\varepsilon}(g^{\alpha} f^{\gamma}))(x;p)}. \tag{11}$$

If f and g are monotonic functions in the same sense, then the inequality (11) *is reverse.*

Proof. Let f, g be monotonic functions in the opposite sense, both positive and continuous. Then for $u, v \in [a,x]$ we obtain

$$[(g(u))^{\alpha} - (g(v))^{\alpha}]\left[(f(v))^{\beta-\gamma} - (f(u))^{\beta-\gamma}\right] \geq 0, \tag{12}$$

that is

$$(g(u))^{\alpha}(f(v))^{\beta-\gamma}+(g(v))^{\alpha}(f(u))^{\beta-\gamma}$$
$$\geq (g(u))^{\alpha}(f(u))^{\beta-\gamma}+(g(v))^{\alpha}(f(v))^{\beta-\gamma}.$$

Multiplying both sides of the above inequality by

$$(x-v)^{\sigma-1}\boldsymbol{E}(\omega(x-v)^{\rho};p)(f(v))^{\gamma}$$

and integrating on $[a,x]$ with respect to the variable v, we get

$$(g(u))^{\alpha}(\boldsymbol{\varepsilon}f^{\beta})(x;p)+(f(u))^{\beta-\gamma}(\boldsymbol{\varepsilon}(g^{\alpha}f^{\gamma}))(x;p)$$
$$\geq (g(u))^{\alpha}(f(u))^{\beta-\gamma}(\boldsymbol{\varepsilon}f^{\gamma})(x;p)+(\boldsymbol{\varepsilon}(g^{\alpha}f^{\beta}))(x;p).$$

Further multiplying by

$$(x-u)^{\sigma-1}\boldsymbol{E}(\omega(x-u)^{\rho};p)(f(u))^{\gamma}$$

and then integrating on $[a,x]$ with respect to the variable u, we have

$$(\boldsymbol{\varepsilon}(g^{\alpha}f^{\gamma}))(x;p)(\boldsymbol{\varepsilon}f^{\beta})(x;p) \;\geq\; (\boldsymbol{\varepsilon}(g^{\alpha}f^{\beta}))(x;p)(\boldsymbol{\varepsilon}f^{\gamma})(x;p)$$

from which follows (11).

If f and g are monotonic in the same sense, then the reverse inequality of (11) can be proved analogously. □

For a special case of an increasing function on $[a,b]$, $g(x)=x-a$, we have the following corollary.

Corollary 2.2. *Let $w\in\mathbb{R}$, $\rho,\sigma,\tau>0$, $c>\delta>0$ with $p\geq 0$, $r>0$ and $0<q\leq r+\rho$. Let $\alpha>0$, $\beta\geq\gamma>0$ and $x\in(a,b]$. Let $f\in L_{\beta}[a,b]$ be a positive continuous decreasing function. Then the following inequality holds*

$$\frac{(\boldsymbol{\varepsilon}f^{\beta})(x;p)}{(\boldsymbol{\varepsilon}f^{\gamma})(x;p)} \;\geq\; \frac{(\boldsymbol{\varepsilon}((x-a)^{\alpha}f^{\beta}))(x;p)}{(\boldsymbol{\varepsilon}((x-a)^{\alpha}f^{\gamma}))(x;p)}. \tag{13}$$

If f is increasing, then the inequality (13) *is reverse.*

Remark 2.3. If we consider special case of Mittag-Leffler function and its corresponding generalized fractional integral operator for $p=w=0$, given in (7)-(10), and if we set $\sigma=1$, then the inequality (11) implies Theorem 1.1. By verifying the condition (12), it is easy to see that although Theorem 1.1 is stated only for the case of decreasing function f and increasing function g, inequality (2) remains valid even if f is increasing and g is decreasing function g, hence monotone in the opposite sense.

Similarly, for $p=w=0$ and $\sigma=1$, the inequality (13) implies [5, Theorem 3].

