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# ON MULTIPLIER WEIGHTED-SPACE OF SEQUENCES

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Abstract. We consider the weighted spaces $\ell^p(\mathbb{S},\varphi)$ and $\ell^p(\mathbb{S},\psi)$ for $1 < p < +\infty$, where $\varphi$ and $\psi$ are weights on $\mathbb{S}$ ($=\mathbb{N}$ or $\mathbb{Z}$). We obtain a sufficient condition for $\ell^p(\mathbb{S},\psi)$ to be multiplier weighted-space of $\ell^p(\mathbb{S},\varphi)$ and $\ell^p(\mathbb{S},\psi)$. Our condition characterizes the last multiplier weighted-space in the case where $\mathbb{S}=\mathbb{Z}$. As a consequence, in the particular case where $\psi=\varphi$, the weighted space $\ell^p(\mathbb{Z},\psi)$ is a convolutive algebra.

## 1. Preliminaries and introduction

Let $\mathbb{S}$ ($\mathbb{S}=\mathbb{N}$ or $\mathbb{S}=\mathbb{Z}$) and $p\in\,]1,+\infty[$. We say that $\omega$ is a weight on $\mathbb{S}$ if $\omega:\mathbb{S}\longrightarrow[1,+\infty[$, is a map satisfying:

$$\sum_{n\in\mathbb{S}}\omega(n)^{\frac{1}{1-p}}<+\infty.$$

We consider the weighted space:

$$\ell^p(\mathbb{S},\omega)=\left\{(a(n))_{n\in\mathbb{S}}\in\mathbb{C}^{\mathbb{S}}:\sum_{n\in\mathbb{S}}|a(n)|^p\,\omega(n)<+\infty\right\}.$$

Endowed with the norm $|\cdot|_{p,\omega}$ defined by:

$$|a|_{p,\omega}=\left(\sum_{n\in\mathbb{S}}|a(n)|^p\,\omega(n)\right)^{\frac{1}{p}}\ \text{ for every }\ (a(n))_{n\in\mathbb{S}}\in\ell^p(\mathbb{S},\omega),$$

the space $\ell^p(\mathbb{S},\omega)$ becomes a Banach subspace of $\ell^p(\mathbb{S})$. We say that the weight $\omega$ is $m$-convolutive if a positive constant $\gamma=\gamma(\omega)$ exists such that:

$$\omega^{\frac{1}{1-p}}*\omega^{\frac{1}{1-p}}\leq\gamma\,\omega^{\frac{1}{1-p}},$$

where $*$ denotes the convolution product. If $a=(a(n))_{n\in\mathbb{S}}\in\ell^p(\mathbb{S},\omega)$, we define the complex function $\mathcal{F}(a)$ by

$$\mathcal{F}(a)(t)=\sum_{n\in\mathbb{S}}a(n)e^{int}\ \text{ for every }\ t\in\mathbb{R}.$$

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$\mathcal{F}(a)$ is called the Fourier transform of $a$. If $\omega$ is an $m$-convolutive weight, then the space $\left(\ell^p(\mathbb{S},\omega), |\cdot|_{p,\omega}\right)$ is a convolutive Banach algebra with unit ([2] and [4]). The converse is true in the case where $\mathbb{S}=\mathbb{Z}$ ([3] and [5]). For the sequence spaces $\ell^p(\mathbb{S},\varphi)$ and $\ell^p(\mathbb{S},\psi)$, the set $\mathcal{M}^p(\mathbb{S},\varphi,\psi)$ defined by:

$$\mathcal{M}^p(\mathbb{S},\varphi,\psi)=\left\{b=(b_n)_{n\in\mathbb{S}}\in\mathbb{C}^{\mathbb{S}} : a*b\in\ell^p(\mathbb{S},\psi),\ \forall a=(a_n)_{n\in\mathbb{S}}\in\ell^p(\mathbb{S},\varphi)\right\}$$

is called the multiplier weighted-space of $\ell^p(\mathbb{S},\varphi)$ and $\ell^p(\mathbb{S},\psi)$.

In this paper, we study the multiplier weighted-space $\mathcal{M}^p(\mathbb{S},\varphi,\psi)$. We prove that if the weights $\varphi$ and $\psi$ satisfy property:

$$\varphi^{\frac{1}{1-p}}*\psi^{\frac{1}{1-p}}\leq\gamma\ \psi^{\frac{1}{1-p}} \tag{1}$$

for some constant $\gamma>0$, then $\ell^p(\mathbb{S},\psi)=\mathcal{M}^p(\mathbb{S},\varphi,\psi)$. The converse is true in the case where $\mathbb{S}=\mathbb{Z}$. As a consequence, the weighted space $\ell^p(\mathbb{Z},\psi)$ is an algebra if and only if $\psi$ is an $m$-convolutive weight on $\mathbb{Z}$, i.e., $\varphi=\psi$ in (1).

## 2. Link between $\ell^p(\mathbb{S},\psi)$ and $\mathcal{M}^p(\mathbb{S},\varphi,\psi)$

We first prove that the property (1) results in $\mathcal{M}^p(\mathbb{S},\varphi,\psi)=\ell^p(\mathbb{S},\psi)$.

**Theorem 2.1.** *Let $p\in]1,+\infty[$. If $\varphi$ and $\psi$ are two weights on $\mathbb{S}$ satisfying property* (1)*, then we have*:

1) $\mathcal{M}^p(\mathbb{S},\varphi,\psi)=\ell^p(\mathbb{S},\psi)$.
2) $|a*b|_{p,\psi}\leq\gamma^{\frac{p-1}{p}}|a|_{p,\varphi}|b|_{p,\psi}$ *for every* $a\in\ell^p(\mathbb{S},\varphi)$ *and* $b\in\ell^p(\mathbb{S},\psi)$.

