Commun. Korean Math. Soc. **35** (2020), No. 4, pp. 1143–1158
https://doi.org/10.4134/CKMS.c200035
pISSN: 1225-1763 / eISSN: 2234-3024

BLOW-UP TIME AND BLOW-UP RATE FOR PSEUDO-PARABOLIC EQUATIONS WITH WEIGHTED SOURCE

Huafei Di and Yadong Shang

Abstract. In this paper, we are concerned with the blow-up phenomena for a class of pseudo-parabolic equations with weighted source $u_t - \triangle u - \triangle u_t = a(x)f(u)$ subject to Dirichlet (or Neumann) boundary conditions in any smooth bounded domain $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$. Firstly, we obtain the upper and lower bounds for blow-up time of solutions to these problems. Moreover, we also give the estimates of blow-up rate of solutions under some suitable conditions. Finally, three models are presented to illustrate our main results. In some special cases, we can even get some exact values of blow-up time and blow-up rate.

1. Introduction

In this paper, we deal with the blow-up time and blow-up rate estimates of solutions to the following initial boundary value problems

$$u_t - \triangle u - \triangle u_t = a(x)f(u),\ x \in \Omega,\ t > 0, \tag{1.1}$$

$$u(x,0) = g(x) \geq 0,\ x \in \Omega, \tag{1.2}$$

subject to null Dirichlet boundary condition

$$u(x,t) = 0,\ x \in \partial\Omega,\ t > 0, \tag{1.3}$$

or homogeneous Neumann boundary condition

$$\frac{\partial u}{\partial \nu} = 0,\ x \in \partial\Omega,\ t > 0, \tag{1.4}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ is a smooth bounded domain, ν is the outward normal vector, $g(x)$ is a continuous nonnegative function and satisfies the compatible condition. Here, the nonlinear function f satisfies

Received February 5, 2020; Accepted July 30, 2020.

2010 *Mathematics Subject Classification.* Primary 35K61, 35K70, 35A23, 35B44.

Key words and phrases. Pseudo-parabolic equation, upper and lower bounds, blow-up rate, weighted source.

This work was financially supported by the NSF of China (11801108, 11701116), the Scientific Program (2016A030310262) of Guangdong Province, and the College Scientific Research Project (YG2020005) of Guangzhou University.

©2020 Korean Mathematical Society

(f_1) $f(s) \geq 0$ for all $s \geq 0$;
And the weight function $a(x) \in C^1(\Omega) \cap C^0(\bar{\Omega})$ satisfies
(a_1) $a(x) \geq C > 0$ for some constant C on $\bar{\Omega}$ or
(a_2) $a(x) > 0$ in Ω and $a(x) = 0$ on $\partial\Omega$.

It is well known that the nonlinear pseudo-parabolic equations often appear in the study of various problems of the hydrodynamics, thermodynamics and filtration theory etc (see [1, 6, 9] and references therein). Also, Eq. (1.1) has extensive physical background and rich theoretical connotation. This type of equations can be regarded as the regularization of semilinear heat equations with weighted source by adding a dispersion term $\triangle u_t$. Especially, if $a(x) \equiv 1$, then Eq. (1.1) reduces to the following semilinear pseudo-parabolic equations

$$u_t - \triangle u - \triangle u_t = f(u),\ x \in \Omega,\ t > 0, \tag{1.5}$$

which also can be considered as Sobolev type equations [24]. In the past decades, much effort in mathematics have been devoted to the study of semilinear pseudo-parabolic equations about the existence and uniqueness [1,3,23], asymptotic behavior [3, 12, 28], blow-up phenomena [3, 4, 13, 20, 28], maximum principle [5] and homogenization [21] and so on. More works on this type of Eq. (1.5) can be found in the monograph [1] and references therein.

In the absence of dispersion term $\triangle u_t$, and weight function $a(x) \equiv 1$, Eq. (1.1) becomes the semilinear heat equations

$$u_t - \triangle u = f(u),\ x \in \Omega,\ t > 0. \tag{1.6}$$

About this model, many results for the blow-up phenomenon of solutions have been obtained, we can refer to [11, 16–19, 25, 27] and references therein. In [18, 19], Payne and Schaefer obtained the lower bound on blow-up time of solutions to Eq. (1.6) under null Dirichlet boundary condition and homogeneous Neumann boundary condition, respectively. Later, Payne et al. [16, 17] studied the blow-up phenomenon of solutions for Eq. (1.6) with nonlinear boundary conditions. When the nonlinear source term $f(u) = \int_\Omega u^q dx - ku^s$, Song [25] obtained the lower bounds for blow-up time of solutions with either homogeneous Dirichlet or homogeneous Neumann boundary conditions in three dimensional space. Afterwards, Liu [11] studied the lower bounds for blow-up time under nonlinear boundary conditions in three dimensional space. In [27], Tang et al. extended the results of literature [11] in higher dimensional space.

Very recently, the study on the blow-up phenomenon had some new development, where more attention was paid on the parabolic equations with weighted source. These models can be used to illustrate the processes of heat transfer arising in physical and engineering applications, such as a model of phase separation in binary alloys [2, 22]. The existence and nonexistence of global solutions, bounds for blow-up time, blow-up rate, blow-up sets and asymptotic behavior for this type of equations were investigated by many authors. We refer the reader to see [8, 14, 15, 26] and papers cited therein. For example, Song and Lv [14, 26] studied the following semilinear parabolic equations with weighted source

$$u_t - \triangle u = a(x)f(u),\ x \in \Omega,\ t > 0. \tag{1.7}$$

The initial boundary value problem for above equations with nonlinear Neumann boundary condition were considered in [14], where they derived the upper and lower bounds for blow-up time in three dimensional space. In [26], they further investigated the estimates of blow-up rate and bounds for blow-up time of solutions with homogeneous Dirichlet or Neumann boundary conditions in higher dimensional space. Ma and Fang [15] changed the diffusion term $\triangle u$ into nonlinear divergence form reaction-diffusion term $\sum_{i,j=1}^{N}\left(a^{i,j}(x)u_{x_i}\right)_{x_j}$ of Eq. (1.7), where the upper and lower bounds for blow-up time were derived under appropriate measure in higher dimensional space. In [8], a blow-up analysis for nonlinear divergence form of parabolic equation with time-dependent coefficients was given under nonlinear boundary flux.