Theorem 2.4. *Let $w\in\mathbb{R}$, $\rho,\sigma,\tau>0$, $c>\delta>0$ with $p\geq 0$, $r>0$ and $0<q\leq r+\rho$. Let $\alpha>0$, $\beta\geq\gamma>0$ and $x\in[a,b]$. Let f,g be positive continuous functions, $f\in L_{\alpha+\beta}[a,b]$, $g\in L_{\alpha}[a,b]$, such that for all $u,v\in[a,x]$*

$$\left[(g(u))^{\alpha}(f(v))^{\alpha}-(g(v))^{\alpha}(f(u))^{\alpha}\right]\left[(f(v))^{\beta-\gamma}-(f(u))^{\beta-\gamma}\right]\geq 0. \tag{14}$$

Then the following inequality holds

$$\frac{(\boldsymbol{\varepsilon} f^{\alpha+\beta})(x;p)}{(\boldsymbol{\varepsilon} f^{\alpha+\gamma})(x;p)} \geq \frac{(\boldsymbol{\varepsilon}(g^{\alpha} f^{\beta}))(x;p)}{(\boldsymbol{\varepsilon}(g^{\alpha} f^{\gamma}))(x;p)}. \tag{15}$$

If the condition (14) *is reverse, then the inequality* (15) *is reverse.*

Proof. According to condition (14) we arrive at

$$\begin{aligned}&(g(u))^{\alpha}(f(v))^{\alpha+\beta-\gamma} + (g(v))^{\alpha}(f(u))^{\alpha+\beta-\gamma}\\ &\geq (g(u))^{\alpha}(f(v))^{\alpha}(f(u))^{\beta-\gamma} + (g(v))^{\alpha}(f(u))^{\alpha}(f(v))^{\beta-\gamma}.\end{aligned}$$

Multiplying the above by

$$(x-v)^{\sigma-1}\boldsymbol{E}(\omega(x-v)^{\rho};p)(f(v))^{\gamma}$$

and integrating on $[a,x]$ with respect to the variable v, we get

$$\begin{aligned}&(g(u))^{\alpha}(\boldsymbol{\varepsilon} f^{\alpha+\beta})(x;p) + (f(u))^{\alpha+\beta-\gamma}(\boldsymbol{\varepsilon}(g^{\alpha} f^{\gamma}))(x;p)\\ &\geq (g(u))^{\alpha}(f(u))^{\beta-\gamma}(\boldsymbol{\varepsilon} f^{\alpha+\gamma})(x;p) + (f(u))^{\alpha}(\boldsymbol{\varepsilon}(g^{\alpha} f^{\beta}))(x;p).\end{aligned}$$

Once more, multiplying the above by

$$(x-u)^{\sigma-1}\boldsymbol{E}(\omega(x-u)^{\rho};p)(f(u))^{\gamma}$$

and then integrating on $[a,x]$ with respect to the variable u, we obtain

$$(\boldsymbol{\varepsilon}(g^{\alpha} f^{\gamma}))(x;p)(\boldsymbol{\varepsilon} f^{\alpha+\beta})(x;p) \geq (\boldsymbol{\varepsilon}(g^{\alpha} f^{\beta}))(x;p)(\boldsymbol{\varepsilon} f^{\alpha+\gamma})(x;p)$$

from which follows (15).

If the condition (14) is reverse, then the reverse inequality of (15) can be proved analogously. □

Again, we have the following corollary for a special case $g(x) = x - a$.

Corollary 2.5. *Let* $w \in \mathbb{R}$, $\rho, \sigma, \tau > 0$, $c > \delta > 0$ *with* $p \geq 0$, $r > 0$ *and* $0 < q \leq r + \rho$. *Let* $\alpha > 0$, $\beta \geq \gamma > 0$ *and* $x \in (a,b]$. *Let* $f \in L_{\alpha+\beta}[a,b]$ *be a positive continuous function such that for all* $u, v \in [a,x]$

$$[(u-a)^{\alpha}(f(v))^{\alpha} - (v-a)^{\alpha}(f(u))^{\alpha}]\left[(f(v))^{\beta-\gamma} - (f(u))^{\beta-\gamma}\right] \geq 0. \tag{16}$$

Then the following inequality holds

$$\frac{(\boldsymbol{\varepsilon} f^{\alpha+\beta})(x;p)}{(\boldsymbol{\varepsilon} f^{\alpha+\gamma})(x;p)} \geq \frac{(\boldsymbol{\varepsilon}((x-a)^{\alpha} f^{\beta}))(x;p)}{(\boldsymbol{\varepsilon}((x-a)^{\alpha} f^{\gamma}))(x;p)}. \tag{17}$$

If the condition (16) *is reverse, then the inequality* (17) *is reverse.*

Remark 2.6. For $p = w = 0$ and $\sigma = 1$, inequalities (15) and (17) imply [5, Theorem 6] and [5, Theorem 5], respectively.