*Proof.* It's clear that $\mathcal{M}^p(\mathbb{S},\varphi,\psi)\subset\ell^p(\mathbb{S},\psi)$. Conversely let $b=(b(n))_{n\in\mathbb{S}}\in\ell^p(\mathbb{S},\psi)$. We put $T_b(a)=a*b$ for every $a=(a(n))_{n\in\mathbb{S}}\in\ell^p(S,\varphi)$ and,

$$\begin{aligned}c(m)&=\sum_{n\in\mathbb{S}}a(m-n)\,b(n)\\&=\sum_{n\in\mathbb{S}}a(m-n)\,b(n)\left(\frac{\psi(n)^{\frac{1}{p}}\varphi(m-n)^{\frac{1}{p}}}{\psi(n)^{\frac{1}{p}}\varphi(m-n)^{\frac{1}{p}}}\right)\ \text{for every } m\in\mathbb{S}.\end{aligned}$$

Using Hölder's inequality we obtain, for every $m\in\mathbb{S}$,

$$|c(m)|\leq\left[\sum_{n\in\mathbb{S}}|a(m-n)|^p\,\varphi(m-n)\,|b(n)|^p\,\psi(n)\right]^{\frac{1}{p}}W^{\frac{-1}{p}}(m),$$

where $W=\left(\varphi^{\frac{1}{1-p}}*\psi^{\frac{1}{1-p}}\right)^{1-p}$. Therefore, for every $m\in\mathbb{S}$,

$$|c(m)|^p\,W(m)\leq\sum_{n\in\mathbb{S}}|a(m-n)|^p\,\varphi(m-n)\,|b(n)|^p\,\psi(n).$$

Since $\gamma^{1-p}\psi\leq W$, the inequality (1) implies that:

$$\gamma^{1-p}\sum_{m\in\mathbb{S}}|c(m)|^p\,\psi(m)\leq\sum_{m\in\mathbb{S}}\left(\sum_{n\in\mathbb{S}}|a(m-n)|^p\,\varphi(m-n)\,|b(n)|^p\,\psi(n)\right)$$

$$\leq |a|_{p,\varphi}|b|_{p,\psi},$$

i.e.,

$$T_b(a) \in \ell^p(\mathbb{Z},\psi) \text{ for every } a \in \ell^p(\mathbb{S},\varphi).$$

Hence

$$\ell^p(\mathbb{S},\psi) \subset \mathcal{M}^p(\mathbb{S},\varphi,\psi) \text{ and } |a*b|_{p,\psi} \leq \gamma^{\frac{p-1}{p}}|a|_{p,\varphi}|b|_{p,\psi}$$

for every $(a,b) \in \ell^p(\mathbb{S},\varphi)\times\ell^p(\mathbb{S},\psi)$. As a result:

$$\ell^p(\mathbb{S},\psi) = \mathcal{M}^p(\mathbb{S},\varphi,\psi) \text{ and } |a*b|_{p,\psi} \leq \gamma^{\frac{p-1}{p}}|a|_{p,\varphi}|b|_{p,\psi}$$

for every $(a,b) \in \ell^p(\mathbb{S},\varphi)\times\ell^p(\mathbb{S},\psi)$. □

*Remark* 2.2. Replacing in Theorem 2.1 $|\cdot|_{p,\varphi}$ by $\gamma^{\frac{p-1}{p}}|\cdot|_{p,\varphi}$ and $|\cdot|_{p,\psi}$ by $\gamma^{\frac{p-1}{p}}|\cdot|_{p,\psi}$, we can suppose, without loss of generality, that:

$$|a*b|_{p,\psi} \leq |a|_{p,\varphi}|b|_{p,\psi} \text{ for every } a \in \ell^p(\mathbb{S},\varphi) \text{ and } b \in \ell^p(\mathbb{S},\psi).$$

If $\psi=\varphi$, the property (1) becomes:

$$\psi^{\frac{1}{1-p}} * \psi^{\frac{1}{1-p}} \leq \gamma\ \psi^{\frac{1}{1-p}}.$$

Therefore $\psi$ is an $m$-convolutive weight and $\ell^p(\mathbb{S},\psi) = \mathcal{M}^p(\mathbb{S},\psi,\psi)$, i.e., $(\ell^p(\mathbb{S},\psi),|\cdot|_{p,\psi})$, endowed with convolution product, is a commutative Banach algebra with unit. Thus, we have the following result:

**Corollary 2.3** ([3] and [5])**.** *Let $p\in]1,+\infty[$. If $\omega$ is an $m$-convolutive weight on $\mathbb{S}$, then $(\ell^p(\mathbb{S},\omega),|\cdot|_{p,\omega})$ is a commutative Banach algebra with unit.*

Now we examine the converse of Theorem 2.1 in the case where $\mathbb{S}=\mathbb{Z}$. The proof will be based on several results.

Let $p\in\,]1,+\infty[$, $\varphi$ and $\psi$ be two weight maps on $\mathbb{Z}$. We put:

$$W = \left(\varphi^{\frac{1}{1-p}} * \psi^{\frac{1}{1-p}}\right)^{1-p}$$

and assume that:

$$\ell^p(\mathbb{Z},\psi) = \mathcal{M}^p(\mathbb{Z},\varphi,\psi).$$

It is clear that $W$ and $W^{\frac{1}{1-p}}$ are weight maps on $\mathbb{Z}$. Let $a \in \ell^p(\mathbb{Z},\varphi)$ and $b\in\ell^p(\mathbb{Z},\psi)$. Since $\ell^p(\mathbb{Z},\psi) = \mathcal{M}^p(\mathbb{Z},\varphi,\psi)$, we have $a*b\in\ell^p(\mathbb{Z},\psi)$ and

$$a*b(m) = \sum_{n\in\mathbb{Z}} a(n)\varphi^{\frac{1}{p}}(n)b(m-n)\psi^{\frac{1}{p}}(m-n)\frac{1}{\varphi^{\frac{1}{p}}(n)\psi^{\frac{1}{p}}(m-n)} \text{ for every } m\in\mathbb{Z}.$$