Motivated by the above researches, in the present work we main study the blow-up phenomena for pseudo-parabolic equations with weighted nonlinear source. As far as we known, there is litter infirmation on the blow-up results of solutions for problem (1.1)-(1.2) with either null Dirichlet boundary condition (1.3) or homogeneous Neumann boundary condition (1.4). Obviously, the existence and uniqueness of local solutions for these problems can be obtained by Faedo-Galerkin methods and Contraction Mapping Principle. The interested reader is referred to $[1, 7, 10]$ for details. Naturally, we would like to study the estimates of blow-up rate and bounds for blow-up time of solutions to Eq. (1.1) in any smooth bounded domain $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$. Especially, we can even give the exact estimates about the blow-up rate and blow-up time of solutions to Eq. (1.1) with some types of nonlinearities.

In detail, this paper is organized as follows: In Section 2, the upper bounds of blow-up time and blow-up rate will be established for problem (1.1)-(1.2) under boundary condition (1.3) or (1.4). In Section 3, we will use two methods to give the lower bounds for blow-up time and blow-up rate to problem (1.1)-(1.2) with boundary condition (1.3) or (1.4). In Section 4, three models and some remarks will be given to illustrate the results of Sections 2 and 3.

2. Upper estimates for blow-up time and blow-up rate

In this section, we will establish some estimates about the upper bounds for blow-up time and blow-up rate of solutions to problem (1.1)-(1.2) under Dirichlet boundary condition or Neumann boundary condition, respectively.

2.1. Under Dirichlet boundary condition (1.3)

To obtain the results of this subsection, we first assume that

(f_2) there exists a positive constant $C_1 > 2$ such that

$$\int_\Omega a(x)s(x)f(s(x))dx \geq C_1 \int_\Omega a(x)F(s(x))dx,$$

for any function $s(x) \geq 0$, where $F(s(x)) = \int_0^{s(x)} f(\theta)d\theta$;

(g_1) the initial data $g(x)$ satisfies

$$\int_\Omega |\nabla g|^2 dx < 2 \int_\Omega a(x)F(g)dx.$$

Then, we further define the following auxiliary function

$$\varphi(t) = \int_\Omega u^2 dx + \int_\Omega |\nabla u|^2 dx. \tag{2.1}$$

Theorem 2.1. *Assume that the conditions* (f_1), (f_2), (g_1), (a_1), (a_2) *hold, and* u *is a nonnegative solution of problem* (1.1)-(1.2) *subject to Dirichlet boundary condition* (1.3). *Then, we conclude that the solutions* u *become unbounded in* H^1*-norm at* $t = t^*$. *Moreover, an upper bound for blow-up time* t^* *is given by*

$$t^* \leq \frac{2\varphi(0)}{(C_1 - 2)\phi_1(0)},$$

and the upper estimate of blow-up rate can be given by

$$\|u\|_{H^1} \leq \left(\frac{(C_1-2)\phi_1(0)}{2}\right)^{-\frac{1}{C_1-2}} [\varphi(0)]^{\frac{C_1}{2(C_1-2)}} (t^* - t)^{-\frac{1}{C_1-2}},$$

where $\varphi(0) = \|g\|_{H^1}^2$ *and* $\phi_1(0) = -C_1 \int_\Omega |\nabla g|^2 dx + 2C_1 \int_\Omega a(x)F(g)dx$.

Proof. Firstly, differentiating (2.1) with respect to t and using Eq. (1.1), then we have

$$\begin{aligned} \varphi'(t) &= 2\int_\Omega u u_t dx + 2\int_\Omega \nabla u \cdot \nabla u_t dx \\ &= -2\int_\Omega |\nabla u|^2 dx + 2\int_\Omega a(x) u f(u) dx. \end{aligned} \tag{2.2}$$

By the combination of (2.2) and condition (f_2), we obtain

$$\varphi'(t) \geq -2\int_\Omega |\nabla u|^2 dx + 2C_1 \int_\Omega a(x)F(u)dx \geq \phi_1(t), \tag{2.3}$$

where

$$\phi_1(t) = -C_1 \int_\Omega |\nabla u|^2 dx + 2C_1 \int_\Omega a(x)F(u)dx. \tag{2.4}$$

On the other hand, a simple computation yields

$$\begin{aligned} \phi_1'(t) &= -2C_1 \int_\Omega \nabla u \cdot \nabla u_t dx + 2C_1 \int_\Omega a(x) u_t f(u) dx \\ &= 2C_1 \int_\Omega u_t \big[\triangle u + a(x) f(u)\big] dx \\ &= 2C_1 \int_\Omega u_t \big[u_t - \triangle u_t\big] dx \\ &= 2C_1 \int_\Omega u_t^2 dx + 2C_1 \int_\Omega |\nabla u_t|^2 dx. \end{aligned} \tag{2.5}$$

Here, we have used the fact that $u(x,t)=0$ on $\partial\Omega$ implies that $u_t(x,t)=0$ on $\partial\Omega$. Multiplying $\varphi(t)$ by $\phi_1'(t)$, then we have

$$\varphi(t)\phi_1'(t)=2C_1\int_\Omega\left[u_t^2+|\nabla u_t|^2\right]dx\int_\Omega\left[u^2+|\nabla u|^2\right]dx. \tag{2.6}$$

Using Schwarz's inequality and Young's inequality, we get

$$\left(\int_\Omega uu_t dx\right)^2\leq\int_\Omega u^2dx\int_\Omega u_t^2dx, \tag{2.7}$$

$$\left(\int_\Omega \nabla u\cdot\nabla u_t dx\right)^2\leq\int_\Omega |\nabla u^2|dx\int_\Omega |\nabla u_t|^2dx, \tag{2.8}$$

and

$$2\int_\Omega uu_tdx\int_\Omega\nabla u\cdot\nabla u_tdx\leq\int_\Omega u^2dx\int_\Omega|\nabla u_t|^2dx+\int_\Omega|\nabla u|^2dx\int_\Omega u_t^2dx. \tag{2.9}$$