Next we present an essential integral inequality that we need in order to easily obtain Theorem 2.9.

Theorem 2.7. *Let $w \in \mathbb{R}$, $\rho, \sigma, \tau > 0$, $c > \delta > 0$ with $p \geq 0$, $r > 0$ and $0 < q \leq r + \rho$. Let $x \in [a, b]$. Let $f, g, h \in L_1[a, b]$ be positive continuous functions such that f/h and g are monotonic in the opposite sense. Then the following inequality holds*

$$\frac{(\boldsymbol{\varepsilon} f)(x;p)}{(\boldsymbol{\varepsilon} h)(x;p)} \geq \frac{(\boldsymbol{\varepsilon}(fg))(x;p)}{(\boldsymbol{\varepsilon}(hg))(x;p)}. \tag{18}$$

If f/h and g are monotonic in the same sense, then the inequality (18) *is reverse.*

Proof. From hypotheses on functions, for $u, v \in [a, x]$ we have

$$[g(u) - g(v)] \left[\frac{f(v)}{h(v)} - \frac{f(u)}{h(u)} \right] \geq 0,$$

that is

$$g(u)\frac{f(v)}{h(v)} + g(v)\frac{f(u)}{h(u)} \geq g(u)\frac{f(u)}{h(u)} + g(v)\frac{f(v)}{h(v)}.$$

Multiplying both sides of the above inequality by

$$(x - v)^{\sigma-1}\boldsymbol{E}(\omega(x - v)^{\rho}; p)h(v)$$

and integrating on $[a, x]$ with respect to the variable v, we get

$$\begin{aligned} & g(u)(\boldsymbol{\varepsilon} f)(x;p) + \frac{f(u)}{h(u)}(\boldsymbol{\varepsilon}(gh))(x;p) \\ \geq\ & g(u)\frac{f(u)}{h(u)}(\boldsymbol{\varepsilon} h)(x;p) + (\boldsymbol{\varepsilon}(gf))(x;p). \end{aligned}$$

Again, multiplying the above by

$$(x - u)^{\sigma-1}\boldsymbol{E}(\omega(x - u)^{\rho}; p)h(u)$$

and then integrating on $[a, x]$ with respect to the variable u, we arrive at

$$(\boldsymbol{\varepsilon}(gh)(x;p)(\boldsymbol{\varepsilon} f)(x;p) \geq (\boldsymbol{\varepsilon}(gf))(x;p)(\boldsymbol{\varepsilon} h)(x;p)$$

from which follows (18).

If f/h and g are monotonic functions in the same sense, then the reverse inequality of (18) can be proved analogously. □

The counterpart of the previous result follows, where we assume $f(x) \leq h(x)$. Hence, inequality (18) remains satisfied if g is replaced by $f^{\alpha-1}$.

Theorem 2.8. *Let $w \in \mathbb{R}$, $\rho, \sigma, \tau > 0$, $c > \delta > 0$ with $p \geq 0$, $r > 0$ and $0 < q \leq r + \rho$. Let $\alpha \geq 1$ and $x \in [a, b]$. Let $f, h \in L_\alpha[a, b]$ be positive continuous functions such that f/h and f are monotonic in the opposite sense, with $f(x) \leq h(x)$ on $[a, b]$. Then the following inequality holds*

$$\frac{(\boldsymbol{\varepsilon} f)(x;p)}{(\boldsymbol{\varepsilon} h)(x;p)} \geq \frac{(\boldsymbol{\varepsilon} f^{\alpha})(x;p)}{(\boldsymbol{\varepsilon} h^{\alpha})(x;p)}. \tag{19}$$

If f/h and f are monotonic functions in the same sense, then the inequality (19) *is reverse.*

Proof. Assume that f/h is a decreasing function and f an increasing one. Then for $\alpha \geq 1$ function $f^{\alpha-1}$ is also increasing. By applying Theorem 2.7 we obtain

$$\frac{(\boldsymbol{\varepsilon} f)(x;p)}{(\boldsymbol{\varepsilon} h)(x;p)} \geq \frac{(\boldsymbol{\varepsilon}(f^{\alpha}))(x;p)}{(\boldsymbol{\varepsilon}(hf^{\alpha-1}))(x;p)}.$$

This together with the assumption $f(x) \leq h(x)$ lead to (19). Analogously we can prove the case when f/h is increasing and f decreasing, and obtain reversed inequality if f/h and f are monotonic in the same sense. □

In the last theorem of this section we involve a convex function in the inequality.