Using Hölder's inequality, one has:

$$a*b\in\ell^p(\mathbb{Z},W) \text{ and } |a*b|_{p,W} \leq |a|_{p,\varphi}|b|_{p,\psi}.$$

Thus, each element $b\in\ell^p(\mathbb{Z},\psi)$ define a continuous linear map $T_b$, of $\ell^p(\mathbb{Z},\varphi)$ in $\ell^p(\mathbb{Z},W)$, by:

$$T_b(a) = a*b \text{ for every } a\in\ell^p(\mathbb{Z},\varphi).$$

For $c = (c(n,m))_{(n,m)\in\mathbb{Z}^2} \in \ell^p(\mathbb{Z}^2)$, let:

$$L(c) = \left(\sum_{n\in\mathbb{Z}} \frac{c\,(n, m-n)}{[\varphi(n)\psi(m-n)]^{\frac{1}{p}}}\right)_{m\in\mathbb{Z}}.$$

If $a \in \ell^p(\mathbb{Z},\varphi)$ and $b \in \ell^p(\mathbb{Z},\psi)$, a simple calculation shows that:

$$c = \left(a(n)b(m)\,[\varphi(n)\psi(m)]^{\frac{1}{p}}\right)_{(n,m)\in\mathbb{Z}^2} \in \ell^p(\mathbb{Z}^2) \quad \text{and} \quad L(c) = a * b.$$

The properties of the map $L$ will be useful later.

**Proposition 2.4.** *Let $p \in \,]1,+\infty[$, $\varphi$ and $\psi$ be two weight maps on $\mathbb{Z}$. We suppose that*

$$\ell^p(\mathbb{Z},\psi) = \mathcal{M}^p\left(\mathbb{Z},\varphi,\psi\right).$$

*Then*

1) *$L$ is a continuous linear map from $\ell^p(\mathbb{Z}^2)$ into $\ell^p(\mathbb{Z},W)$.*
2) *For $c \in \ell^p(\mathbb{Z}^2)$,*

$$\left|L(c)\right|_{p,W} \leq \left|c\right|_p.$$

*Proof.* 1) First, we introduce the transpose $S$ of $L$, given by:

$$S(x) = \left(x\,(n+m)\,\varphi^{\frac{-1}{p}}(n)\psi^{\frac{-1}{p}}(m)\right)_{(n,m)\in\mathbb{Z}^2} \quad \text{for every } x \in \ell^q(\mathbb{Z}, W^{\frac{1}{1-p}}),$$

where $q = \frac{p-1}{p}$. It is clear that $S$ is linear. Let prove that it is an isometric map of $\ell^q(\mathbb{Z}, W^{\frac{1}{1-p}})$ into $\ell^q(\mathbb{Z}^2)$. Let $x \in \ell^q(\mathbb{Z}, W^{\frac{1}{1-p}})$. Then, for every $n, m \in \mathbb{Z}$, we can write:

$$|x(n+m)\varphi^{\frac{-1}{p}}(n)\psi^{\frac{-1}{p}}(m)|^q = |x(n+m)|^q\varphi^{\frac{1}{1-p}}(n)\psi^{\frac{1}{1-p}}(m).$$

Thus

$$\begin{aligned}
|S(x)|_q^q &= \sum_{m\in\mathbb{Z}}\left[\sum_{n\in\mathbb{Z}}|x(n+m)|^q\varphi^{\frac{1}{1-p}}(n)\psi^{\frac{1}{1-p}}(m)\right] \\
&= \sum_{k\in\mathbb{Z}}\left[\sum_{n\in\mathbb{Z}}|x(k)|^q\varphi^{\frac{1}{1-p}}(n)\psi^{\frac{1}{1-p}}(k-n)\right] \\
&= \sum_{k\in\mathbb{Z}}|x(k)|^q W^{\frac{1}{1-p}}(k) = |x|^q_{q,W^{\frac{1}{1-p}}}.
\end{aligned}$$

This yields,

$$\left|S\left(x\right)\right|_q = \left|x\right|_{q,W^{\frac{1}{1-p}}} \quad \text{for every } x \in \ell^q(\mathbb{Z}, W^{\frac{1}{1-p}}).$$

It follows that $L = {}^tS$. Since the spaces $\ell^p(\mathbb{Z}^2)$ and $\ell^p(\mathbb{Z},W)$ are reflexive, the result follows from [1].

2) Let $c = (c(n,m))_{(n,m)\in\mathbb{Z}^2} \in \ell^p(\mathbb{Z}^2)$. Then, for every $m \in \mathbb{Z}$,

$$|L(c)(m)|^p = \left|\sum_{n\in\mathbb{Z}} \frac{c\,(n,m-n)}{[\varphi(n)\psi(m-n)]^{\frac{1}{p}}}\right|^p \le \left(\sum_{n\in\mathbb{Z}} |c\,(n,m-n)|^p\right) \frac{1}{W(m)},$$

i.e.,

$$|L(c)(m)|^p\, W(m) \le \sum_{n\in\mathbb{Z}} |c\,(n,m-n)|^p \quad \text{for every } m \in \mathbb{Z}.$$

Hence,

$$\sum_{m\in\mathbb{Z}} |L(c)(m)|^p\, W(m) \le \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} |c\,(n,m-n)|^p = |c|_p^p\,.$$

Thus,

$$|L(c)|_{p,W} \le |c|_p\,. \qquad \square$$

In the following, we assume that $p \in ]1,+\infty[$, $q = \frac{p-1}{p}$, $\varphi$ and $\psi$ are two weight maps on $\mathbb{Z}$, satisfying:

1) $\ell^p(\mathbb{Z},\psi) = \mathcal{M}^p(\mathbb{Z},\varphi,\psi)$.