Inserting (2.7)-(2.9) into (2.6), we have from (2.2) and (2.3) that

$$\begin{aligned}\varphi(t)\phi_1'(t)&\geq 2C_1\left(\int_\Omega uu_tdx+\int_\Omega\nabla u\cdot\nabla u_tdx\right)^2\\&=\frac{C_1}{2}\left[\varphi'(t)\right]^2\geq\frac{C_1}{2}\varphi'(t)\phi_1(t).\end{aligned} \tag{2.10}$$

Thus, the above inequality implies that

$$\left(\phi_1(t)\left[\varphi(t)\right]^{-\frac{C_1}{2}}\right)'=\left[\varphi(t)\right]^{-\frac{C_1+2}{2}}\left\{\varphi(t)\phi_1'(t)-\frac{C_1}{2}\varphi'(t)\phi_1(t)\right\}\geq 0. \tag{2.11}$$

Under the assumption (g_1) and using (2.1) and (2.5), we know that

$$\varphi(0)=\|g\|_{H^1}^2>0, \tag{2.12}$$

and

$$\phi_1(t)\geq\phi_1(0)=-C_1\int_\Omega|\nabla g|^2dx+2C_1\int_\Omega a(x)F(g)dx>0. \tag{2.13}$$

Integrating (2.11) from 0 to t, we obtain

$$\phi_1(t)\left[\varphi(t)\right]^{-\frac{C_1}{2}}\geq\phi_1(0)\left[\varphi(0)\right]^{-\frac{C_1}{2}}=M>0. \tag{2.14}$$

By (2.3) and (2.14), we get

$$\begin{aligned}\frac{1}{C_1(2-C_1)}\left(\left[\varphi(t)\right]^{\frac{2-C_1}{2}}\right)'&=\frac{1}{2C_1}\varphi'(t)\left[\varphi(t)\right]^{-\frac{C_1}{2}}\\&\geq\frac{1}{2C_1}\phi_1(t)\left[\varphi(t)\right]^{-\frac{C_1}{2}}\geq\frac{1}{2C_1}M.\end{aligned} \tag{2.15}$$

Integrating (2.15) from 0 to t, we have

$$\left[\varphi(t)\right]^{\frac{2-C_1}{2}}\leq\left[\varphi(0)\right]^{\frac{2-C_1}{2}}-\frac{C_1-2}{2}Mt, \tag{2.16}$$

which implies that

$$\varphi(t) \geq \frac{1}{\left[(\varphi(0))^{\frac{2-C_1}{2}} - \frac{C_1-2}{2}Mt\right]^{\frac{2}{C_1-2}}}. \tag{2.17}$$

Clearly, the above inequality cannot hold for all $t > 0$. Consequently, u blow up at some finite time t^* and

$$t^* \leq \frac{2\varphi(0)}{(C_1-2)\phi_1(0)}. \tag{2.18}$$

Furthermore, from (2.3) and (2.14) again, we have

$$\varphi'(t) \geq \phi_1(t) \geq \phi_1(0)\,[\varphi(0)]^{-\frac{C_1}{2}}\,[\varphi(t)]^{\frac{C_1}{2}}. \tag{2.19}$$

Integrating (2.19) from t to t^*, we obtain

$$\varphi(t) \leq \left(\frac{2\varphi(0)^{\frac{C_1}{2}}}{(C_1-2)\phi_1(0)}\right)^{\frac{2}{C_1-2}} (t^*-t)^{-\frac{2}{C_1-2}}, \tag{2.20}$$

which means that the upper estimate of blow-up rate is given by

$$\|u\|_{H^1} \leq \left(\frac{2\varphi(0)^{\frac{C_1}{2}}}{(C_1-2)\phi_1(0)}\right)^{\frac{1}{C_1-2}} (t^*-t)^{-\frac{1}{C_1-2}}. \tag{2.21}$$

□

2.2. Under Neumann boundary condition (1.4)

To obtain the results of this subsection, we first assume that
(f_3) there exists a positive function $G(\theta)$ such that

$$\int_\Omega a(x)f(s(x))dx \geq G\left[\int_\Omega s(x)dx\right] \text{ with } \int_0^{+\infty}\frac{d\theta}{G(\theta)} < +\infty$$

for any function $s(x) \geq 0$. Then, we define the following auxiliary function

$$\phi_2(t) = \int_\Omega u dx. \tag{2.22}$$

Theorem 2.2. *Assume that the conditions (f_1), (f_3), (a_1), (a_2) hold, and u is a nonnegative solution of problem* (1.1)-(1.2) *subject to Neumann boundary condition* (1.4). *Then, we conclude that the solutions u become unbounded in L^1-norm at $t = t^*$. Moreover, an upper bound for blow-up time t^* is given by*

$$t^* \leq \int_0^{+\infty}\frac{d\theta}{G(\theta)} < +\infty,$$

and the upper estimate of the blow-up rate can be given by

$$\|u\|_{L^1} \leq Y_1^{-1}(t^*-t),$$

where the function $Y_1(s) := \int_s^{+\infty}\frac{d\theta}{G(\theta)}$ for any $s \geq 0$.

Proof. Integrating Eq. (1.1) by parts, from condition (f_3) and (2.22) we have

$$\int_\Omega u_t dx = \int_\Omega a(x)f(u)dx \geq G\left[\int_\Omega u dx\right], \tag{2.23}$$

which means that

$$\frac{d\phi_2(t)}{dt} \geq G\left[\phi_2(t)\right] > 0. \tag{2.24}$$

Here, we have used the fact that $\frac{\partial u}{\partial \nu} = 0$ and $\frac{\partial u_t}{\partial \nu} = 0$ on $\partial\Omega$. It then follows from (2.24) that $\phi_2(t)$ is an increasing function, so we have

$$\phi_2(t) > \phi_2(0) = \int_\Omega g(x)dx \geq 0. \tag{2.25}$$

Integrating (2.24) from 0 to t and using (2.25), (f_3), we discover

$$t \leq \int_{\phi_2(0)}^{\phi_2(t)} \frac{d\theta}{G(\theta)} \leq \int_0^{+\infty} \frac{d\theta}{G(\theta)} < +\infty. \tag{2.26}$$

Obviously, (2.26) cannot hold for all time t. Consequently, we can derive an upper bound t^* such that

$$t^* \leq \int_0^{+\infty} \frac{d\theta}{G(\theta)} < +\infty, \tag{2.27}$$

and

$$\lim_{t\to t^*} \phi_2(t) = +\infty, \tag{2.28}$$

where $(0, t^*)$ is the interval of existence of solutions u in L^1-norm. In fact, if the equality (2.28) doesn't hold, then there exists a time $t_1 > t^*$ such that $\phi_2(t^*) < \phi_2(t_1) < +\infty$ and t_1 satisfies the inequalities (2.26), (2.27), which contradict the maximum existence of t^*.