Theorem 2.9. *Let $w \in \mathbb{R}$, $\rho, \sigma, \tau > 0$, $c > \delta > 0$ with $p \geq 0$, $r > 0$ and $0 < q \leq r + \rho$. Let $x \in [a,b]$. Let $f, g, h \in L_1[a,b]$ be positive continuous functions such that f/h is a decreasing function and f, g are increasing, with $f(x) \leq h(x)$ on $[a,b]$. Let ϕ be a convex function on $[0,\infty]$ with $\phi(0) = 0$. Then the following inequality holds*

$$\frac{(\boldsymbol{\varepsilon} f)(x;p)}{(\boldsymbol{\varepsilon} h)(x;p)} \geq \frac{(\boldsymbol{\varepsilon}(\phi(f)g))(x;p)}{(\boldsymbol{\varepsilon}(\phi(h)g))(x;p)}. \tag{20}$$

Proof. The function $\frac{\phi(x)}{x}$ is increasing since ϕ is a convex function on $[0,\infty]$ with $\phi(0) = 0$. From the assumption $f(x) \leq h(x)$ with the positivity of f and h, we get

$$\frac{\phi(f(x))}{f(x)} \leq \frac{\phi(h(x))}{h(x)}.$$

Further, since f, g and $\frac{\phi(x)}{x}$ are increasing then the following function

$$\frac{\phi(f(x))}{f(x)} g(x)$$

is also increasing. Hence

$$\begin{aligned}\frac{(\boldsymbol{\varepsilon}(\phi(f)g))(x;p)}{(\boldsymbol{\varepsilon}(\phi(h)g))(x;p)} &= \frac{(\boldsymbol{\varepsilon}(\frac{\phi(f)}{f} fg))(x;p)}{(\boldsymbol{\varepsilon}(\frac{\phi(h)}{h} hg))(x;p)} \\ &\leq \frac{(\boldsymbol{\varepsilon}(\frac{\phi(f)}{f} fg))(x;p)}{(\boldsymbol{\varepsilon}(\frac{\phi(f)}{f} hg))(x;p)},\end{aligned}$$

and by applying Theorem 2.7 for f, h, $\frac{\phi(f)}{f} g$ we obtain

$$\leq \frac{(\boldsymbol{\varepsilon}(f))(x;p)}{(\boldsymbol{\varepsilon}(h))(x;p)}.$$

□

Corollary 2.10. *Let $w \in \mathbb{R}$, $\rho, \sigma, \tau > 0$, $c > \delta > 0$ with $p \geq 0$, $r > 0$ and $0 < q \leq r + \rho$. Let $x \in [a, b]$. Let $f, h \in L_1[a, b]$ be positive continuous functions such that f/h is a decreasing function and f is increasing, with $f(x) \leq h(x)$ on $[a, b]$. Let ϕ be a convex function on $[0, \infty]$ with $\phi(0) = 0$. Then the following inequality holds*

$$\frac{(\boldsymbol{\varepsilon} f)(x;p)}{(\boldsymbol{\varepsilon} h)(x;p)} \geq \frac{(\boldsymbol{\varepsilon}(\phi(f))(x;p)}{(\boldsymbol{\varepsilon}(\phi(h))(x;p)}. \tag{21}$$

Remark 2.11. If we set $p = w = 0$ and $\sigma = 1$, then Theorem 2.7, Theorem 2.8, Theorem 2.9 and Corollary 2.10 generalize Theorem 7, Theorem 8, Theorem 10 and Theorem 9 from [5], respectively.

3. Extensions of certain integral inequalities involving Mittag-Leffler function

We continue to further extend the previously presented integral inequalities. Again we use our extended generalized Mittag-Leffler function $\boldsymbol{E}$ with the corresponding fractional integral operator $\boldsymbol{\varepsilon}$ (in real domain) applied on $(f_i)_{i=1,2,\ldots,n}$.

First theorem is an extension of Theorem 2.1.