2) There exists a constant $M > 0$ such that:

$$|a * b|_{p,\psi} \le M\ |a|_{p,\varphi}\ |b|_{p,\psi} \quad \text{for every } (a,b) \in \ell^p(\mathbb{Z},\varphi)\times\ \ell^p(\mathbb{Z},\psi).$$

By the same argument in Remark 2.2, we can assume without loss of generality that:

$$(2) \qquad |a * b|_{p,\psi} \le |a|_{p,\varphi}|b|_{p,\psi} \quad \text{for every } (a,b) \in \ell^p(\mathbb{Z},\varphi)\times\ \ell^p(\mathbb{Z},\psi).$$

We first prove that $\ell^p(\mathbb{Z},W) = \mathcal{M}^p(\mathbb{Z},\varphi,W)$, where $W = \left[\varphi^{\frac{1}{1-p}} * \psi^{\frac{1}{1-p}}\right]^{1-p}$. The following lemmas will be needed:

**Lemma 2.5.** *Let $x \in \ell^p(\mathbb{Z},\varphi)$ and $c = (c(n,m))_{(n,m)\in\mathbb{Z}^2} \in \ell^p(\mathbb{Z}^2)$. If*

$$d(n,m) = \sum_{k\in\mathbb{Z}} \frac{c(n,k)x\,(m-k)}{\psi^{\frac{1}{p}}\,(k)} \psi^{\frac{1}{p}}(m) \quad \textit{for every } (n,m) \in \mathbb{Z}^2,$$

*then $d \in \ell^p(\mathbb{Z}^2)$ and $|d|_p \le |c|_p\,|x|_{p,\varphi}$.*

*Proof.* Observe first that, for every $n \in \mathbb{Z}$, we have:

$$\left(|c(n,k)|\,\psi^{\frac{-1}{p}}\,(k)\right)_{k\in\mathbb{Z}} \in \ell^p(\mathbb{Z},\psi).$$

Since $x \in \ell^p(\mathbb{Z},\varphi)$, then

$$d(n,m) = \left[\left(\left(|c(n,k)|\,\psi^{\frac{-1}{p}}\,(k)\right)_{k\in\mathbb{Z}}\right) * x\right](m)\psi^{\frac{1}{p}}(m) \quad \text{for every } m \in \mathbb{Z}.$$

Using (2),

$$\sum_{m\in\mathbb{Z}} |d(n,m)|^p \le |x|_{p,\varphi}^p \sum_{k\in\mathbb{Z}} |c(n,k)|^p$$

and

$$\sum_{n\in\mathbb{Z}}\left(\sum_{m\in\mathbb{Z}}|d(n,m)|^p\right)\leq |c|_p^p\,|x|_{p,\varphi}^p\,.$$

Thus

$$d\in\ell^p\left(\mathbb{Z}^2\right)\quad\text{and}\quad |d|_p\leq |c|_p\,|x|_{p,\varphi}\,.\qquad\square$$

**Lemma 2.6.** *Let* $x\in\ell^p\left(\mathbb{Z},\varphi\right)$ *and* $y\in\ell^q\left(\mathbb{Z},W^{\frac{1}{1-p}}\right)$*. Then*

$$\widetilde{x}*y\in\ell^q\left(\mathbb{Z},W^{\frac{1}{1-p}}\right)\quad\textit{and}\quad |\widetilde{x}*y|_{q,W^{\frac{1}{1-p}}}\leq |x|_{p,\varphi}\,|y|_{q,W^{\frac{1}{1-p}}}\,,$$

*where* $\widetilde{x}(n)=x(-n)$ *for every* $n\in\mathbb{Z}$*.*

*Proof.* We first prove that $S(\widetilde{x}*y)$ define a continuous linear form on $\ell^p\left(\mathbb{Z}^2\right)$. Indeed, for every $c=(c(n,m))_{(n,m)\in\mathbb{Z}^2}\in\ell^p\left(\mathbb{Z}^2\right)$, if

$$A=\left|\sum_{n,m\in\mathbb{Z}}\left(\widetilde{x}*y\right)(n+m)\frac{c(n,m)}{\varphi^{\frac{1}{p}}(n)\psi^{\frac{1}{p}}(m)}\right|,$$

then,

$$\begin{aligned}A&\leq\sum_{n,m\in\mathbb{Z}}\left(|\widetilde{x}|*|y|\right)(n+m)\frac{|c(n,m)|}{\varphi^{\frac{1}{p}}(n)\psi^{\frac{1}{p}}(m)}\\&\leq\sum_{n,m\in\mathbb{Z}}\left(\sum_{k\in\mathbb{Z}}|\widetilde{x}|\,(n+m-k)\,|y|\,(k)\right)\frac{|c(n,m)|}{\varphi^{\frac{1}{p}}(n)\psi^{\frac{1}{p}}(m)}.\end{aligned}$$

Using the fact that $\widetilde{x}(n)=x(-n)$ for every $n\in\mathbb{Z}$, and the change of variable $k=j+n$, we obtain:

$$\begin{aligned}A&\leq\sum_{n,m\in\mathbb{Z}}\left(\sum_{j\in\mathbb{Z}}|x|\,(j-m)\,|y|\,(j+n)\right)\frac{|c(n,m)|}{\varphi^{\frac{1}{p}}(n)\psi^{\frac{1}{p}}(m)}\\&\leq\sum_{n,j\in\mathbb{Z}}\frac{|y(j+n)|}{\psi^{\frac{1}{p}}(j)\varphi^{\frac{1}{p}}(n)}\left(\sum_{m\in\mathbb{Z}}|x|\,(j-m)\frac{|c(n,m)|}{\psi^{\frac{1}{p}}(m)}\right)\psi^{\frac{1}{p}}(j)\\&\leq\sum_{n,j\in\mathbb{Z}}|S(y)(n,j)|\left(\sum_{m\in\mathbb{Z}}|x|\,(j-m)\frac{|c(n,m)|}{\psi^{\frac{1}{p}}(m)}\psi^{\frac{1}{p}}(j)\right).\end{aligned}$$