Furthermore, integrating (2.24) from t to t^*, it follows that

$$t^* - t \leq \int_{\phi_2(t)}^{+\infty} \frac{d\theta}{G(\theta)} := Y_1(\phi_2(t)). \tag{2.29}$$

We note that Y_1 is a decreasing function, which means its inverse function Y_1^{-1} exists and is also a decreasing function. Therefore, we have

$$\phi_2(t) \leq Y_1^{-1}(t^* - t), \tag{2.30}$$

that is

$$\|u\|_{L^1} \leq Y_1^{-1}(t^* - t). \tag{2.31}$$

□

Remark 2.3. This result can be generalized to the case of problem (1.1)-(1.2) subject to $\frac{\partial u}{\partial \nu} = b(x,t)$, where $b(x,t)$ is a nonnegative increasing function of time t, on $x \in \partial\Omega$. In this case, we can also obtain the inequalities (2.27) and (2.31).

3. Lower estimates for blow-up time and blow-up rate

In this section, we will give two methods to establish the lower bounds for blow-up time and blow-up rate of solutions to problem (1.1)-(1.2) under Dirichlet (or Neumann) boundary condition.

3.1. The first method

This method can not only be used to deal with the problem (1.1)-(1.2) under null Dirichlet boundary condition (1.3) but also be applied to discuss (1.1)-(1.2) subject to homogeneous Neumann boundary condition (1.4).

Firstly, let us assume that

(f_4): there exist positive constants C_2, C_3 such that

$$a(x)f(s(x)) \leq C_2 + C_3 s(x)^l$$

for any function $s(x) \geq 0$, where $1 < l < +\infty$ if $n \leq 2$, $1 < l < \frac{n+2}{n-2}$ if $n \geq 3$. And then we introduce the auxiliary function $\varphi(t) = \int_\Omega u^2 dx + \int_\Omega |\nabla u|^2 dx$ as (2.1).

Next, we shall state and prove the main results of this subsection as follows:

Theorem 3.1. *Assume that the conditions* (f_1), (f_4), (a_1), (a_2) *hold, and* u *is a nonnegative solution of problem* (1.1)-(1.2) *subject to Dirichlet* (*or Neumann*) *boundary condition which becomes unbounded in* H^1*-norm at* $t = t^*$*. Then, we conclude that a lower bound for blow-up time* t^* *is given by*

$$t^* \geq \int_{\varphi(0)}^{+\infty} \frac{d\eta}{k_1 \eta^{\frac{1}{2}} + k_2 \eta^{\frac{l+1}{2}}},$$

and the lower estimate of the blow-up rate is

$$\|u\|_{H^1} \geq \left[k_2(l-1)\right]^{-\frac{1}{l-1}} (t^* - t)^{-\frac{1}{l-1}},$$

where $k_1 = \frac{2C_2|\Omega|^{\frac{1}{2}}}{\sqrt{1+\lambda_1}}$, $k_2 = 2C_3 B^{l+1}$, *and* B *is the optimal constant satisfying the Sobolev's inequality* $\|u\|_{l+1} \leq B\|u\|_{H^1}$.

Proof. Multiplying u on two sides of Eq. (1.1) and integrating by parts, we have

$$\int_\Omega u u_t dx + \int_\Omega \nabla u \cdot \nabla u_t dx = -\int_\Omega |\nabla u|^2 dx + \int_\Omega a(x) u f(u) dx. \tag{3.1}$$

Differentiating (2.1) with respect to t, we have from condition (f_4) and (3.1) that

$$\begin{aligned} \varphi'(t) &= -2\int_\Omega |\nabla u|^2 dx + 2\int_\Omega a(x) u f(u) dx \\ &\leq -2\int_\Omega |\nabla u|^2 dx + 2C_2 \int_\Omega u dx + 2C_3 \int_\Omega u^{l+1} dx. \end{aligned} \tag{3.2}$$

Using Schwarz's inequality and Sobolev's inequality, we obtain

$$\varphi'(t) \leq -2\int_\Omega |\nabla u|^2 dx + 2C_2|\Omega|^{\frac{1}{2}}\left(\int_\Omega u^2 dx\right)^{\frac{1}{2}} + 2C_3B^{l+1}\|u\|_{H^1}^{l+1}, \tag{3.3}$$

where $|\Omega|$ denotes the volume of Ω, B is the embedding constant for spaces $H_0^1(\Omega) \hookrightarrow L^{l+1}(\Omega)$.

The Poincáre's inequality gives $\|\nabla u\|^2 \geq \lambda_1\|u\|^2$, where λ_1 is the first eigenvalue of the problem

$$\begin{cases} \triangle w + \lambda w = 0, \text{ in } \Omega, \\ w = 0, \text{ on } \partial\Omega. \end{cases}$$

Thus, we have

$$\|\nabla u\|^2 = \frac{1}{1+\lambda_1}\|\nabla u\|^2 + \frac{\lambda_1}{1+\lambda_1}\|\nabla u\|^2 \geq \frac{\lambda_1}{1+\lambda_1}\|u\|_{H^1}^2, \tag{3.4}$$

$$\|u\|^2 = \frac{1}{1+\lambda_1}\|u\|^2 + \frac{\lambda_1}{1+\lambda_1}\|u\|^2 \leq \frac{1}{1+\lambda_1}\|u\|_{H^1}^2. \tag{3.5}$$

Then, (2.1), (3.3) and (3.5) imply

$$\varphi'(t) \leq \frac{2C_2|\Omega|^{\frac{1}{2}}}{\sqrt{1+\lambda_1}}\big[\varphi(t)\big]^{\frac{1}{2}} + 2C_3B^{l+1}\big[\varphi(t)\big]^{\frac{l+1}{2}}. \tag{3.6}$$