Theorem 3.1. *Let $w \in \mathbb{R}$, $\rho, \sigma, \tau > 0$, $c > \delta > 0$ with $p \geq 0$, $r > 0$ and $0 < q \leq r + \rho$. Let $\alpha > 0$, $\beta \geq \gamma_i > 0$ for $i = 1, 2, \ldots, n$ and let $x \in [a, b]$. Let $(f_i)_{i=1,2,\ldots,n}$ and g be positive continuous functions, such that $(f_i)_{i=1,2,\ldots,n}$ are decreasing and g is increasing with $(f_i)_{i=1,2,\ldots,n} \in L_\beta[a, b]$ and $g \in L_\alpha[a, b]$. Then for the fixed integer $s \in \{1, 2, \ldots, n\}$ the following inequality holds*

$$\frac{\left(\boldsymbol{\varepsilon}\left(\prod_{i\neq s}^{n} f_i^{\gamma_i} f_s^{\beta}\right)\right)(x;p)}{\left(\boldsymbol{\varepsilon}\left(\prod_{i=1}^{n} f_i^{\gamma_i}\right)\right)(x;p)} \geq \frac{\left(\boldsymbol{\varepsilon}\left(g^{\alpha}\prod_{i\neq s}^{n} f_i^{\gamma_i} f_s^{\beta}\right)\right)(x;p)}{\left(\boldsymbol{\varepsilon}\left(g^{\alpha}\prod_{i=1}^{n} f_i^{\gamma_i}\right)\right)(x;p)}. \tag{22}$$

If $(f_i)_{i=1,2,\ldots,n}$ are increasing and g is decreasing, then the inequality (22) also holds.

If all functions are monotonic in the same sense, then the inequality (22) is reverse.

Proof. Let $(f_i)_{i=1,2,\ldots,n}$ be decreasing and g increasing, all positive and continuous. Let $s \in \{1, 2, 3, ..., n\}$. Then for $u, v \in [a, x]$ we obtain

$$[(g(u))^{\alpha} - (g(v))^{\alpha}]\left[(f_s(v))^{\beta-\gamma_s} - (f_s(u))^{\beta-\gamma_s}\right] \geq 0,$$

that is

$$\begin{aligned}&(g(u))^{\alpha}(f_s(v))^{\beta-\gamma_s} + (g(v))^{\alpha}(f_s(u))^{\beta-\gamma_s}\\ &\geq (g(u))^{\alpha}(f_s(u))^{\beta-\gamma_s} + (g(v))^{\alpha}(f_s(v))^{\beta-\gamma_s}.\end{aligned}$$

Multiplying both sides of the above inequality by

$$(x-v)^{\sigma-1}\boldsymbol{E}(\omega(x-v)^{\rho};p)\prod_{i=1}^{n}(f_i(v))^{\gamma_i}$$

and integrating on $[a,x]$ with respect to the variable v, we get

$$(g(u))^{\alpha}\left(\boldsymbol{\varepsilon}\left(\prod_{i\neq s}^{n} f_i^{\gamma_i} f_s^{\beta}\right)\right)(x;p)+(f_s(u))^{\beta-\gamma_s}\left(\boldsymbol{\varepsilon}\left(g^{\alpha}\prod_{i=1}^{n} f_i^{\gamma_i}\right)\right)(x;p)$$
$$\geq (g(u))^{\alpha}(f_s(u))^{\beta-\gamma_s}\left(\boldsymbol{\varepsilon}\left(\prod_{i=1}^{n} f_i^{\gamma_i}\right)\right)(x;p)+\left(\boldsymbol{\varepsilon}\left(g^{\alpha}\prod_{i\neq s}^{n} f_i^{\gamma_i} f_s^{\beta}\right)\right)(x;p).$$

Further multiplying by

$$(x-u)^{\sigma-1}\boldsymbol{E}(\omega(x-u)^{\rho};p)\prod_{i=1}^{n}(f_i(u))^{\gamma_i}$$

and then integrating on $[a,x]$ with respect to the variable u, we have

$$\left(\boldsymbol{\varepsilon}\left(g^{\alpha}\prod_{i=1}^{n} f_i^{\gamma_i}\right)\right)(x;p)\left(\boldsymbol{\varepsilon}\left(\prod_{i\neq s}^{n} f_i^{\gamma_i} f_s^{\beta}\right)\right)(x;p)$$
$$\geq \left(\boldsymbol{\varepsilon}\left(g^{\alpha}\prod_{i\neq s}^{n} f_i^{\gamma_i} f_s^{\beta}\right)\right)(x;p)\left(\boldsymbol{\varepsilon}\left(\prod_{i=1}^{n} f_i^{\gamma_i}\right)\right)(x;p),$$

from which follows (22).