As $S$ is a linear map from $\ell^q(\mathbb{Z},W^{\frac{1}{1-p}})$ into $\ell^q(\mathbb{Z}^2)$ and $d\in\ell^p\left(\mathbb{Z}^2\right)$, by Lemma 2.5, $S(y)\in\ell^q\left(\mathbb{Z}^2\right)$. Then, using Hölder's inequality:

$$\begin{aligned}A=\left|\sum_{n,m\in\mathbb{Z}}\left(\widetilde{x}*y\right)(n+m)\frac{c(n,m)}{\varphi^{\frac{1}{p}}(n)\psi^{\frac{1}{p}}(m)}\right|&\leq\sum_{n,j\in\mathbb{Z}}|S(y)(n,j)|\,|d(n,j|\\&\leq |S(y)|_q\,|d|_p\leq |S(y)|_q\,|x|_{p,\varphi}\,|c|_p\,.\end{aligned}$$

Since $|S(y)|_q = |y|_{q,W^{\frac{1}{1-p}}}$, we obtain:

$$A = \left| \sum_{n,m\in\mathbb{Z}} (\widetilde{x} * y)(n+m) \frac{c(n,m)}{\varphi^{\frac{1}{p}}(n)\psi^{\frac{1}{p}}(m)} \right| \leq |y|_{q,W^{\frac{1}{1-p}}} |x|_{p,\varphi} |c|_p .$$

As $\ell^q(\mathbb{Z}^2)$ is the dual space of $\ell^p(\mathbb{Z}^2)$, we have $S(\widetilde{x} * y) \in \ell^q(\mathbb{Z}^2)$ and

$$\begin{aligned} |S(\widetilde{x} * y)|_q &= \sup_{|c|_p \leq 1} \left| \sum_{n,m\in\mathbb{Z}} (\widetilde{x} * y)(n+m) \frac{c(n,m)}{\varphi^{\frac{1}{p}}(n)\psi^{\frac{1}{p}}(m)} \right| \\ &\leq \sup_{|c|_p \leq 1} |y|_{q,W^{\frac{1}{1-p}}} |x|_{p,\varphi} |c|_p \\ &\leq |y|_{q,W^{\frac{1}{1-p}}} |x|_{p,\varphi} . \end{aligned}$$

Finally, $\widetilde{x} * y \in \ell^q(\mathbb{Z}, W^{\frac{1}{1-p}})$. Indeed:

$$\begin{aligned} \sum_{k\in\mathbb{Z}} |(\widetilde{x} * y)(k)|^q W^{\frac{1}{1-p}}(k) &= \sum_{k\in\mathbb{Z}} |(\widetilde{x} * y)(k)|^q \left(\varphi^{\frac{1}{1-p}} * \psi^{\frac{1}{1-p}}\right)(k) \\ &= \sum_{k\in\mathbb{Z}} |(\widetilde{x} * y)(k)|^q \left( \sum_{m\in\mathbb{Z}} \varphi^{\frac{1}{1-p}}(k-m)\psi^{\frac{1}{1-p}}(m) \right) \\ &= \sum_{k,m\in\mathbb{Z}} |(\widetilde{x} * y)(k)|^q \varphi^{\frac{1}{1-p}}(k-m)\psi^{\frac{1}{1-p}}(m) \\ &= \sum_{n,m\in\mathbb{Z}} |(\widetilde{x} * y)(n+m)|^q \varphi^{\frac{1}{1-p}}(n)\psi^{\frac{1}{1-p}}(m) \\ &= |S(\widetilde{x} * y)|_q^q < +\infty. \end{aligned}$$

Thus

$$\widetilde{x} * y \in \ell^q(\mathbb{Z}, W^{\frac{1}{1-p}}) \text{ and } |\widetilde{x} * y|_{q,W^{\frac{1}{1-p}}} = |S(\widetilde{x} * y)|_q \leq |x|_{p,\varphi} |y|_{q,W^{\frac{1}{1-p}}} . \qquad \square$$

**Proposition 2.7.** *Assume that* $p \in ]1,+\infty[$, $q = \frac{p-1}{p}$, $\varphi$ *and* $\psi$ *are two weight maps on* $\mathbb{Z}$, *satisfying*:

1) $\ell^p(\mathbb{Z},\psi) = \mathcal{M}^p(\mathbb{Z},\varphi,\psi)$, *and*
2) $|a * b|_{p,\psi} \leq |a|_{p,\varphi}|b|_{p,\psi}$ *for every* $(a,b) \in \ell^p(\mathbb{Z},\varphi)\times \ell^p(\mathbb{Z},\psi)$.