Integrating (3.6) from 0 to t, we get

$$\int_{\varphi(0)}^{\varphi(t)} \frac{d\eta}{k_1\eta^{\frac{1}{2}} + k_2\eta^{\frac{l+1}{2}}} \leq t, \tag{3.7}$$

where $k_1 = \frac{2C_2|\Omega|^{\frac{1}{2}}}{\sqrt{1+\lambda_1}}$, $k_2 = 2C_3B^{l+1}$. If u blow up in the measure $\varphi(t)$ as $t \to t^*$, then we can obtain the lower bound

$$t^* \geq \int_{\varphi(0)}^{+\infty} \frac{d\eta}{k_1\eta^{\frac{1}{2}} + k_2\eta^{\frac{l+1}{2}}}. \tag{3.8}$$

Furthermore, integrating the inequality (3.6) from t to t^*, we obtain

$$t^* - t \geq \int_{\varphi(t)}^{+\infty} \frac{d\eta}{k_1\eta^{\frac{1}{2}} + k_2\eta^{\frac{l+1}{2}}} := Y_2(\varphi(t)). \tag{3.9}$$

We note that Y_2 is a decreasing function, which means its inverse function Y_2^{-1} exists and is also a decreasing function. Therefore, we have

$$\varphi(t) \geq Y_2^{-1}(t^* - t), \tag{3.10}$$

which gives the lower estimate of blow-up rate. In fact, if t closes t^* enough such that $\varphi(t) \gg 1$ and $k_2\eta^{\frac{l+1}{2}} > k_1\eta^{\frac{1}{2}}$ in the inequality (3.9), then we have

$$t^* - t \geq \frac{1}{k_2(l-1)}\big[\varphi(t)\big]^{-\frac{l-1}{2}}, \tag{3.11}$$

which means that

$$\varphi(t) \geq \left[k_2(l-1)\right]^{-\frac{2}{l-1}}(t^*-t)^{-\frac{2}{l-1}},$$

or

$$\|u\|_{H^1} \geq \left[k_2(l-1)\right]^{-\frac{1}{l-1}}(t^*-t)^{-\frac{1}{l-1}}. \tag{3.12}$$

□

3.2. The second method

In this subsection, we will use the second method to establish the lower bounds for blow-up time and blow-up rate of solutions to problem (1.1)-(1.2) under null Dirichlet boundary condition (1.3) or homogeneous Neumann boundary condition (1.4).

Firstly, we need the following assumption:

(f_5) there exist positive constants C_4, C_5 and Q such that

$$a(x)f(s(x)) \leq C_4 + C_5 s(x)^p \left(\int_\Omega s(x)^{q+1}dx\right)^Q$$

for any function $s(x) \geq 0$.

(e_1) we also assume that

$$1 < q < +\infty \text{ if } n \leq 2,\ 1 < q < \frac{n+2}{n-2} \text{ if } n \geq 3,$$
$$0 \leq p \leq 1 \text{ and } (q+1)Q + p > 1.$$

To obtain the main results, we define the auxiliary function $\varphi(t) = \int_\Omega u^2 dx + \int_\Omega |\nabla u|^2 dx$ again.

Next, we will state our results below:

Theorem 3.2. *Assume that the conditions* (f_1), (f_5), (e_1), (a_1), (a_2) *hold, and u is a nonnegative solution of problem* (1.1)-(1.2) *subject to Dirichlet* (*or Neumann*) *boundary condition which becomes unbounded in H^1-norm at $t = t^*$. Then, we conclude that a lower bound for blow-up time t^* is given by*

$$t^* \geq \int_{\varphi(0)}^{+\infty} \frac{d\eta}{k_3\eta^{\frac{1}{2}} + k_4\eta^{\frac{(q+1)Q+p+1}{2}}},$$

and the lower estimate of the blow-up rate is

$$\|u\|_{H^1} \geq \left[k_4((q+1)Q+p-1)\right]^{-\frac{1}{(q+1)Q+p-1}}(t^*-t)^{-\frac{1}{(q+1)Q+p-1}},$$

where $k_3 = \frac{2C_4|\Omega|^{\frac{1}{2}}}{\sqrt{1+\lambda_1}}$, $k_4 = 2C_5 B_1^{(q+1)Q}|\Omega|^{\frac{1-p}{2}}\left(\frac{1}{1+\lambda_1}\right)^{\frac{p+1}{2}}$, *and B_1 is the optimal constant satisfying the Sobolev's inequality* $\|u\|_{q+1} \leq B_1\|u\|_{H^1}$.

Proof. From condition (f_5) and (3.1), we obtain

$$\varphi'(t) = -2\int_\Omega |\nabla u|^2 dx + 2\int_\Omega a(x)uf(u)dx$$

$$\leq -2\int_\Omega |\nabla u|^2dx + 2C_4\int_\Omega udx + 2C_5\left(\int_\Omega u^{q+1}dx\right)^Q\int_\Omega u^{p+1}dx. \tag{3.13}$$

Using Schwarz's inequality and Sobolev's inequality, it follows that

$$\begin{aligned}\varphi'(t) \leq\ & -2\int_\Omega |\nabla u|^2dx + 2C_4|\Omega|^{\frac{1}{2}}\left(\int_\Omega u^2dx\right)^{\frac{1}{2}}\\ &+ 2C_5B_1^{(q+1)Q}|\Omega|^{\frac{1-p}{2}}\|u\|_{H^1}^{(q+1)Q}\left(\int_\Omega u^2dx\right)^{\frac{p+1}{2}},\end{aligned} \tag{3.14}$$

where B_1 is the embedding constant for spaces $H_0^1(\Omega)\hookrightarrow L^{q+1}(\Omega)$.