Analogously we can prove the case when $(f_i)_{i=1,2,\ldots,n}$ are increasing and g is decreasing, and obtain reversed inequality if all functions are monotonic in the same sense. □

Remark 3.2. As mentioned in Introduction, for $p=w=0$ we obtain the left-sided Riemann-Liouville fraction integral J^{σ}_{a+} of order σ, i.e., (8), a special case of Mittag-Leffler function and its corresponding generalized fractional integral operator. Therefore, Theorem 3.1 generalizes Theorem 1.2.

On the other hand, if we set $n=1$, then $s=1$ which implies Theorem 2.1.

For $g(x)=x-a$, which is an increasing function on $[a,b]$, we have the following corollary. In this case, the inequality (23) implies [3, Theorem 3.1].

Corollary 3.3. *Let $w\in\mathbb{R}$, $\rho,\sigma,\tau>0$, $c>\delta>0$ with $p\geq 0$, $r>0$ and $0<q\leq r+\rho$. Let $\alpha>0$, $\beta\geq\gamma_i>0$ for $i=1,2,\ldots,n$ and let $x\in(a,b]$. Let $(f_i)_{i=1,2,\ldots,n}$ be positive continuous decreasing functions with $(f_i)_{i=1,2,\ldots,n}\in L_{\beta}[a,b]$. Then for the fixed integer $s\in\{1,2,\ldots,n\}$ the following inequality holds*

$$\frac{\left(\boldsymbol{\varepsilon}\left(\prod_{i\neq s}^{n} f_i^{\gamma_i} f_s^{\beta}\right)\right)(x;p)}{\left(\boldsymbol{\varepsilon}\left(\prod_{i=1}^{n} f_i^{\gamma_i}\right)\right)(x;p)}\geq\frac{\left(\boldsymbol{\varepsilon}\left((x-a)^{\alpha}\prod_{i\neq s}^{n} f_i^{\gamma_i} f_s^{\beta}\right)\right)(x;p)}{\left(\boldsymbol{\varepsilon}\left((x-a)^{\alpha}\prod_{i=1}^{n} f_i^{\gamma_i}\right)\right)(x;p)}. \tag{23}$$

If $(f_i)_{i=1,2,\ldots,n}$ are increasing, then the inequality (23) is reverse.

Theorem 3.4. *Let $w \in \mathbb{R}$, $\rho, \sigma, \tau > 0$, $c > \delta > 0$ with $p \geq 0$, $r > 0$ and $0 < q \leq r + \rho$. Let $\alpha > 0$, $\beta \geq \gamma_i > 0$ for $i = 1, 2, \ldots, n$ and let $x \in [a, b]$. Let $(f_i)_{i=1,2,\ldots,n}$ and g be positive continuous functions, $(f_i)_{i=1,2,\ldots,n} \in L_{\alpha+\beta}[a, b]$ and $g \in L_\alpha[a, b]$. Let $s \in \{1, 2, \ldots, n\}$ be fixed integer and for all $u, v \in [a, x]$*

$$[(g(u))^\alpha (f_s(v))^\alpha - (g(v))^\alpha (f_s(u))^\alpha] \left[(f_s(v))^{\beta-\gamma_s} - (f_s(u))^{\beta-\gamma_s}\right] \geq 0. \tag{24}$$

Then the following inequality holds

$$\frac{\left(\boldsymbol{\varepsilon}\left(\prod_{i\neq s}^n f_i^{\gamma_i} f_s^{\alpha+\beta}\right)\right)(x;p)}{\left(\boldsymbol{\varepsilon}\left(\prod_{i\neq s}^n f_i^{\gamma_i} f_s^{\alpha+\gamma_s}\right)\right)(x;p)} \geq \frac{\left(\boldsymbol{\varepsilon}\left(g^\alpha \prod_{i\neq s}^n f_i^{\gamma_i} f_s^{\beta}\right)\right)(x;p)}{\left(\boldsymbol{\varepsilon}\left(g^\alpha \prod_{i=1}^n f_i^{\gamma_i}\right)\right)(x;p)}. \tag{25}$$

If the condition (24) *is reverse, then the inequality* (25) *is reverse.*

Proof. From the (24) we obtain

$$\begin{aligned}&(g(u))^\alpha (f_s(v))^{\alpha+\beta-\gamma_s} + (g(v))^\alpha (f_s(u))^{\alpha+\beta-\gamma_s}\\ &\geq (g(u))^\alpha (f_s(v))^\alpha (f_s(u))^{\beta-\gamma_s} + (g(v))^\alpha (f_s(u))^\alpha (f_s(v))^{\beta-\gamma_s}.\end{aligned}$$