*Then* $\ell^p(\mathbb{Z},W) = \mathcal{M}^p(\mathbb{Z},\varphi,W)$ *and*

$$|a * b|_{p,W} \leq |a|_{p,\varphi} |b|_{p,W} \quad \text{for every } (a,b) \in \ell^p(\mathbb{Z},\varphi) \times \ell^p(\mathbb{Z},W).$$

*Proof.* It's clear that $\mathcal{M}^p(\mathbb{Z},\varphi,W) \subset \ell^p(\mathbb{Z},W)$. Conversely, let $b \in \ell^p(\mathbb{Z},W)$ and $T_b(a) = a * b$ for every $a \in \ell^p(\mathbb{Z},\varphi)$. We will show that $T_b(a) \in \ell^p(\mathbb{Z},W)$ for every $a \in \ell^p(\mathbb{Z},\varphi)$ and $|a * b|_{p,W} \leq |a|_{p,\varphi} |b|_{p,W}$. For $y \in \ell^q\left(\mathbb{Z}, W^{\frac{1}{1-p}}\right)$, we have $\breve{a} * y \in \ell^q\left(\mathbb{Z}, W^{\frac{1}{1-p}}\right)$ by Lemma 2.6, where $\widetilde{a}(n) = a(-n)$ for every $n \in \mathbb{Z}$. Therefore,

$$\left|\sum_{m\in\mathbb{Z}}(\breve{a} * y)(m)b(m)\right| < +\infty.$$

Moreover,

$$\begin{aligned}\left|\sum_{m\in\mathbb{Z}}(\breve{a} * y)(m)b(m)\right| &= \left|\sum_{m\in\mathbb{Z}}\left(\sum_{n\in\mathbb{Z}}a\,(n-m)\,y\,(n)\right)b\,(m)\right| \\ &= \left|\sum_{n,m\in\mathbb{Z}}a\,(n-m)\,b\,(m)\,y\,(n)\right| \\ &= \left|\sum_{n\in\mathbb{Z}}(a*b)(n)y\,(n)\right|.\end{aligned}$$

Thus

$$a * b \in \left(\ell^q(\mathbb{Z}, W^{\frac{1}{1-p}})\right)' = \ell^p(\mathbb{Z}, W).$$

Furthermore, by Hölder's inequality and Lemma 2.6, we obtain:

$$\begin{aligned}\left|\sum_{n\in\mathbb{Z}}(a*b)(n)y\,(n)\right| &= \left|\sum_{m\in\mathbb{Z}}(\breve{a} * y)(m)b\,(m)\right| \\ &\le |\breve{a} * y|_{q,W^{\frac{1}{1-p}}}\,|b|_{p,W} \\ &\le |a|_{p,\varphi}\;|y|_{q,W^{\frac{1}{1-p}}}\,|b|_{p,W}\,.\end{aligned}$$

Consequently:

$$T_b(a) \in \ell^p(\mathbb{Z}, W) \quad \text{and} \quad |a*b|_{p,W} \le |a|_{p,\varphi}\,|b|_{p,W}.$$ □

Let $b \in \ell^p(\mathbb{Z}, \psi)$. Since $\ell^p(\mathbb{Z}, \psi) = \mathcal{M}^p(\mathbb{Z}, \varphi, \psi)$, one has $a * b \in \ell^p(\mathbb{Z}, \psi)$ for every $a \in \ell^p(\mathbb{Z}, \varphi)$. Using Hölder's inequality, we obtain:

$$|a * b(m)|^p W(m) \le \sum_{n\in\mathbb{Z}}|a(n)|^p\varphi(n)|b(m-n)|^p\psi(m-n) \text{ for every } m \in \mathbb{Z},$$

this yields, $a * b \in \ell^p(\mathbb{Z}, W)$ and $T_b : \ell^p(\mathbb{Z}, \varphi) \to \ell^p(\mathbb{Z}, W)$, $a \mapsto a * b$ is a continuous map satisfying:

$$|T_b(a)|_{p,W} \le |a|_{p,\varphi}\,|b|_{p,\psi}.$$

Now, we consider the map $T$ defined by:

$$T : \ell^p(\mathbb{Z}, \psi) \to \mathcal{L}_c\left[\ell^p(\mathbb{Z}, \varphi), \ell^p(\mathbb{Z}, W)\right] : b \mapsto T_b,$$

where $\mathcal{L}_c[\ell^p(\mathbb{Z}, \varphi), \ell^p(\mathbb{Z}, W)]$ is the space of all continuous linear maps of $\ell^p(\mathbb{Z}, \varphi)$ into $\ell^p(\mathbb{Z}, W)$. We have the following result:

**Proposition 2.8.** *$T$ is a one-to-one continuous linear map and its inverse is also continuous for the operator-norm.*

*Proof.* Observe first that $\ell^p(\mathbb{Z}, W)$ is a Banach space. Thus $\mathcal{L}_c[\ell^p(\mathbb{Z}, \varphi), \ell^p(\mathbb{Z}, W)]$ is also a Banach space. And we have:

$$\|T_b\| = \sup_{|a|_{p,\varphi} \leq 1} |T_b(a)|_{p,W} \leq |b|_{p,\psi} \text{ for every } b \in \ell^p(\mathbb{Z}, \psi).$$

Hence $T$ is continuous. Moreover, $T_b = 0$ if and only if $a * b = 0$ for every $a \in \ell^p(\mathbb{Z}, \varphi)$. Applying the last relation with $a = (\delta_{0,n})_{n\in\mathbb{Z}}$, where $\delta_{0,n} = 0$ or 1 according to whether $n \neq 0$ or $n = 0$, we obtain:

$$b_n = \sum_{m\in\mathbb{Z}} \delta_{0,n-m} b_m = 0 \text{ for every } n \in \mathbb{Z},$$

so $b = 0$. Thus $T$ is one-to-one. Thereby $T$ is a continuous map from $\ell^p(\mathbb{Z}, \psi)$ into $T(\ell^p(\mathbb{Z}, \psi))$. It remains to show that:

$$T^{-1} : T\left(\ell^p(\mathbb{Z}, \psi)\right) \to \ell^p(\mathbb{Z}, \psi) : T_b \mapsto b$$

is also continuous. To proceed, we show that $T\left(\ell^p(\mathbb{Z}, \psi)\right)$ is closed in the space $\mathcal{L}_c\left[\ell^p(\mathbb{Z}, \varphi), \ell^p(\mathbb{Z}, W)\right]$ and conclude by Banach theorem. Indeed, let $(b^{(k)})_{k\geq 0} \subset \ell^p(\mathbb{Z}, \psi)$ be a sequence such that $\left(T(b^{(k)})\right)_{k\geq 0}$ converges to $u$ in

$$\mathcal{L}_c\left[\ell^p(\mathbb{Z}, \varphi), \ell^p(\mathbb{Z}, W)\right].$$