Utilizing the Poincáre's inequality, we have from (3.5) and (3.14) that

$$\begin{aligned}\varphi'(t) \leq\ & \frac{2C_4|\Omega|^{\frac{1}{2}}}{\sqrt{1+\lambda_1}}\big[\varphi(t)\big]^{\frac{1}{2}} + 2C_5B_1^{(q+1)Q}|\Omega|^{\frac{1-p}{2}}\\ &\times\left(\frac{1}{1+\lambda_1}\right)^{\frac{p+1}{2}}\big[\varphi(t)\big]^{\frac{(q+1)Q+p+1}{2}}.\end{aligned} \tag{3.15}$$

Integrating the above inequality from 0 to t, we get

$$\int_{\varphi(0)}^{\varphi(t)}\frac{d\eta}{k_3\eta^{\frac{1}{2}}+k_4\eta^{\frac{(q+1)Q+p+1}{2}}} \leq t, \tag{3.16}$$

where $k_3 = \frac{2C_4|\Omega|^{\frac{1}{2}}}{\sqrt{1+\lambda_1}}$, $k_4 = 2C_5B_1^{(q+1)Q}|\Omega|^{\frac{1-p}{2}}\left(\frac{1}{1+\lambda_1}\right)^{\frac{p+1}{2}}$ and $(q+1)Q+p>1$. If u blows up in the measure $\varphi(t)$ as $t\to t^*$, then we can obtain the lower bound

$$t^* \geq \int_{\varphi(0)}^{+\infty}\frac{d\eta}{k_3\eta^{\frac{1}{2}}+k_4\eta^{\frac{(q+1)Q+p+1}{2}}}. \tag{3.17}$$

Furthermore, integrating the inequality (3.15) from t to t^*, we obtain

$$t^*-t \geq \int_{\varphi(t)}^{+\infty}\frac{d\eta}{k_3\eta^{\frac{1}{2}}+k_4\eta^{\frac{(q+1)Q+p+1}{2}}} := Y_3(\varphi(t)). \tag{3.18}$$

We note that Y_3 is a decreasing function, which means its inverse function Y_3^{-1} exists and it is also a decreasing function. Therefore, we have

$$\varphi(t) \geq Y_3^{-1}(t^*-t), \tag{3.19}$$

which gives the lower estimate of blow-up rate. In fact, if t closes t^* enough such that $\varphi(t)\gg 1$ and $k_4\eta^{\frac{(q+1)Q+p+1}{2}} > k_3\eta^{\frac{1}{2}}$, then we have from (3.18) that

$$t^*-t \geq \frac{1}{k_4[(q+1)Q+p-1]}\big[\varphi(t)\big]^{-\frac{(q+1)Q+p-1}{2}}, \tag{3.20}$$

which means that

$$\varphi(t) \geq \big[k_4((q+1)Q+p-1)\big]^{-\frac{2}{(q+1)Q+p-1}}(t^*-t)^{-\frac{2}{(q+1)Q+p-1}}, \tag{3.21}$$

or

$$\|u\|_{H^1} \geq \left[k_4((q+1)Q+p-1)\right]^{-\frac{1}{(q+1)Q+p-1}}(t^*-t)^{-\frac{1}{(q+1)Q+p-1}}. \qquad \square \tag{3.22}$$

4. Some related models and remarks

In this section, we will present three models of problem (1.1)-(1.2) with homogeneous Dirichlet (or Neumann) boundary condition to illustrate the main results which have been obtained in Sections 2, 3 and make some discussions.

Model 4.1. *We consider a special model as follows*:

$$u_t - \triangle u - \triangle u_t = a(x)u^l, \ x \in \Omega, \ t > 0, \tag{4.1}$$

$$u(x,0) = g(x) \geq 0, \ x \in \Omega, \tag{4.2}$$

$$u(x,t) = 0 \ \text{or} \ \frac{\partial u}{\partial \nu} = 0, \ x \in \partial\Omega, \ t > 0, \tag{4.3}$$

where the parameter l satisfy $1 < l < +\infty$ if $n \leq 2$, $1 < l < \frac{n+2}{n-2}$ if $n \geq 3$; And the initial data $g(x)$ satisfy $(l+1)\int_\Omega |\nabla g|^2 dx < 2\int_\Omega a(x)g^{l+1}dx$.

As a particular case $f(u) = u^l$, $l > 1$ of problem (1.1)-(1.4), applying Theorems 2.1 and 3.1, then we have:

Theorem 4.2. *Assume that the conditions (a_1), (a_2) hold, and u is a non-negative solution of problem (4.1)-(4.3). Then, we conclude that the solutions u become unbounded in H^1-norm at $t = t^*$. Moreover, the bounds for blow-up time t^* are given by*

$$\frac{2\varphi(0)^{\frac{1-l}{2}}}{k_5(l-1)} \leq t^* \leq \frac{2}{l-1}\frac{\varphi(0)}{\phi_3(0)},$$

and the estimates of the blow-up rate can be given by

$$\begin{aligned}&\left[k_5(l-1)\right]^{-\frac{1}{l-1}}(t^*-t)^{-\frac{1}{l-1}} \leq \|u\|_{H^1}\\ \leq &\left(\frac{(l-1)\phi_3(0)}{2}\right)^{-\frac{1}{l-1}}\left[\varphi(0)\right]^{\frac{l+1}{2(l-1)}}(t^*-t)^{-\frac{1}{l-1}},\end{aligned}$$

where $k_5 = 2\sup_{x\in\bar{\Omega}} a(x)B^{l+1}$, $\phi_3(0) = -(l+1)\int_\Omega |\nabla g|^2 dx + 2\int_\Omega a(x)g^{l+1}dx$, $\varphi(0) = \|g\|_{H^1}^2$, and B is the optimal constant satisfying the Sobolev's inequality $\|u\|_{l+1} \leq B\|u\|_{H^1}$.

Model 4.3. *Let us consider the initial boundary value problem of a pseudo-parabolic equations with weighted nonlocal source*:

$$u_t - \triangle u - \triangle u_t = a(x)\left(\int_\Omega u^r dx\right)^{\frac{p}{r}}, \ x \in \Omega, \ t > 0, \tag{4.4}$$

$$u(x,0) = g(x) \geq 0, \ x \in \Omega, \tag{4.5}$$

$$\frac{\partial u}{\partial \nu} = 0, \ x \in \partial\Omega, \ t > 0, \tag{4.6}$$

where the parameters $1 \le r < +\infty$ *if* $n \le 2$, $1 \le r < \frac{2n}{n-2}$ *if* $n \ge 3$, *and* $p > 1$.