Multiplying both sides of the above inequality by

$$(x-v)^{\sigma-1}\boldsymbol{E}(\omega(x-v)^\rho; p)\prod_{i=1}^n (f_i(v))^{\gamma_i}$$

and integrating on $[a, x]$ with respect to the variable v, we get

$$\begin{aligned}&(g(u))^\alpha \left(\boldsymbol{\varepsilon}\left(\prod_{i\neq s}^n f_i^{\gamma_i} f_s^{\alpha+\beta}\right)\right)(x;p) + (f_s(u))^{\alpha+\beta-\gamma_s}\left(\boldsymbol{\varepsilon}\left(g^\alpha \prod_{i=1}^n f_i^{\gamma_i}\right)\right)(x;p)\\ &\geq (g(u))^\alpha (f_s(u))^{\beta-\gamma_s}\left(\boldsymbol{\varepsilon}\left(\prod_{i\neq s}^n f_i^{\gamma_i} f_s^{\alpha+\gamma_s}\right)\right)(x;p)\\ &\quad + (f_s(u))^\alpha \left(\boldsymbol{\varepsilon}\left(g^\alpha \prod_{i\neq s}^n f_i^{\gamma_i} f_s^{\beta}\right)\right)(x;p).\end{aligned}$$

Further multiplying by

$$(x-u)^{\sigma-1}\boldsymbol{E}(\omega(x-u)^\rho; p)\prod_{i=1}^n (f_i(u))^{\gamma_i}$$

and then integrating on $[a, x]$ with respect to the variable u, we have

$$\begin{aligned}&\left(\boldsymbol{\varepsilon}\left(g^\alpha \prod_{i=1}^n f_i^{\gamma_i}\right)\right)(x;p)\left(\boldsymbol{\varepsilon}\left(\prod_{i\neq s}^n f_i^{\gamma_i} f_s^{\alpha+\beta}\right)\right)(x;p)\\ &\geq \left(\boldsymbol{\varepsilon}\left(g^\alpha \prod_{i\neq s}^n f_i^{\gamma_i} f_s^{\beta}\right)\right)(x;p)\left(\boldsymbol{\varepsilon}\left(\prod_{i\neq s1}^n f_i^{\gamma_i} f_s^{\alpha+\gamma_s}\right)\right)(x;p),\end{aligned}$$

from which follows (25).

If the condition (24) is reverse, then the reverse inequality of (25) can be proved analogously. □

Remark 3.5. For $p = w = 0$, Theorem 3.4 generalizes [3, Theorem 3.10].

Setting $n = 1$, Theorem 2.4 follows.

Corollary 3.6. *Let $w \in \mathbb{R}$, $\rho, \sigma, \tau > 0$, $c > \delta > 0$ with $p \geq 0$, $r > 0$ and $0 < q \leq r + \rho$. Let $\alpha > 0$, $\beta \geq \gamma_i > 0$ for $i = 1, 2, \ldots, n$ and let $x \in (a, b]$. Let $(f_i)_{i=1,2,\ldots,n} \in L_{\alpha+\beta}[a, b]$ be positive continuous functions. Let $s \in \{1, 2, \ldots, n\}$ be fixed integer and for all $u, v \in [a, x]$*

$$[(u-a)^\alpha (f_s(v))^\alpha - (u-v)^\alpha (f_s(u))^\alpha] \left[(f_s(v))^{\beta-\gamma_s} - (f_s(u))^{\beta-\gamma_s} \right] \geq 0. \tag{26}$$

Then the following inequality holds

$$\frac{\left(\boldsymbol{\varepsilon} \left(\prod_{i \neq s}^n f_i^{\gamma_i} f_s^{\alpha+\beta} \right) \right)(x;p)}{\left(\boldsymbol{\varepsilon} \left(\prod_{i \neq s}^n f_i^{\gamma_i} f_s^{\alpha+\gamma_s} \right) \right)(x;p)} \geq \frac{\left(\boldsymbol{\varepsilon} \left((x-a)^\alpha \prod_{i \neq s}^n f_i^{\gamma_i} f_s^{\beta} \right) \right)(x;p)}{\left(\boldsymbol{\varepsilon} \left((x-a)^\alpha \prod_{i=1}^n f_i^{\gamma_i} \right) \right)(x;p)}. \tag{27}$$