Then, for every $a \in \ell^p(\mathbb{Z}, \varphi)$, the sequence $\left(T_{b^{(k)}}(a)\right)_{k\geq 0} = \left(b^{(k)} * a\right)_{k\geq 0}$ converges to $u(a)$ in $\ell^p(\mathbb{Z}, W)$. So, for every $a \in \ell^p(\mathbb{Z}, \varphi)$, the sequence of functions $\left(\mathcal{F}(b^{(k)})\mathcal{F}(a)\right)_{k\geq 0}$ converges uniformly to $\mathcal{F}(u(a))$, in the space $\mathcal{C}(\mathbb{R}, \mathbb{C})$. It follows that $\left(\mathcal{F}(b^{(k)})\right)_{k\geq 0}$ converges pointwise to a function $h \in \mathcal{C}(\mathbb{R}, \mathbb{C})$. Given Proposition 2.4, for every $a \in \ell^p(\mathbb{Z}, \varphi)$, there exists $c_a \in \ell^p(\mathbb{Z}^2)$ such that:

$$u(a) = L(c_a).$$

Since, for every $a \in \ell^p(\mathbb{Z}, \varphi)$, we have:

$$\mathcal{F}(u(a)) = \mathcal{F}(L(c_a)) = \lim_{k\to\infty} \mathcal{F}\left(a * b^{(k)}\right) = \lim_{k\to\infty} \mathcal{F}(a)\mathcal{F}(b^{(k)}) = \mathcal{F}(a)h,$$

$c_a$ is of the form:

$$c_a = a\varphi^{\frac{1}{p}} \otimes d, \text{ where } d \in \ell^p(\mathbb{Z}),$$

i.e.,

$$c_a = \left(a(n)\,\varphi^{\frac{1}{p}}(n) d(m)\right)_{(n,m)\in\mathbb{Z}^2} \in \ell^p(\mathbb{Z}) \otimes \ell^p(\mathbb{Z}).$$

By definition of $L$, we have:

$$L(c_a) = \left(\sum_{n\in\mathbb{Z}} \frac{a(n)\varphi^{\frac{1}{p}}(n) d(m-n)}{[\varphi(n)\psi(m-n)]^{\frac{1}{p}}} = \sum_{n\in\mathbb{Z}} a(n) \frac{d(m-n)}{\psi(m-n)^{\frac{1}{p}}}\right)_{m\in\mathbb{Z}}.$$

Then, for every $m \in \mathbb{Z}$,

$$L(c_a)(m) = \sum_{n\in\mathbb{Z}} \frac{a(n)\varphi^{\frac{1}{p}}(n)d(m-n)}{[\varphi(n)\psi(m-n)]^{\frac{1}{p}}} = \sum_{n\in\mathbb{Z}} a(n)\frac{d(m-n)}{\psi(m-n)^{\frac{1}{p}}} = \left(a * \frac{d}{\psi^{\frac{1}{p}}}\right)(m),$$

i.e.,

$$L(c_a) = \left(a * \frac{d}{\psi^{\frac{1}{p}}}\right) \text{ for every } a \in \ell^p(\mathbb{Z}, \varphi).$$

Hence, for every $a \in \ell^p(\mathbb{Z}, \varphi)$,

$$\mathcal{F}(L(c_a)) = \mathcal{F}(a * \frac{d}{\psi^{\frac{1}{p}}}) = \mathcal{F}(a)\mathcal{F}\left(\frac{d}{\psi^{\frac{1}{p}}}\right) = \mathcal{F}(a)h.$$

Thus, for every $a \in \ell^p(\mathbb{Z}, \varphi)$,

$$h = \mathcal{F}\left(\frac{d}{\psi^{\frac{1}{p}}}\right) \quad \text{and} \quad u(a) = a * \left(\frac{d}{\psi^{\frac{1}{p}}}\right) = T_{d\psi^{\frac{-1}{p}}}(a).$$

Consequently:

$$u = T_{d\psi^{\frac{-1}{p}}} \in T\left(\ell^p(\mathbb{Z}, \psi)\right).$$

This proves that $T\left(\ell^p(\mathbb{Z}, \psi)\right)$ is closed in $\mathcal{L}_c\left[\ell^p(\mathbb{Z}, \varphi), \ell^p(\mathbb{Z}, W)\right]$ and the proof is complete. □

Now, let's prove the converse of Theorem 2.1 in the case where $\mathbb{S} = \mathbb{Z}$.

**Theorem 2.9.** *Let $p \in ]1, +\infty[$, $\varphi$ and $\psi$ be two weight maps, on $\mathbb{Z}$, satisfying*:

1) $\ell^p(\mathbb{Z}, \psi) = \mathcal{M}^p(\mathbb{Z}, \varphi, \psi)$.
2) $|a * b|_{p,\psi} \le |a|_{p,\varphi}|b|_{p,\psi}$ *for every* $a \in \ell^p(\mathbb{Z}, \varphi)$ *and* $b \in \ell^p(\mathbb{Z}, \psi)$.
*Then*

$$\exists \gamma > 0 : \varphi^{\frac{1}{1-p}} * \psi^{\frac{1}{1-p}} \le \gamma\, \psi^{\frac{1}{1-p}}.$$