In the particular case $f(u) = \left(\int_\Omega u^r dx\right)^{\frac{p}{r}}$, $r \ge 1$ and $p > 1$ for problem (1.1)-(1.4), using the analogous proof as Theorems 2.2 and 3.2, we can obtain the following results:

Theorem 4.4. *Assume that the conditions* (a_1), (a_2) *hold, and* u *is a non-negative solution of problem* (4.4)-(4.6). *Then, we conclude that the solutions* u *blow up in finite time* $t = t^*$. *Moreover, the bounds for blow-up time* t^* *are given by*

$$\frac{2\varphi(0)^{\frac{1-p}{2}}}{k_7(p-1)} \le t^* \le \frac{\phi_2(0)^{1-p}}{k_6(p-1)},$$

and the estimates of the blow-up rate can be given by

$$\|u\|_{L^1} \le \left[k_6(p-1)\right]^{-\frac{1}{p-1}}(t^*-t)^{-\frac{1}{p-1}}, \tag{4.7}$$

$$\|u\|_{H^1} \ge \left[k_7(p-1)\right]^{-\frac{1}{p-1}}(t^*-t)^{-\frac{1}{p-1}}, \tag{4.8}$$

where $k_6 = \int_\Omega a(x)dx|\Omega|^{\frac{p(1-r)}{r}}$, $k_7 = \left(\int_\Omega a(x)^2 dx\right)^{\frac{1}{2}} \frac{B_2^p}{\sqrt{1+\lambda_1}}$, $\phi_2(0) = \int_\Omega g(x)dx$, $\varphi(0) = \|g\|_{H^1}^2$, *and* B_2 *is the optimal constant satisfying the Sobolev's inequality* $\|u\|_r \le B_2\|u\|_{H^1}$.

Model 4.5. *We will consider a related model as follows*:

$$u_t - \triangle u - \triangle u_t = a(x)\int_\Omega u^p dx - u^q,\ x \in \Omega,\ t > 0, \tag{4.9}$$

$$u(x,0) = g(x) \ge 0,\ x \in \Omega, \tag{4.10}$$

$$\frac{\partial u}{\partial \nu} = 0,\ x \in \partial\Omega,\ t > 0, \tag{4.11}$$

where the parameters p, q *satisfy* $p > q$ *and* $1 < p < +\infty$ *if* $n \le 2$, $1 < p < \frac{2n}{n-2}$ *if* $n \ge 3$.

Due to the present of the weighted nonlocal source $a(x)\int_\Omega u^p dx$ and absorbtion term $-u^q$ in (4.9), we cannot directly apply the results which were obtained in the sections above. However, we can use the similar arguments with slight modification to established the following estimates for blow-up time and blow-up rate:

Theorem 4.6. *Assume that the conditions* (a_1), (a_2) *hold, and* u *is a non-negative solution of problem* (4.9)-(4.11). *Then, we conclude that the solutions* u *blow up in finite time* $t = t^*$. *Moreover, the bounds for blow-up time* t^* *are given by*

$$\int_{\varphi(0)}^{+\infty} \frac{d\eta}{k_9\eta^{\frac{p+1}{2}} - k_{10}\eta^{\frac{q+1}{2}}} \le t^* \le \int_{\phi_2(0)}^{+\infty} \frac{d\eta}{k_8\eta^p - C(\varepsilon)|\Omega|},$$

and the estimates of the blow-up rate can be given by

$$\|u\|_{L^1} \leq Y_4^{-1}(t^* - t), \tag{4.12}$$

$$\|u\|_{H^1} \geq \left[k_9(p-1)\right]^{-\frac{1}{p-1}}(t^* - t)^{-\frac{1}{p-1}}, \tag{4.13}$$

where $k_8 = \left(\int_\Omega a(x)dx - \varepsilon\right)|\Omega|^{-(p-1)}$, $k_9 = \left(\int_\Omega a(x)^2 dx\right)^{\frac{1}{2}} \frac{B_3^p}{\sqrt{1+\lambda_1}}$, $k_{10} = |\Omega|^{-\frac{q-1}{2}} \left(\frac{1}{1+\lambda_1}\right)^{\frac{q+1}{2}}$, $Y_4(\phi_2(t)) = \int_{\phi_2(t)}^{+\infty} \frac{d\eta}{k_8\eta^p - C(\varepsilon)|\Omega|}$ *and* B_3 *is the optimal constant satisfying the Sobolev's inequality* $\|u\|_p \leq B_3\|u\|_{H^1}$.

Remark 4.7. The estimates on the lower bound for blow-up time and blow-up rate of Theorem 4.4 (or Theorem 4.6) can be applied to discuss Eq. (4.4) (or Eq. (4.9)) subject to Dirichlet boundary condition $u(x,t) = 0$, $x \in \partial\Omega$. We can also obtain the inequality (4.8) (or inequality (4.13)).

Acknowledgements. The authors would like to express their thanks to the editors and the referees for their helpful comments. Dr. Huafei Di also specially appreciates Prof. Yue Liu for his invitation of visiting to the University of Texas at Arlington.

References

[1] A. B. Al'shin, M. O. Korpusov, and A. G. Sveshnikov, *Blow-up in nonlinear Sobolev type equations*, De Gruyter Series in Nonlinear Analysis and Applications, 15, Walter de Gruyter & Co., Berlin, 2011. https://doi.org/10.1515/9783110255294

[2] J. Bebernes and A. Bressan, *Thermal behavior for a confined reactive gas*, J. Differential Equations **44** (1982), no. 1, 118–133. https://doi.org/10.1016/0022-0396(82)90028-6

[3] H. Di and Y. Shang, *Global well-posedness for a nonlocal semilinear pseudo-parabolic equation with conical degeneration*, J. Differential Equations **269** (2020), no. 5, 4566–4597. https://doi.org/10.1016/j.jde.2020.03.030

[4] H. Di, Y. Shang, and X. Peng, *Blow-up phenomena for a pseudo-parabolic equation with variable exponents*, Appl. Math. Lett. **64** (2017), 67–73. https://doi.org/10.1016/j.aml.2016.08.013

[5] E. DiBenedetto and M. Pierre, *On the maximum principle for pseudoparabolic equations*, Indiana Univ. Math. J. **30** (1981), no. 6, 821–854. https://doi.org/10.1512/iumj.1981.30.30062

[6] E. S. Dzektser, *Generalization of equations of motion of underground water with free surface*, Dokl. Akad. Nauk SSSR. **202** (1972), no. 5, 1031–1033.