If the condition (26) *is reverse, then the inequality* (27) *is reverse.*

Theorem 3.7. *Let $w \in \mathbb{R}$, $\rho, \sigma, \tau > 0$, $c > \delta > 0$ with $p \geq 0$, $r > 0$ and $0 < q \leq r + \rho$. Let $\alpha > 0$, $\beta \geq \gamma_i > 0$ for $i = 1, 2, \ldots, n$ and let $x \in [a, b]$. Let $f, g, (h_i)_{i=1,2,\ldots,n} \in L_1[a, b]$ be positive continuous functions such that f/h_s and g are monotonic in the opposite sense, for $s \in \{1, 2, \ldots, n\}$. Then the following inequality holds*

$$\frac{\left(\boldsymbol{\varepsilon} \left(f \prod_{i \neq s}^n h_i \right) \right)(x;p)}{\left(\boldsymbol{\varepsilon} \left(\prod_{i=1}^n h_i \right) \right)(x;p)} \geq \frac{\left(\boldsymbol{\varepsilon} \left(g f \prod_{i \neq s}^n h_i \right) \right)(x;p)}{\left(\boldsymbol{\varepsilon} \left(g \prod_{i=1}^n h_i \right) \right)(x;p)}. \tag{28}$$

If f/h_s and g are monotonic in the same sense for $s \in \{1, 2, \ldots, n\}$, the inequality (28) *is reverse.*

Proof. From hypotheses on functions, for $u, v \in [a, x]$ we have

$$[g(u) - g(v)] \left[\frac{f(v)}{h_s(v)} - \frac{f(u)}{h_s(u)} \right] \geq 0,$$

that is

$$g(u)\frac{f(v)}{h_s(v)} + g(v)\frac{f(u)}{h_s(u)} \geq g(u)\frac{f(u)}{h_s(u)} + g(v)\frac{f(v)}{h_s(v)}.$$

Multiplying both sides of the above inequality by

$$(x-v)^{\sigma-1} \boldsymbol{E}(\omega(x-v)^\rho; p) \prod_{i=1}^n h_i(v)$$

and integrating on $[a, x]$ with respect to the variable v, we get

$$g(u) \left(\boldsymbol{\varepsilon} \left(f \prod_{i \neq s}^n h_i \right) \right)(x;p) + \frac{f(u)}{h_s(u)} \left(\boldsymbol{\varepsilon} \left(g \prod_{i=1}^n h_i \right) \right)(x;p)$$

$$\geq g(u)\frac{f(u)}{h_s(u)}\left(\boldsymbol{\varepsilon}\left(\prod_{i=1}^{n}h_i\right)\right)(x;p)+\left(\boldsymbol{\varepsilon}\left(gf\prod_{i\neq s}^{n}h_i\right)\right)(x;p).$$

Again, multiplying the above by

$$(x-u)^{\sigma-1}\boldsymbol{E}(\omega(x-u)^{\rho};p)\prod_{i=1}^{n}h_i(u)$$

and then integrating on $[a,x]$ with respect to the variable u, we arrive at

$$\left(\boldsymbol{\varepsilon}\left(g\prod_{i=1}^{n}h_i\right)\right)(x;p)\left(\boldsymbol{\varepsilon}\left(f\prod_{i\neq s}^{n}h_i\right)\right)(x;p)$$
$$\geq\left(\boldsymbol{\varepsilon}\left(\prod_{i=1}^{n}h_i\right)\right)(x;p)\left(\boldsymbol{\varepsilon}\left(gf\prod_{i\neq s}^{n}h_i\right)\right)(x;p),$$

from which follows (28).

If f/h_s and g are monotonic in the same sense for $s\in\{1,2,\ldots,n\}$, then the reverse inequality of (28) can be proved analogously. □

Remark 3.8. For $p=w=0$, Theorem 3.7 generalizes [3, Theorem 3.14].

Setting $n=1$, Theorem 2.7 follows.

4. Concluding remarks

In this paper, we have presented certain integral inequalities using our extended generalized Mittag-Leffler function with the corresponding fractional integral operator (in real domain), thereby generalizing and extending known integral inequalities. Our work was motivated with researches of integral inequalities by W. Liu et al. [5] and by Z. Dahmani [3], in a way that *Section 2* generalizes Liu's results, and *Section 3* those by Dahmani.

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