*Proof.* We will first show that the space $\ell^p(\mathbb{Z}, \psi) \cap \ell^p(\mathbb{Z}, W)$ is dense in $\ell^p(\mathbb{Z}, W)$. Indeed, for every $a, b \in \ell^p(\mathbb{Z})$, we have:

$$L\left((a \otimes b)_{(n,m)\in\mathbb{Z}^2}\right) = \left(a\varphi^{-\frac{1}{p}}\right) * \left(b\psi^{-\frac{1}{p}}\right) = T_{b\psi^{-\frac{1}{p}}}\left(a\varphi^{-\frac{1}{p}}\right) \in \ell^p(\mathbb{Z}, \psi).$$

Therefore,

$$L(\ell^p(\mathbb{Z})\widehat{\otimes}\ell^p(\mathbb{Z})) \subset \ell^p(\mathbb{Z}, \psi).$$

Since

$$L(\ell^p(\mathbb{Z}^2)) = \ell^p(\mathbb{Z}, W)$$

and

$$L\left(\left[\ell^p(\mathbb{Z})\widehat{\otimes}\ell^p(\mathbb{Z})\right] \cap \ell^p(\mathbb{Z}^2)\right) \subset L(\ell^p(\mathbb{Z})\widehat{\otimes}\ell^p(\mathbb{Z})) \cap L(\ell^p(\mathbb{Z}^2)),$$

we have:

$$L\left(\left[\ell^p(\mathbb{Z})\widehat{\otimes}\ell^p(\mathbb{Z})\right] \cap \ell^p(\mathbb{Z}^2)\right) \subset \ell^p(\mathbb{Z}, \psi) \cap \ell^p(\mathbb{Z}, W) \subset \ell^p(\mathbb{Z}, W) = L(\ell^p(\mathbb{Z}^2)).$$

We conclude by the continuity of $L$ and the density of $\left[\ell^p(\mathbb{Z})\widehat{\otimes}\ell^p(\mathbb{Z})\right] \cap \ell^p(\mathbb{Z}^2)$ in $\ell^p(\mathbb{Z}^2)$. Now we consider the identity map $I$ defined from $\ell^p(\mathbb{Z}, \psi) \cap \ell^p(\mathbb{Z}, W)$

(endowed with the norm of $\ell^p(\mathbb{Z},W)$) into $\ell^p(\mathbb{Z},\psi)$. By the continuity of $T^{-1}$, there exists a constant $K>0$, such that:

$$|b|_{p,\psi} \leq K \sup_{|a|_{p,\varphi}\leq 1} |T_b(a)|_{p,W} \text{ for every } b \in \ell^p(\mathbb{Z},\psi)\cap \ell^p(\mathbb{Z},W).$$

Moreover, since $\ell^p(\mathbb{Z},W) = \mathcal{M}^p(\mathbb{Z},\varphi,W)$, then for every $a\in \ell^p(\mathbb{Z},\varphi)$ and $b\in \ell^p(\mathbb{Z},W)$, we have:

$$T_b(a)\in \ell^p(\mathbb{Z},W) \quad\text{and}\quad |T_b(a)|_{p,W} \leq |a|_{p,\varphi}\ |b|_{p,W}.$$

Therefore $|b|_{p,\psi} \leq K\ |b|_{p,W}$ for every $b\in \ell^p(\mathbb{Z},\psi)\cap \ell^p(\mathbb{Z},W)$. Hence the identity $I$ is continuous on $\ell^p(\mathbb{Z},\psi)\cap \ell^p(\mathbb{Z},W)$ to $\ell^p(\mathbb{Z},\psi)$. Thus $I$ extends to a continuous map on $\ell^p(\mathbb{Z},W)$ and $\ell^p(\mathbb{Z},W)\subset \ell^p(\mathbb{Z},\psi)$. And so, there exists a constant $C>0$ such that:

$$|b|_{p,\psi} \leq C\ |b|_{p,W} \text{ for every } b\in \ell^p(\mathbb{Z},W),$$

which implies that $\psi \leq CW$, and

$$\varphi^{\frac{1}{1-p}} * \psi^{\frac{1}{1-p}} \leq \gamma\psi^{\frac{1}{1-p}}, \text{ where } \gamma = C^{\frac{1}{1-p}}. \qquad \square$$

As a consequence, we obtain the following result.

**Corollary 2.10** ([3] and [5])**.** *Let $p\in ]1,+\infty[$ and $\omega$ be a weight map on $\mathbb{Z}$. The following assertions are equivalent*:

1) *$\omega$ is an $m$-convolutive weight.*
2) *The space $(\ell^p(\mathbb{Z},\omega), |\cdot|_{p,\omega})$ is a commutative algebra, for the convolution product.*

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## References

[1] R. E. Edwards, *Functional Analysis. Theory and applications*, Holt, Rinehart and Winston, New York, 1965.

[2] A. El Kinani, *A version of Wiener's and Lévy's theorems*, Rend. Circ. Mat. Palermo (2) **57** (2008), no. 3, 343–352. https://doi.org/10.1007/s12215-008-0025-4

[3] A. El Kinani and A. Benazzouz, *Structure m-convexe dans l'espace à poids $L^p_\Omega(\mathbb{R}^n)$*, Bull. Belg. Math. Soc. Simon Stevin **10** (2003), no. 1, 49–57. http://projecteuclid.org/euclid.bbms/1047309412

[4] A. El Kinani and L. Bouchikhi, *Wiener's and Lévy's theorems for some weighted power series*, Rend. Circ. Mat. Palermo (2) **63** (2014), no. 2, 301–309. https://doi.org/10.1007/s12215-014-0159-5

[5] A. El Kinani, A. Roukbi, and A. Benazzouz, *Structure d'algèbre de Banach sur l'espace à poids $L^p_\omega(G)$*, Matematiche (Catania) **64** (2009), no. 1, 179–193.

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