[7] M. Escobedo and M. A. Herrero, *A semilinear parabolic system in a bounded domain*, Ann. Mat. Pura Appl. **165** (1993), no. 1, 315–336. https://doi.org/10.1007/BF01765854

[8] Z. Fang and Y. Wang, *Blow-up analysis for a semilinear parabolic equation with time-dependent coefficients under nonlinear boundary flux*, Z. Angew. Math. Phys. **66** (2015), no. 5, 2525–2541. https://doi.org/10.1007/s00033-015-0537-7

[9] M. O. Korpusov and A. G. Sveshnikov, *Three-dimensional nonlinear evolution equations of pseudo-parabolic type in problems of mathematicial physics*, Comput. Math. Math. Phys. **43** (2003), no. 12, 1765–1797; translated from Zh. Vychisl. Mat. Mat. Fiz. **43** (2003), no. 12, 1835–1869.

[10] J.-L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, 1969.

[11] Y. Liu, *Lower bounds for the blow-up time in a non-local reaction diffusion problem under nonlinear boundary conditions*, Math. Comput. Modelling **57** (2013), no. 3-4, 926–931. https://doi.org/10.1016/j.mcm.2012.10.002

[12] Y. Liu, W. Jiang, and F. Huang, *Asymptotic behaviour of solutions to some pseudoparabolic equations*, Appl. Math. Lett. **25** (2012), no. 2, 111–114. https://doi.org/10.1016/j.aml.2011.07.012

[13] P. Luo, *Blow-up phenomena for a pseudo-parabolic equation*, Math. Methods Appl. Sci. **38** (2015), no. 12, 2636–2641. https://doi.org/10.1002/mma.3253

[14] X. Lv and X. Song, *Bounds of the blowup time in parabolic equations with weighted source under nonhomogeneous Neumann boundary condition*, Math. Methods Appl. Sci. **37** (2014), no. 7, 1019–1028. https://doi.org/10.1002/mma.2859

[15] L. Ma and Z. Fang, *Blow-up analysis for a reaction-diffusion equation with weighted nonlocal inner absorptions under nonlinear boundary flux*, Nonlinear Anal. Real World Appl. **32** (2016), 338–354. https://doi.org/10.1016/j.nonrwa.2016.05.005

[16] L. E. Payne, G. A. Philippin, and S. Vernier Piro, *Blow-up phenomena for a semilinear heat equation with nonlinear boundary condition, I*, Z. Angew. Math. Phys. **61** (2010), no. 6, 999–1007. https://doi.org/10.1007/s00033-010-0071-6

[17] ______, *Blow-up phenomena for a semilinear heat equation with nonlinear boundary condition, II*, Nonlinear Anal. **73** (2010), no. 4, 971–978. https://doi.org/10.1016/j.na.2010.04.023

[18] L. E. Payne and P. W. Schaefer, *Lower bounds for blow-up time in parabolic problems under Neumann conditions*, Appl. Anal. **85** (2006), no. 10, 1301–1311. https://doi.org/10.1080/00036810600915730

[19] ______, *Lower bounds for blow-up time in parabolic problems under Dirichlet conditions*, J. Math. Anal. Appl. **328** (2007), no. 2, 1196–1205. https://doi.org/10.1016/j.jmaa.2006.06.015

[20] X. Peng, Y. Shang, and X. Zheng, *Blow-up phenomena for some nonlinear pseudo-parabolic equations*, Appl. Math. Lett. **56** (2016), 17–22. https://doi.org/10.1016/j.aml.2015.12.005

[21] M. Peszyńska, R. Showalter, and S.-Y. Yi, *Homogenization of a pseudoparabolic system*, Appl. Anal. **88** (2009), no. 9, 1265–1282. https://doi.org/10.1080/00036810903277077

[22] J. Rubinstein and P. Sternberg, *Nonlocal reaction-diffusion equations and nucleation*, IMA J. Appl. Math. **48** (1992), no. 3, 249–264. https://doi.org/10.1093/imamat/48.3.249

[23] R. E. Showalter, *Existence and representation theorems for a semilinear Sobolev equation in Banach space*, SIAM J. Math. Anal. **3** (1972), 527–543. https://doi.org/10.1137/0503051

[24] S. L. Sobolev, *On a new problem of mathematical physics*, Izvestiya Rossiiskoi Akademii Nauk. Seriya Matematicheskaya **18** (1954), no. 1, 3–50.

[25] J. C. Song, *Lower bounds for the blow-up time in a non-local reaction-diffusion problem*, Appl. Math. Lett. **24** (2011), no. 5, 793–796. https://doi.org/10.1016/j.aml.2010.12.042

[26] X. Song and X. Lv, *Bounds for the blowup time and blowup rate estimates for a type of parabolic equations with weighted source*, Appl. Math. Comput. **236** (2014), 78–92. https://doi.org/10.1016/j.amc.2014.03.023

[27] G. Tang, Y. Li, and X. Yang, *Lower bounds for the blow-up time of the nonlinear nonlocal reaction diffusion problems in $\mathbb{R}^N$ $(N \geq 3)$*, Bound. Value Probl. **2014** (2014), 265, 5 pp. https://doi.org/10.1186/s13661-014-0265-5

[28] R. Xu and J. Su, *Global existence and finite time blow-up for a class of semilinear pseudo-parabolic equations*, J. Funct. Anal. **264** (2013), no. 12, 2732–2763. https://doi.org/10.1016/j.jfa.2013.03.010

Huafei Di
School of Mathematics and Information Science
Guangzhou University
Guangzhou 510006, P. R. China
and
Department of Mathematics
University of Texas at Arlington
Texas 76019, USA
Email address: dihuafei@yeah.net

Yadong Shang
School of Mathematics and Information Science
Guangzhou University
Guangzhou 510006, P. R. China
Email address: gzydshang@126.com